On the Usage of Orthogonal Polynomials with Picard Iteration Method to Find Numerical Solutions of Neutrosophic Non-Linear Ordinary and Partial Differential Equations

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Abstract

This research aims to modify the Picard iteration method by hybridizing it with some orthogonal polynomials and then applying the hybrid method in solving neutrosophic nonlinear elementary value problems. This method is based on modifying the Picard iteration method by approximating the right-hand side of the neutrosophic differential equation of the studied problem either by Legendre polynomials or by Chebyshev polynomials of the first kind to obtain two different hybrids of the Picard iteration method. Also, we apply this modification to neutrosophic elementary value problems represented by neutrosophic nonlinear and right-handed nonlinear differential equations to demonstrate the reliability and efficiency of the proposed modified method. For this goal, we prove how effective this method is, we calculate the neutrosophic absolute error of approximate solutions resulting from the application of the proposed modification of the Picard iteration method and with the exact solution.

Keywords: Orthogonal polynomials; Picard Method; neutrosophic partial differential equation; numerical solutions; neutrosophic approximation.

1. Introduction

The Picard iteration method was proposed by the French researcher Joseph Liouville in 1838 and later developed by Picard to solve elementary value problems represented by ordinary differential equations [1]. Many researchers have worked to modify and improve this method as it has been used to solve higher limit and elementary value problems [2].

Some methods rely on orthogonal polynomials to solve boundary value problems, and these polynomials are Legendre polynomials [3] and Chebyshev polynomials [4], which have many applications. In [5], authors hybridized the method of Adomian differentiation with Chebyshev polynomials, and in [6] authors hybridized the method of Adomian differentiation with Legendre polynomials, and in [7-8], the authors hybridized both the method of perturbation homotopy and the method of metathesis with Chebyshev polynomials to solve elementary value problems represented by nonlinear Ordinary Differential Equations, by approximating the right-hand side of the differential equation. In [9], the authors modified the Picard iteration method using Chebyshev polynomials of the first kind to solve the Sine-Gordon problem.

Neutrosophic logic was presented by Smarandache [10], and then the idea of inserting an element that refers to indeterminacy into algebraic structures was used by many authors to derive many interesting results that are related to matrix theory, vector space theory, and also algebraic structures [11].

In this work, we propose to modify the Picard iteration method by using Chebyshev polynomials of the first kind and Legendre polynomials of the right-hand side separately, which enables us to overcome the difficulty of calculating neutrosophic integrals related to the Picard iteration method, especially when applied to the nonlinear right-hand side.

2. Main discussion

Definition:

The Neutrosophic Picard iteration method can be summarized as follows:

Consider the neutrosophic differential equation:

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$F(x_1 + x_2I, y(x_1 + x_2I), y'(x_1 + x_2I), \dots, y^{(n)}(x_1 + x_2I)) = 0$ (1)

With the conditions:

$y(a_1 + a_2I) = A_0, y'(a_1 + a_2I) = A_1, \dots, y^{n-1}(a_1 + a_2I) = A_{n-1}$ (2)

If it is solvable concerning the derivative, then:

$y^{(n)}(x_1 + x_2I) = F(x_1 + x_2I, y(x_1 + x_2I), y'(x_1 + x_2I), \dots, y^{(n-1)}(x_1 + x_2I))$ (3)

The Picard formula is:

$y_{m+1}(x_1 + x_2I) = \frac{1}{(n-1)!}\int_a^{x_1+x_2I}(x_1 + x_2I - t)^{n-1}\left(F(t, y_m(t), y'_m(t), \dots, y_m^{(n-1)}(t))\right)dt + \sum_{i=0}^{n-1}\frac{A_i(x_1+x_2I)^i}{i!}$

(4)

$y(x_1 + x_2I) = \lim_{m\to\infty} y_m(x_1 + x_2I)$ (5)

Also,

$y_0(x_1 + x_2I) = \sum_{i=0}^{n-1}\alpha_i\frac{(x_1+x_2I-a)^i}{i!}$ (6)

$Where : \alpha_i, i = 0,1,2,\dots,n-1$

Definition:

