On the Non-Commutative Logical Rings As Novel Extensions of Neutrosophic Rings

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Abstract

This paper uses some logical algebraic elements to extend any ring into a non-commutative ring containing the original ring with many generalized substructures and special elements. On the other hand, we study the substructures of non-commutative logical rings such as AH-homomorphisms and AH-ideals with many examples that explain their algebraic validity. Also, we discuss the possibility of solving a linear Diophantine equation with two variables in the non-commutative logical ring of integers, where we present an easy algorithm to solve this kind of generalized Diophantine equation.

Keywords: neutrosophic ring; non-commutative; AH-ideal; AH-homomorphism.

1. Introduction

Algebraic ring theory is one of modern algebra's most broad and important branches of mathematics.
Ring extension plays a major role in modern algebraic theories and in solving algebraic equations.

It was customary to build ring expansions through algebraic elements and the roots of polynomials until Smarandache et al. [17] introduced the idea of using logical elements defined by specific algebraic properties in constructing new expansions and defining Neutrosophic algebraic structures of all kinds, including vector spaces, integers, modules, and even matrices [7-10, 18-19].These new ideas have been widely used by many researchers in expanding algebraic structures through plithogenic sets, which have been widely used in expanding rings, matrices, and even distance-preserving algebraic functions [1-6, 11-13].

On the other hand, some expansions related to real numbers that depend on fuzzy sets have been studied, and their algebraic properties and geometric constructions have been studied in many recently published research papers [14-16].

As a continuation of the extensive research efforts made previously, we have created a new expansion of rings based on algebraic elements of a logical nature, where we present the concept of non-commutative logical extension of a ring, with many elementary properties of this new generalization. Also, we discuss the solvability of linear Diophantine equations in two variables built over those rings in the integer case.

2. Main Discussion

Definition:

Let R be a ring, the corresponding non-commutative logical ring is defined as follows:
$R_N = \{a + bN_1 + cN_2 \quad ; {N_1}^2 = N_1, {N_2}^2 = N_2, N_1N_2 = N_1, N_2N_1 = N_2, a, b, c \in R\}$.
We call it the non-commutative logical ring of type (1), and we denote it by NCR_1.

Definition:

Addition: $R_N \times R_N \to R_N$ such that:
$(a + bN_1 + cN_2) + (a' + b'N_1 + c'N_2) = (a + a') + (b + b')N_1 + (c + c')N_2$.
Multiplication: : $R_N \times R_N \to R_N$ such that:
$(a + bN_1 + cN_2)(a' + b'N_1 + c'N_2) = aa' + N_1(ab' + ba' + bb' + bc') + N_2(ac' + ca' + cc' + cb')$.