The neutrosophic Legendre polynomials of order n are defined as solutions to the following neutrosophic differential equation:

$(1 - x^2)y''(x_1 + x_2I) - 2xy'(x_1 + x_2I) + n(1 + n)y(x_1 + x_2I) = 0$ (7)

On the other hand, we have:

$f(x_1 + x_2I) \approx \sum_{i=0}^{N}\alpha_iP_i(x_1 + x_2I)$ (8)

and

$\alpha_i = (i + \frac{1}{2})\int_{-1}^{1} f(x_1 + x_2Ix)P_i(x_1 + x_2I)dx \qquad i = 0,1,\dots$ (9)

thus

$P_0(x_1 + x_2I) = 1$

$P_1(x_1 + x_2I) = x_1 + x_2I$ (10)

$P_n(x_1 + x_2I) = \frac{2n-1}{n}xP_{n-1}(x_1 + x_2I) - \frac{n-1}{n}P_{n-2}(x_1 + x_2I) \qquad n \ge 2, \quad -1 \le x \le 1$

$P_n^*(x_1 + x_2I) = P_n(2x_1 + x_2I - 1); \qquad 0 \le x_1 + x_2I \le 1$ (11)

Definition:

The neutrosophic Chybechev polynomials are defined as solutions to the following neutrosophic differential equation:

$(1 - (x_1 + x_2I)^2)y'' - (x_1 + x_2I)y' + P^2y = 0$, where P is a real neutrosophic number.

$f(x_1 + x_2I) \approx \sum_{j=0}^{n} C_jT_j(x_1 + x_2I)$ (12)

and

$C_j = \frac{2}{\pi}\int_{-1}^{1}\frac{f(x_1+x_2I)T_j(x_1+x_2I)}{\sqrt{1-(x_1+x_2I)^2}}d(x_1 + x_2I) \qquad n = 0,1,\dots$ (13)

On the other hand, we have

$T_{n+1}(x_1 + x_2I) = 2[x_1 + x_2I]T_n(x_1 + x_2I) - T_{n-1}(x_1 + x_2I) \quad ; \quad n \ge 1$ (14)

$T_0(x_1 + x_2I) = 1$

$T_n^*(x_1 + x_2I) = T_n(2[x_1 + x_2I] - 1); \qquad 0 \le x_1 + x_2I \le 1$ (15)

3. A proposal to modify the neutrosophic Picard iteration method using some neutrosophic orthogonal polynomials to solve neutrosophic non-linear elementary value problems:

This modification boils down to writing the Picard recurrence relation for the nonlinear differential equation of the studied problem, and then approximating the nonlinear right-hand side with orthogonal polynomials either Chebyshev polynomials of the first kind or Legendre polynomials, and applying successive Picard iterations to obtain approximations $y_1(x_1 + x_2I), \dots, y_m(x_1 + x_2I)$, then the solution is given by:

$y(x_1 + x_2I) = \lim_{m\to\infty} y_m(x_1 + x_2I)$

Problem (1):

Consider the following equation*:*

$u'' + (x_1 + x_2I)u' + (x_1 + x_2I)^2u^3 = f(x_1 + x_2I) = (2 + 6(x_1 + x_2I)^2)e^{(x_1+x_2I)^2} + (x_1 + x_2I)^2e^{3(x_1+x_2I)^2}; \ 0 \le x_1 + x_2I \le 1$

$\begin{cases} u(0) = 1 \\ u'(0) = 0 \end{cases}$ (16)

The exact solution is:

$u(x_1 + x_2I) = e^{(x_1+x_2I)^2}$

The solution by using the new method is:

$f_L(x_1 + x_2I) \approx 9.336081230P_0(x_1 + x_2I) + 14.02513857P_1(x_1 + x_2I) + 10.08707306P_2(x_1 + x_2I) +$

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$4.960989652P_3(x_1+x_2I)+\ldots$

By using (11), we get

$$f_L(x_1+x_2I)\approx 2.006009275-0.5160346220(x_1+x_2I)+19.72835932(x_1+x_2I)^2 -93.37317421(x_1+x_2I)^3+426.4047168(x_1+x_2I)^4-1035.684385(x_1+x_2I)^5 +1467.426417(x_1+x_2I)^6-1089.603457(x_1+x_2I)^7+354.4332362(x_1+x_2I)^8$$