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Remark:
$(R_N.\times)$ is not commutative, that is because for $a \in R$, $(aN_1)(aN_2) = a^2N_1 \neq (aN_2)(aN_1) = a^2N_2$.
Remark:
If R has a unity (1), then R_N has as a unity.
Theorem:
$(R_N.+.\times)$ is a ring.
Proof:
Let $A = a_0 + a_1N_1 + a_2N_2, B = b_0 + b_1N_1 + b_2N_2 , C = c_0 + c_1N_1 + c_2N_2 \in R_N$, we have:
$A + B = B + A . A + O = O + A = A . A + (-A) = (-A) + A = O$.
Also, $(A \times B) \times C = [a_0b_0 + N_1(a_0b_1 + a_1b_0 + a_1b_1 + a_1b_2) + N_2(a_0b_2 + a_2b_0 + a_2b_2 + a_2b_1)] \times (c_0 + c_1N_1 + c_2N_2) = a_0b_0c_0 + a_0b_0c_1N_1 + a_0b_0c_2N_2 + N_1(a_0b_1c_0 + a_1b_0c_0 + a_1b_1c_0 + a_1b_2c_0) + N_1(a_0b_1c_1 + a_1b_0c_1 + a_1b_1c_1 + a_1b_2c_1) + N_1(a_0b_1c_2 + a_1b_0c_2 + a_1b_1c_2 + a_1b_2c_2) + N_2(a_0b_2c_0 + a_2b_0c_0 + a_2b_2c_0 + a_2b_1c_0) + N_2(a_0b_2c_1 + a_2b_0c_1 + a_2b_2c_1 + a_2b_1c_1) + N_2(a_0b_2c_2 + a_2b_0c_2 + a_2b_2c_2 + a_2b_1c_2) = (a_0b_0c_0) + N_1(a_0b_0c_1 + a_0b_1c_0 + a_1b_0c_0 + a_1b_1c_0 + a_1b_2c_0 + a_0b_1c_1 + a_1b_0c_1 + a_1b_1c_1 + a_1b_2c_1 + a_0b_1c_2 + a_1b_0c_2 + a_1b_1c_2 + a_1b_2c_2) + N_2(a_0b_0c_2 + a_0b_2c_0 + a_2b_0c_0 + a_2b_2c_0 + a_2b_1c_0 + a_0b_2c_1 + a_2b_0c_1 + a_2b_2c_1 + a_2b_1c_1 + a_0b_2c_2 + a_2b_0c_2 + a_2b_2c_2 + a_2b_1c_2) = A \times (B \times C)$.
$A \times (B + C) = (a_0 + a_1N_1 + a_2N_2) \times [(b_0 + c_0) + (b_1 + c_1)N_1 + (b_2 + c_2)N_2] = a_0(b_0 + c_0) + N_1[a_0(b_1 + c_1) + a_1(b_0 + c_0) + a_1(b_1 + c_1) + a_1(b_2 + c_2)] + N_2[a_2(b_0 + c_0) + a_0(b_2 + c_2) + a_2(b_1 + c_1) + a_2(b_2 + c_2)] = [a_0b_0 + N_1(a_0b_1 + a_1b_0 + a_1b_1 + a_1b_2) + N_2(a_2b_0 + a_0b_2 + a_2b_1 + a_2b_2)] + [a_0c_0 + N_1(a_0c_1 + a_1c_0 + a_1c_1 + a_1c_2) + N_2(a_2c_0 + a_0c_2 + a_2c_1 + a_2c_2)] = (A \times B) + (A \times C)$.
Remark:
It is clear that $R \subset R_N$.
Example:
Take $R = Z_3 = \{0.1.2\}$. $R_N = \{0.1.2. N_1. N_2. 1 + N_1. 1 + N_2. 2 + N_1. 2 + N_2. 1 + N_1 + N_2. 2 + N_1 + N_2. 2 + 2N_1 + 2N_2 . N_1 + N_2. 1 + 2N_1. 1 + 2N_2. 1 + 2N_1 + N_2. 1 + N_1 + 2N_2.. 2N_1 + 2N_2.. 2N_1 + N_2.. N_1 + 2N_2. 2N_1. 2N_2. 1 + 2N_1 + 2N_2. 2 + 2N_1 + N_2. 2 + N_1 + 2N_2. 2 + 2N_1. 2 + 2N_2\}$.
For example:
$(1 + N_1)(1 + N_2) = 2 + N_2 + 2N_1 + N_1 = 2 + N_2$.
$(2 + N_2)(1 + N_1) = 2 + 2N_1 + N_2 + N_2 = 2 + 2N_1 + 2N_2$.
Remark:
If R is finite with $|R| = n$. then $|R_N| = n^3$.
Example:
Take $R = Z_2 = \{0.1\}$. then $R_N = \{0,1, N_1, N_2, 1 + N_1, 1 + N_2, N_1 + N_2, 1 + N_1 + N_2\}$.
Definition:
If R is a field, then R_N is called the non-commutative logical field, which is not a field in the ordinary meaning. It is only a ring.
Definition:
Let $A = a_0 + a_1N_1 + a_2N_2 \in R_N$, then it is called a unity if and only if there exists $B = b_0 + b_1N_1 + b_2N_2 \in R_N$ such that $AB = BA = 1$.
Under the condition that R_N has a unity (1).
Theorem:
Let R be a commutative ring with unity, R_N be its corresponding $N \subset R_1$, then $A = a_0 + a_1N_1 + a_2N_2 \in R_N$ is a unit if and only if: $a_0. {a_0}^2 + a_0a_2 + a_1a_2$ are units in R.
Proof:
Assume that there exists $B = b_0 + b_1N_1 + b_2N_2 \in R_N$ such that $AB = BA = 1$. This is equivalent to:

$$\begin{cases} a_0b_0 = 1 \\ a_0b_1 + a_1b_0 + a_1b_1 + a_1b_2 = 0 \\ a_0b_2 + a_2b_0 + a_2b_2 + a_0b_1 = 0 \end{cases}$$