Where,

$u_0(x_1+x_2I)=u(0)+u_{x_1+x_2I}(0)(x_1+x_2I)=1 \Rightarrow u_0(x_1+x_2I)=1$

By using Picard formula, we get:

$$u_{m+1}(x_1+x_2I)=u_0(x_1+x_2I)+\int_0^{x_1+x_2I}(x_1+x_2I-t)(f(t)-tu'_m(t)-t^2{u_m}^3(t))dt \quad m\geq 0$$

$$u_1(x_1+x_2I)=u_0(x_1+x_2I)+\int_0^{x_1+x_2I}(x_1+x_2I-t)(f(t)-t\,u'_m-t^2{u_m}^3)dt$$

$$=1+\int_0^{x_1+x_2I}(x_1+x_2I-t)(f(t)-t\,u'_m-t^2{u_0}^3)dt$$

$$u_1(x_1+x_2I)=1+1.003004638(x_1+x_2I)^2-0.08600577035(x_1+x_2I)^3+1.560696609(x_1+x_2I)^4 -4.668658710(x_1+x_2I)^5+14.21349055(x_1+x_2I)^6-24.65915201(x_1+x_2I)^7 +26.20404322(x_1+x_2I)^8-15.25838136(x_1+x_2I)^9+3.938147065(x_1+x_2I)^{10}$$

$$u_2(x_1+x_2I)=1+1.003004638(x_1+x_2I)^2-0.08600577035(x_1+x_2I)^3+1.393529169(x_1+x_2I)^4 -4.655757845(x_1+x_2I)^5+13.90509720(x_1+x_2I)^6-24.09721604(x_1+x_2I)^7 +24.54366665(x_1+x_2I)^8-12.65924767(x_1+x_2I)^9+1.019298567(x_1+x_2I)^{10} +2.186035908(x_1+x_2I)^{11}-1.651362265(x_1+x_2I)^{12}+1.667475110(x_1+x_2I)^{13} -2.381320407(x_1+x_2I)^{14}+4.098899339(x_1+x_2I)^{15}-7.802427477(x_1+x_2I)^{16} +13.69189012(x_1+x_2I)^{17}-20.75306524(x_1+x_2I)^{18}+27.23904657(x_1+x_2I)^{19} -33.13519761x^{20}+42.3878789(x_1+x_2I)^{21}-60.33184336(x_1+x_2I)^{22} +92.68357048(x_1+x_2I)^{23}-136.1053178(x_1+x_2I)^{24}+177.0025245(x_1+x_2I)^{25} -196.3231584(x_1+x_2I)^{26}+181.9507191(x_1+x_2I)^{27}-138.6481922(x_1+x_2I)^{28} +85.30157500(x_1+x_2I)^{29}-41.34095153(x_1+x_2I)^{30}+15.21216318(x_1+x_2I)^{31} -4.001827288(x_1+x_2I)^{32}+0.6722791831(x_1+x_2I)^{33}-0.05443558969(x_1+x_2I)^{34}$$

$$u_3((x_1+x_2I))=1+1.003004638(x_1+x_2I)^2-0.08600577035(x_1+x_2I)^3+1.393539169(x_1+x_2I)^4+\ldots$$

So that, the approximate solution is:

$$u_L((x_1+x_2I))=\lim_{n\to\infty}u_n((x_1+x_2I))=u_3((x_1+x_2I))=1+1.003004638(x_1+x_2I)^2-\ldots$$

By using the other method, we get:

$$f_C((x_1+x_2I))\approx 12.194T_0((x_1+x_2I))+16.097T_1((x_1+x_2I))+8.3083T_2((x_1+x_2I))+3.3455T_3((x_1+x_2I))+\ldots$$

By using (15), we get:

$$f_C(x)\approx 2.0024-0.36160(x_1+x_2I)+18.046(x_1+x_2I)^2-86.480(x_1+x_2I)^3+416.59(x_1+x_2I)^4- 1042.7(x_1+x_2I)^5+1502.7(x_1+x_2I)^6-1134.6(x_1+x_2I)^7+366.62(x_1+x_2I)^8$$

$$u_1((x_1+x_2I))=1+1.001200000(x_1+x_2I)^2-0.06026666668(x_1+x_2I)^3+1.420499999(x_1+x_2I)^4 -4.324000000(x_1+x_2I)^5+13.88633332(x_1+x_2I)^6-24.82619047(x_1+x_2I)^7 +26.83392864(x_1+x_2I)^8-15.75833335(x_1+x_2I)^9+4.073555551(x_1+x_2I)^{10}$$

$$u_2((x_1+x_2I))=1.001200000(x_1+x_2I)^2-0.06026666668(x_1+x_2I)^3+1.253633333(x_1+x_2I)^4 -4.314960000(x_1+x_2I)^5+13.59681332(x_1+x_2I)^6-24.30712380(x_1+x_2I)^7 +25.21630892(x_1+x_2I)^8-13.15948102(x_1+x_2I)^9+1.119353808(x_1+x_2I)^{10} +2.208850831(x_1+x_2I)^{11}-1.640568070(x_1+x_2I)^{12}+1.614123597(x_1+x_2I)^{13} -2.232084719(x_1+x_2I)^{14}+3.777245835(x_1+x_2I)^{15}-7.295962756(x_1+x_2I)^{16} +13.11411141(x_1+x_2I)^{17}-20.25424593(x_1+x_2I)^{18}+26.75490528(x_1+x_2I)^{19} -32.19320732(x_1+x_2I)^{20}+39.89422470(x_1+x_2I)^{21}-56.52523387(x_1+x_2I)^{22} +87.79879258(x_1+x_2I)^{23}-132.0928305(x_1+x_2I)^{24}+176.3608847(x_1+x_2I)^{25} -200.3428474(x_1+x_2I)^{26}+189.5035538(x_1+x_2I)^{27}-146.8655810(x_1+x_2I)^{28} +91.60319237(x_1+x_2I)^{29}-44.87750388(x_1+x_2I)^{30}+16.64977297(x_1+x_2I)^{31} -4.405779101(x_1+x_2I)^{32}+0.7428735704(x_1+x_2I)^{33}-0.06024597916(x_1+x_2I)^{34}$$

$$u_3((x_1+x_2I))=1+1.001200000(x_1+x_2I)^2+1.253633333(x_1+x_2I)^4+\ldots$$

So that, the approximate solution is given by:

$$u_C((x_1+x_2I))=\lim_{n\to\infty}u_n((x_1+x_2I))=u_3((x_1+x_2I))=1+1.001200000(x_1+x_2I)^2+\ldots$$

Table (1)

(x_1+x_2I)	Exact $u((x_1+x_2I))$	Chebyshev $u((x_1+x_2I))$	Legendre $\mathbf{u}((x_1+x_2I))$	Error of MLPI	Error of MCPI

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0	1	1	1	0	0
0.1	1.010050167	1.010045375	0.010048586	$1.579929830 \times 10^{-6}$	$4.79219779 \times 10^{-6}$
0.2	1.040810774	1.040809428	1.040811418	$6.448670927 \times 10^{-7}$	1.3464586×10^{-6}
0.3	1.094174284	1.094169542	1.094177747	$3.465353156 \times 10^{-6}$	$4.741948990 \times 10^{-6}$
0.4	1.173510871	1.173489818	1.173508313	$2.559083359 \times 10^{-6}$	$2.105251811 \times 10^{-5}$
0.5	1.284025417	1.284013221	1.284037987	$1.256847321 \times 10^{-5}$	$1.219580685 \times 10^{-5}$
0.6	1.433329415	1.433369437	1.433405286	$7.587029910 \times 10^{-5}$	$4.002628455 \times 10^{-5}$
0.7	1.632316220	1.632558388	1.632626418	$3.101924850 \times 10^{-4}$	$2.421687506 \times 10^{-4}$
0.8	1.896480879	1.897630710	1.987756407	$1.27552641 \times 10^{-3}$	1.1498213×10^{-3}
0.9	2.247907987	2.25278657	2.253000862	5.093162×10^{-3}	4.879115×10^{-3}
1	2.718281828	2.73982050	2.73981	2.1585472×10^{-2}	2.1428672×10^{-2}