Which is equivalent to:

$$\begin{cases} b_0 = \frac{1}{a_0} \quad (a_0 \text{ is a unit in R}) \\ b_1(a_0 + a_1) + b_2a_1 = -\frac{a_1}{a_0} \\ b_1a_0 + b_2(a_0 + a_2) = -\frac{a_2}{a_0} \end{cases}$$

$$\Delta = \begin{vmatrix} a_0 + a_1 & a_1 \\ a_0 & a_0 + a_2 \end{vmatrix} = (a_0 + a_1)(a_0 + a_2) - a_0a_1 = {a_0}^2 + a_0a_2 + a_1a_2$$

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To get a solution, Δ should be invertible in R.

$$\Delta_{b_1}=\begin{vmatrix} -\frac{a_1}{a_0} & a_1 \\ -\frac{a_2}{a_0} & a_0+a_2 \end{vmatrix}=\frac{a_1a_2}{a_0}-\frac{a_1(a_0+a_2)}{a_0}=\frac{a_1a_2-a_1(a_0+a_2)}{a_0}=\frac{a_1(a_2-a_0-a_2)}{a_0}=-a_1$$

$$\Delta_{b_2}=\begin{vmatrix} a_0+a_1 & -\frac{a_1}{a_0} \\ a_0 & -\frac{a_2}{a_0} \end{vmatrix}=-\frac{a_2(a_0+a_1)}{a_0}+\frac{a_1a_0}{a_0}=-\frac{a_2(a_0+a_1)}{a_0}+a_1$$

Thus: $b_1=\frac{\Delta_{b_1}}{\Delta}=\frac{-a_1}{(a_0+a_1)(a_0+a_2)-a_0a_1}$

$$b_2=\frac{\Delta_{b_2}}{\Delta}=\frac{-\frac{a_2(a_0+a_1)}{a_0}+a_1}{(a_0+a_1)(a_0+a_2)-a_0a_1}$$

Thus: $\begin{cases} b_1=\frac{-a_1}{a_0{}^2+a_0a_2+a_1a_2} \\ b_2=\frac{-a_2a_0-a_2a_1+a_1a_0}{a_0(a_0{}^2+a_0a_2+a_1a_2)} \end{cases}$

And $A^{-1}=\frac{1}{a_0}+N_1\left(\frac{-a_1}{a_0{}^2+a_0a_2+a_1a_2}\right)+N_2\left(\frac{a_1a_0-a_2a_0-a_2a_1}{a_0(a_0{}^2+a_0a_2+a_1a_2)}\right)$.

Remark:

If R is a field, then A is invertible if and only if: $a_0\neq 0. a_0{}^2+a_0a_2+a_1a_2\neq 0$.

Open problem:

What is the classification of the group of units of R_N?

What is the relationship between $U(R_N)$ and $U(R)$?

Definition:

The element $A\in R_N$ is called idempotent if and only if $A^2=A$.

It is called 2- potent if and only if $A^2=0$.

Theorem:

Let $A=a_0+a_1N_1+a_2N_2\in R_N$, where R is commutative ring, then:

1] A is idempotent in R_N if and only if:

a) a_0 is idempotent in R.

b) $(a_1,a_1+a_2+2a_0),(a_2,a_1+a_2+2a_0)$ are duplets in R, with $a_1+a_2+2a_0$ acts as an identity.

2] A is 2-potent in R_N if and only if:

a) a_0 is 2-potent in R.

b) a_1,a_2 are zero divisors in R with: $a_1(a_1+a_2+2a_0)=a_2(a_1+a_2+2a_0)=0$.

Proof:

1] $A^2=A$ is equivalent to:

$$\begin{cases} a_0{}^2=a_0 \\ a_1{}^2+2a_0a_1+a_2a_1=a_1 \\ a_2{}^2+2a_0a_2+a_1a_2=a_2 \end{cases}$$

Thus: $\begin{cases} a_0{}^2=a_0 \\ a_1(a_1+a_2+2a_0)=a_1 \\ a_2(a_1+a_2+2a_0)=a_2 \end{cases}$

Which is equivalent to $\boxed{a}$ and $\boxed{b}$.

2] $A^2=0$ is equivalent to: $\begin{cases} a_0{}^2=0 \\ a_1(a_1+2a_0+a_2)=0 \\ a_2(a_1+2a_0+a_2)=0 \end{cases}$

Which is equivalent to $\boxed{a}$ and $\boxed{b}$.

Definition:

Let R_N be the NCR_1 , then $K=k_0+k_1N_1+k_2N_2=\{r_0+r_1N_1+r_2N_2\ ;r_i\in k_i\}$ is called AH-ideal of R_N if k_i are ideal in R.