Problem (2):

$u" + u'u = f((x_1 + x_2I)) = (x_1 + x_2I)\sin(2(x_1 + x_2I)^2) - 4(x_1 + x_2I)^2 \sin((x_1 + x_2I)^2) + 2\cos((x_1 + x_2I)^2)$; $0 \leq (x_1 + x_2I) \leq 1$

$$\begin{cases} u(0) = 0 \\ u'(0) = 0 \end{cases} \qquad (17)$$

The exact solution is:

$u((x_1 + x_2I)) = \sin((x_1 + x_2I)^2)$

By using the new method, we get:

$f_L((x_1 + x_2I)) \approx 1.434641321P_0((x_1 + x_2I)) - 1.243506674P_1((x_1 + x_2I)) - 1.051935847P_2((x_1 + x_2I)) - \ldots$

By using (11), we get:

$$f_L(x) \approx 1.999985935 + 0.0008505927600x - 0.01048801544x^2 + 2.027262191x^3 - 4.811932251x^4 - 1.359408205x^5 + 3.480443969x^6 - 5.546970002x^7 + 2.844270001x^8$$

$u_{m+1}((x_1 + x_2I)) = u_0((x_1 + x_2I)) + \int_0^{(x_1+x_2I)}[((x_1 + x_2I) - t)(f(t) - u'_m(t)u_m(t)]dt \qquad m \geq 0$

$u_0((x_1 + x_2I)) = u(0) + u_{(x_1+x_2I)}(0)(x_1 + x_2I) = 0 \Longrightarrow u_0((x_1 + x_2I)) = 0$

$u_1((x_1 + x_2I)) = u_0((x_1 + x_2I)) + \int_0^{(x_1+x_2I)}((x_1 + x_2I) - t)(f(t) - u'_0u_0)dt$

$= 0 + \int_0^{(x_1+x_2I)}((x_1 + x_2I) - t)f(t)dt$

$u_1((x_1 + x_2I)) = 0.9999929680(x_1 + x_2I)^2 + 0.0001417647934(x_1 + x_2I)^3 - 0.0008740009530(x_1 + x_2I)^4 + 0.1013631096(x_1 + x_2I)^5 - 0.1603977415(x_1 + x_2I)^6 - 0.03236686201(x_1 + x_2I)^7 + 0.06215078531(x_1 + x_2I)^8 - 0.07704125009(x_1 + x_2I)^9 + 0.03160299998(x_1 + x_2I)^{10}$

$u_2((x_1 + x_2I)) = 0.9999929680(x_1 + x_2I)^2 + 0.0001417647934(x_1 + x_2I)^3 - 0.0008740009530(x_1 + x_2I)^4 + 0.001364515945(x_1 + x_2I)^5 - 0.1604213688(x_1 + x_2I)^6 - 0.03224200703x^7 + 0.04948050117(x_1 + x_2I)^8 - 0.05922104320(x_1 + x_2I)^9 + 0.006129380687(x_1 + x_2I)^{11} + 0.007771835232(x_1 + x_2I)^{12} - 0.003163111394(x_1 + x_2I)^{13} - 0.0008259416240(x_1 + x_2I)^{14} + 0.001152120030(x_1 + x_2I)^{15} - 0.0008468121865(x_1 + x_2I)^{16} + 0.00003788860314(x_1 + x_2I)^{17} + 0.0003228368957(x_1 + x_2I)^{18} - 0.0002595699146(x_1 + x_2I)^{19} + 0.0001217367314(x_1 + x_2I)^{20} - 0.00002377975255(x_1 + x_2I)^{21}$

$u_3((x_1 + x_2I)) = 0.9999929680(x_1 + x_2I)^2 + 0.0001417647934(x_1 + x_2I)^3 + \ldots$

The approximate solution is:

$u_3((x_1 + x_2I)) = \lim_{n\to\infty} u_n((x_1 + x_2I)) = u_3((x_1 + x_2I)) = 0.9999929680(x_1 + x_2I)^2 + \ldots$