If $k_0=k_1=k_2$, then K is called AHS-ideal.

Example:

Take $R=\mathbb{Z}$. $R_N=\{a+bN_1+cN_2\,;a.b.c\ \in\mathbb{Z}\}.k_0=\langle 2\rangle.k_1=\langle 3\rangle.k_2=\langle 6\rangle$, we have:

$K=k_0+k_1N_1+k_2N_2=\{2x_0+3x_1N_1+6x_2N_2\ \ ;x_i\in\mathbb{Z}\}$ is an AH-ideal of R_N.

Also, $k_0+k_0N_1+k_0N_2=\{2x_0+2x_1N_1+2x_2N_2\ \ ;x_i\in\mathbb{Z}\}$ is an AHS-ideal.

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Theorem:
Let $K = k_0 + k_1N_1 + k_2N_2 . R = R_0 + R_1N_1 + R_2N_2$ be two AH-ideals of R_N, then:$K \cap R$ is an AH-ideal.
If $K.R$ are two AHS-ideals, then $K \cap R$ is an AHS-ideal.
Proof:
$K \cap R = (K_0 \cap R_0) + (K_1 \cap R_1)N_1 + (K_2 \cap R_2)N_2$ .where{$K_i \cap R_i$ is an ideal of R.
So that $K \cap R$ is an AH-ideal:
On the other hand, if $K.R$ are two AHS-ideals, then:
$K \cap R = (K_0 \cap R_0) + (K_1 \cap R_1)N_1 + (K_2 \cap R_2)N_2$;$K_0 \cap R_0 = K_1 \cap R_1 = K_2 \cap R_2$. thus $K \cap R$ is an AHS $-$ ideal.
Definition:
Let $R.T$ be two rings, and $f_i: R \to T$; $0 \le i \le 2$ are three homomrphisms, then $f = f_0 + f_1N_1 + f_2N_2: R_N \to T_N$ such that:
$f(a + bN_1 + cN_2) = f_0(a) + f_1(b)N_1 + f_2(c)N_2$ is called an AH-homomorphism.
If $f_0 = f_1 = f_2$, then f is called AHS-homomrphism.
Remark:
If f_0, f_1, f_2 are isomorphisms, then f is called AHS-isomorphism.
Definition:
Let $f = f_0 + f_1N_1 + f_2N_2, g = g_0 + g_1N_1 + g_2N_2 : R_N \to T_N$ be two AH-homomorphisms, we define:
$(f + g) : R_N \to T_N$ such that:
$(f + g)(a + bN_1 + cN_2) = (f_0 + g_0)(a) + (f_1 + g_1)(b)N_1 + (f_2 + g_2)(c)N_2$.
$(f \times g) : R_N \to T_N$ such that:
$(f \times g)(a + bN_1 + cN_2) = (f_0 \cdot g_0)(a) + (f_1 \cdot g_1)(b)N_1 + (f_2 \cdot g_2)(c)N_2$.
The set of all AH-homomorphisms between R_N and T_N is denoted by $AHH(R_N.T_N)$.
Definition:
Let $f = f_0 + f_1N_1 + f_2N_2 : R_N \to T_N. g = g_0 + g_1N_1 + g_2N_2 : T_N \to S_N$ be two AH-homomorphisms, we define:
$g \circ f: R_N \to S_N$; $g \circ f(A) = g \circ f(a + bN_1 + cN_2) = (g_0 \circ f_0) + (g_1 \circ f_1)N_1 + (g_2 \circ f_2)N_2$.
Definition:
Let $f = f_0 + f_1N_1 + f_2N_2 : R_N \to T_N$ be an AH-homomrphism, then:
$AH\_k_{er}(f) = k_{er}(f_0) + k_{er}(f_1)N_1 + k_{er}(f_2)N_2$.
$AH\_I_m(f) = I_m(f_0) + I_m(f_1)N_1 + I_m(f_2)N_2$.
Theorem:
Let $f = f_0 + f_1N_1 + f_2N_2 : R_N \to T_N$ be an AH-homomorphism, then:
1] $f(0) = 0 . f(1) = 1 + N_1 + N_2$; $1 \in \mathbb{R}$.
2] $AH\_k_{er}(f)$ is an AH-ideal of R_N.
3] $AH\_I_m(f)$ is an AH-subring of T_N.
4] If K is an AH-ideal of R_N , then $f(K)$ is an AH-ideal of T_N.
Proof:
1] $f(0) = f_0(0) + f_1(0)N_1 + f_2(0)N_2 = 0 + 0 \cdot N_1 + 0 \cdot N_2 = 0$.
$f(1) = f_0(1) + f_1(1)N_1 + f_2(1)N_2 = 1 + N_1 + N_2$.
2] we have: $k_{er}(f_0), k_{er}(f_1), k_{er}(f_2)$ are ideals of R, thus $AH\_k_{er}(f) = k_{er}(f_0) + k_{er}(f_1)N_1 + k_{er}(f_2)N_2$ is an AH-ideal of R_N.
3] we have: $I_m(f_0), I_m(f_1), I_m(f_2)$ are subrings of T, hence: $AH\_I_m(f)$ is an AH-subring of T_N.
4] Assume that $K = k_0 + k_1N_1 + k_2N_2$; k_i are ideals of R for $0 \le i \le 2$, then: $f(K) = f_0(k_0) + f_1(k_1)N_1 + f_2(k_2)N_2$ is an AH-ideal of T_N, that is because $f_i(k_i)$ is an ideal of T.
Theorem:
$(AHH(R_N.T_N), +, \times)$ is a ring.
Proof:
Suppose that $f = f_0 + f_1N_1 + f_2N_2$, $g = g_0 + g_1N_1 + g_2N_2$, $h = h_0 + h_1N_1 + h_2N_2$ be three arbitrary elements of $AHH(R_N.T_N)$, then:
$f + (g + h) = (f_0 + g_0 + h_0) + (f_1 + h_1 + g_1)N_1 + (f_2 + h_2 + g_2)N_2 = (f_0 + g_0) + h_0 + [(f_1 + h_1) + g_1]N_1 + [(f_2 + h_2) + g_2]N_2 = (f + g) + h$.
$f \times (g \times h) = (f_0g_0h_0) + (f_1g_1h_1)N_1 + (f_2g_2h_2)N_2 = (f_0g_0)h_0 + [(f_1g_1)h_1]N_1 + [(f_2g_2)h_2]N_2 = (f \times g) \times h$.
$f + 0 = 0 + f = f$, $f + (-f) = (-f) + f = 0$, $f + g = g + f$.
On the other hand, we have:
$f \times (g + h) = f_0(g_0 + h_0) + [f_1(g_1 + h_1)]N_1 + [f_2(g_2 + h_2)]N_2 = f_0g_0 + f_0h_0 + [f_1g_1 + f_1h_1]N_1 + [f_2g_2 + f_2h_2]N_2 = (f_0g_0 + f_1g_1N_1 + f_2g_2N_2) + (f_0h_0 + f_1h_1N_1 + f_2h_2N_2) = (f \times g) + (f \times h)$.
So that, our proof is complete.