By using the other method:

$f_C((x_1 + x_2I)) \approx 1.1614T_0((x_1 + x_2I)) - 1.4118T_1((x_1 + x_2I)) - 0.81172T_2((x_1 + x_2I)) - \ldots$

From (15), we can get:

$f_C((x_1 + x_2I)) \approx 2.0024 - 0.36160(x_1 + x_2I) + 18.046(x_1 + x_2I)^2 - 96.480(x_1 + x_2I)^3 - 1042.7(x_1 + x_2I)^5 + 1502.7(x_1 + x_2I)^6 - 1134.6(x_1 + x_2I)^7 + 366.62(x_1 + x_2I)^8$

$u_1((x_1 + x_2I)) = (x_1 + x_2I)^2 + 0.00002033333334(x_1 + x_2I)^3 - 0.0006269999997(x_1 + x_2I)^4 + 0.09458500000(x_1 + x_2I)^5 - 0.1435699999(x_1 + x_2I)^6 - 0.05712619045(x_1 + x_2I)^7 + 0.08361071449(x_1 + x_2I)^8 - 0.08716666674(x_1 + x_2I)^9 + 0.03360999997(x_1 + x_2I)^{10}$

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$$u_2((x_1+x_2I)) = (x_1+x_2I)^2 + 0.00002033333334(x_1+x_2I)^3 - 0.0006269999997(x_1+x_2I)^4 + 0.09458500000(x_1+x_2I)^5 - 0.1435713554(x_1+x_2I)^6 - 0.05715604762(x_1+x_2I)^7 + 0.08023267717(x_1+x_2I)^8 - 0.08317884672(x_1+x_2I)^9 + 0.03487383231(x_1+x_2I)^{10} - 0.001918556732(x_1+x_2I)^{11} + 0.002455180946(x_1+x_2I)^{12} - 0.0008119392230(x_1+x_2I)^{13} - 0.001146854034(x_1+x_2I)^{14} + 0.001239725010(x_1+x_2I)^{15} - 0.0006823224118(x_1+x_2I)^{16} - 0.0002146757473(x_1+x_2I)^{17} + 0.0005115599183(x_1+x_2I)^{18} - 0.0003478510531(x_1+x_2I)^{19} + 0.0001464835835(x_1+x_2I)^{20} - 0.00002689600230(x_1+x_2I)^{21}$$

$$u_3((x_1+x_2I)) = (x_1+x_2I)^2 + 0.005415000000(x_1+x_2I)^5 + \ldots$$

Hence, the approximate solution is:

$$u_C((x_1+x_2I)) = \lim_{n\to\infty} u_n((x_1+x_2I)) = u_3((x_1+x_2I)) = (x_1+x_2I)^2 + 0.005415000000(x_1+x_2I)^5 + \ldots$$

Table (2)

$(x_1 + x_2I)$	Exact $u((x_1+x_2I))$	Chebshev $u((x_1+x_2I))$	Legendre $\mathbf{u}((x_1+x_2I))$	Error of MLPI	Error of MCPI
0	0	0	0	0	0
0.1	0.009999833334	0.009999880239	0.009999834608	1.2747777×10^{-9}	4.6905×10^{-8}
0.2	0.03998933419	0.03998966323	0.03998934326	9.07465×10^{-9}	3.2903×10^{-7}
0.3	0.08987854920	0.08987882174	0.08987853331	1.58994×10^{-8}	2.7255×10^{-7}
0.4	0.1593182066	0.1593123753	0.1593182523	4.57260×10^{-8}	5.8310×10^{-6}
0.5	0.2474039593	0.2473597072	0.2474045291	5.69702×10^{-7}	4.42523×10^{-5}
0.6	0.3522742333	0.3520753678	0.3522782188	3.985574×10^{-6}	1.988654×10^{-4}
0.7	0.4706258882	0.4699456719	0.4706470020	2.111374×10^{-5}	6.802162×10^{-4}
0.8	0.5971954414	0.5952669513	0.5972831418	8.770031×10^{-5}	1.9284903×10^{-3}
0.9	0.7242871744	0.7195576557	0.7245865513	2.9937685×10^{-4}	4.7295188×10^{-3}
1	0.8414709848	0.8311847630	0.8423378984	8.669154×10^{-4}	1.02862218×10^{-2}