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Theorem:
Let $f: R_N \to T_N . g: T_N \to S_N$ be two AH-homomrphisms, then $g \circ f$ is an AH-homomrphism.
Proof:
$g \circ f = (g_0 \circ f_0) + (g_1 \circ f_1)N_1 + (g_2 \circ f_2)N_2$ .and $g_i \circ f_i$ is a ring homomorphism between R and S, thus $g \circ f$ is an AH-homomrphism between R_N and S_N.
Definition:
Let $R_N . T_N$ be two ($N \subset R_1$), then:
$R_N \times T_N = \{(x.y);\ x \in R_N . y \in T_N\}$.
Theorem:
$R_N \times T_N = (R \times T)_N$ is a ring.
Proof:
$R_N \times T_N = \{(x.y);\ x \in R_N . y \in T_N\}$; $\begin{cases} x = x_0 + x_1N_1 + x_2N_2 \\ y = y_0 + y_1N_1 + y_2N_2 \end{cases}$ with $x_i \in$ R . $y_i \in$ T ; $0 \le$ i ≤ 2.
There for, $R_N \times T_N = \{(x_0.y_0) + (x_1.y_1)N_1 + (x_2.y_2)N_2\} = (R \times T)_N$which is the non-commutative logical extension of $R \times T$.
Special types of NCR_1:
$\mathbb{Z}_N = \{a + bN_1 + cN_2\ ; a.b.c \in \mathbb{Z}\}$ is called the integer NCR_1.
$\mathbb{Q}_N = \{a + bN_1 + cN_2\ ; a.b.c \in \mathbb{Q}\}$ is called the rational NCR_1.
$\mathbb{R}_N = \{a + bN_1 + cN_2\ ; a.b.c \in \mathbb{R}\}$ is called the real NCR_1.
$\mathbb{C}_N = \{a + bN_1 + cN_2\ ; a.b.c \in \mathbb{C}\}$ is called the complex NCR_1.
$\mathbb{W}_N = \{a + bN_1 + cN_2\ ; a.b.c \in F_J\}$ is called the weak fuzzy complex NCR_1.
$\mathbb{S}_N = \{a + bN_1 + cN_2\ ; a.b.c \in \mathbb{S}\}$ is called the split-complex NCR_1.
Where $\begin{cases} F_J = \{x + yJ\ ; x.y \in \mathbb{R}\ .J^2 = t \in]0.1[\ \} \cdot \\ \mathbb{S} = \{x + yJ\ ; x.y \in \mathbb{R}\ .J^2 = 1\ \} \cdot \end{cases}$
$\mathbb{P}_N = \{a + bN_1 + cN_2\ ; a.b.c \in n - sp_R\}$ is called the symbolic n-plithogenic NCR_1.
$R_{N_N} = \{a + bN_1 + cN_2\ ; a.b.c \in R_n(I)\}$ is called the n-refined neutrosophic NCR_1.
For definitions of n-refined neutrosophic rings and symbolic n-plithogenic rings.