Problem (3):

Consider the following problem:

$$u' - u = f((x_1+x_2I)) = (x_1+x_2I)\cos((x_1+x_2I)) - (x_1+x_2I)\sin((x_1+x_2I)) + \sin((x_1+x_2I)) \quad ; \quad 0 \le (x_1+x_2I) \le 1 \qquad (18)$$

$u(0) = 0$

The exact solution is:

$$u((x_1+x_2I)) = (x_1+x_2I)\sin((x_1+x_2I))$$

We use (MLPI) method:

$$f_L((x_1+x_2I)) \approx 0.5403023060P_0((x_1+x_2I)) + 0.2814412647P_1((x_1+x_2I)) - 0.2737136145P_2((x_1+x_2I)) + \ldots$$

By using (11) , we get:

$$f_L((x_1+x_2I)) \approx -0.002542924450 + 2.012337265(x_1+x_2I) - 1.083834683(x_1+x_2I)^2 + 0.2682142469(x_1+x_2I)^3 - 2.874245062(x_1+x_2I)^4 + 3.778263720(x_1+x_2I)^5 - 1.911034125(x_1+x_2I)^6 + 0.7722000000(x_1+x_2I)^7 - 0.4273242188(x_1+x_2I)^8$$

The corresponding Picard formula is:

$$u_{m+1}((x_1+x_2I)) = u_0((x_1+x_2I)) + \int_0^{(x_1+x_2I)} (f(t) + u_m(t))dt \quad m \ge 0$$

So that:

$$u_0((x_1+x_2I)) = u(0) = 0 \Longrightarrow u_0(x_1+x_2I)) = 0$$

$$u_1((x_1+x_2I)) = u_0((x_1+x_2I)) + \int_0^{(x_1+x_2I)} (f(t) + u_1(t))dt$$

$$u_1((x_1+x_2I)) = -0.0025429(x_1+x_2I) + 1.0062(x_1+x_2I)^2 - 0.36128(x_1+x_2I)^3 + 0.067054(x_1+x_2I)^4 - 0.57485(x_1+x_2I)^5 + 0.62971(x_1+x_2I)^6 - 0.27300(x_1+x_2I)^7 + 0.096525(x_1+x_2I)^8 - 0.047480(x_1+x_2I)^9$$

$$u_2((x_1+x_2I)) = -0.0025429x + 1.0049x^2 - 0.025867(x_1+x_2I)^3 - 0.023268(x_1+x_2I)^4 - 0.56142(x_1+x_2I)^5 + 0.53390x^6 - 0.18304(x_1+x_2I)^7 + 0.062400(x_1+x_2I)^8 - 0.036756x^9 - 0.0047480(x_1+x_2I)^{10}$$

$$u_3((x_1+x_2I)) = -0.0025429(x_1+x_2I) + 1.0049(x_1+x_2I)^2 - 0.026300(x_1+x_2I)^3 - 0.060585(x_1+x_2I)^4 - 0.57950(x_1+x_2I)^5 + 0.53615(x_1+x_2I)^6 - 0.19673(x_1+x_2I)^7 + 0.073645(x_1+x_2I)^8 - 0.040547(x_1+x_2I)^9 - 0.0036756(x_1+x_2I)^{10} - 0.00043164(x_1+x_2I)^{11}$$

The approximate solution is:

$$u_L((x_1+x_2I)) = \lim_{n\to\infty} u_n((x_1+x_2I)) = u_3((x_1+x_2I)) = -0.0025429(x_1+x_2I) + 1.0049(x_1+x_2I)^2 -$$

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$0.026300(x_1+x_2I)^3+\ldots$
$f_C((x_1+x_2I)) \approx 0.47237544T_0((x_1+x_2I)) + 0.27719746T_1((x_1+x_2I)) - 0.20417054T_2((x_1+x_2I))+\ldots$
$f_C((x_1+x_2I)) \approx 0.00017737487 + 1.9978716(x_1+x_2I) - 0.98935752(x_1+x_2I)^2 - 0.69504986(x_1+x_2I)^3 + 0.20925732(x_1+x_2I)^4 + 0.015878182(x_1+x_2I)^5 - 0.0031401222(x_1+x_2I)^6 - 0.0017211795(x_1+x_2I)^7 + 0.00028363064(x_1+x_2I)^8$

$u_1(x) = 0.00017737487(x_1+x_2I) + 0.99893580(x_1+x_2I)^2 - 0.32978584(x_1+x_2I)^3 - 0.17376246(x_1+x_2I)^4 + 0.041851464x^5 + 0.0026463637(x_1+x_2I)^6 + 0.00044858889(x_1+x_2I)^7 - 0.00021514744(x_1+x_2I)^8 + 0.0000315141516(x_1+x_2I)^9$
$u_2(x) = 0.00017737487(x_1+x_2I) + 0.99902450(x_1+x_2I)^2 + 0.0031927600(x_1+x_2I)^3 - 0.25620892(x_1+x_2I)^4 + 0.0070989720(x_1+x_2I)^5 + 0.0096216077(x_1+x_2I)^6 + 0.00082664084(x_1+x_2I)^7 - 0.00015907382(x_1+x_2I)^8 + 0.0000076092444(x_1+x_2I)^9 + 0.0000031514516(x_1+x_2I)^{10}$
$u_3((x_1+x_2I)) = 0.00017737487(x_1+x_2I) + 0.99902450x^2 + 0.0032223267(x_1+x_2I)^3 - 0.17296428(x_1+x_2I)^4 - 0.0093903200(x_1+x_2I)^5 + 0.0038295257x^6 + 0.0018231043(x_1+x_2I)^7 - 0.00011181733(x_1+x_2I)^8 + 0.000013839647(x_1+x_2I)^9 + 0.03487383231(x_1+x_2I)^{10} + 7.6092444 \times 10^{-7}(x_1+x_2I)^{10} + 2.8649560 \times 10^{-7}(x_1+x_2I)^{11}$
So that the approximate solution is:
$u_C((x_1+x_2I)) = \lim_{n\to\infty} u_n((x_1+x_2I)) = u_3((x_1+x_2I)) = 0.00017737487(x_1+x_2I) + 0.99902450(x_1+x_2I)^2+\ldots$
Table (3)

(x_1+x_2I)	Exact u$((x_1+x_2I))$	Chebshev u$((x_1+x_2I))$	Legendre **u**$((x_1+x_2I))$	Error of MLPI	Error of MCPI
0	0	0	0	0	0
0.1	0.0099833417	0.0099938185	0.0097691907	2.1415102×10^{-4}	1.0476794×10^{-5}
0.2	0.039733866	0.039742754	0.039420480	3.1338716×10^{-4}	8.8879791×10^{-6}
0.3	0.088656063	0.088631775	0.088402423	2.5363970×10^{-4}	2.4288563×10^{-5}
0.4	0.15576734	0.15561566	0.15561151	1.5582470×10^{-4}	1.5168052×10^{-4}
0.5	0.23971277	0.23921755	0.23938831	3.2445959×10^{-4}	4.9521694×10^{-4}
0.6	0.33878548	0.33753287	0.33765933	1.1261489×10^{-3}	1.2526021×10^{-3}
0.7	0.45095238	0.44823929	0.44812283	2.8295585×10^{-3}	2.7130945×10^{-3}
0.8	0.57388487	0.56861364	0.56833409	5.5507784×10^{-3}	5.2712245×10^{-3}
0.9	0.70499422	0.69555656	0.69548126	9.5129657×10^{-3}	9.4376742×10^{-3}
1	0.84147098	0.82562524	0.82555286	1.591812×10^{-2}	1.584574×10^{-2}

4. Conclusion

Through this research, it is clear to us how important the proposed methods are, which allowed solving neutrosophic ordinary and partial differential problems represented by linear and nonlinear neutrosophic differential equations with a strongly nonlinear second party by representing this party either with Chebyshev polynomials of the first type or with Legendre polynomials.

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