3. Diophantine equations in two variables:

Definition:
Let $A = a_0 + a_1N_1 + a_2N_2$, $B = b_0 + b_1N_1 + b_2N_2$, $C = c_0 + c_1N_1 + c_2N_2$ be three elements in $\mathbb{Z}_n = \{x + yN_1 + zN_2\ ; x, y, z \in \mathbb{Z}, {N_1}^2 = N_1, {N_2}^2 = N_2, N_1N_2 = N_1, N_2N_1 = N_2\}$.
We define the right non-commutative linear Diophantine equation in two variables $X = x_0 + x_1N_1 + x_2N_2$, $Y = y_0 + y_1N_1 + y_2N_2$ as follows:
$A \cdot X + B \cdot Y = C$, we denote it by NCD_R.
Definition:
The left non-commutative linear Diophantine equation in two variables as follows:
$X \cdot A + Y \cdot B = C$, we denote it by NCD_L.
Definition:
The right - left non-commutative linear Diophantine equation in two variables as follows:
$A \cdot X + Y \cdot B = C$, we denote it by $NCD_{(R.L)}$.
The discussion of NCD_R:
$A \cdot X + B \cdot Y = C$ is equivalent to:
$(a_0x_0 + b_0y_0) + N_1[a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + b_0y_1 + b_1y_0 + b_1y_1 + b_1y_2] + N_2[a_0x_2 + a_2x_0 + a_2x_2 + a_2x_1 + b_0y_2 + b_2y_0 + b_2y_2 + b_2y_1] = c_0 + c_1N_1 + c_2N_2$.
This is equivalent to:
$$\begin{cases} a_0x_0 + b_0y_0 = c_0 \quad (1) \\ a_0x_1 + b_0y_1 + a_1(x_0 + x_1 + x_2) + b_1(y_0 + y_1 + y_2) = c_1 \quad (2) \\ a_0x_2 + b_0y_2 + a_2(x_0 + x_1 + x_2) + b_2(y_0 + y_1 + y_2) = c_2 \quad (3) \end{cases}$$
Equation (1) is solvable if and only if $\gcd(a_0, b_0) | c_0$.
Assume that (1) is solvable, and (k_0, l_0) is a solution.
By adding (1),(2),(3) we get:
$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$ (3)
Assume that (3) is solvable i.e. $\gcd(a_0 + a_1 + a_2 . b_0 + b_1 + b_2) | c_0 + c_1 + c_2$ and (k_1, l_1) is a solution of (3).
Then, we get from (2):
$a_0x_1 + b_0y_1 + a_1k_1 + b_1l_1 = c_1$, thus:

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$a_0x_1 + b_0y_1 = c_1 - a_1k_1 - b_1l_1$
It is solvable if and only if:
$\gcd(a_0.,)|\, c_1 - a_1k_1 - b_1l_1$.
If (II) is solvable, with $(k_2.l_2)$ as a solution, then we get:

$$\begin{cases} x_0 = k_0 \\ y_0 = l_0 \\ x_0 + x_1 + x_2 = k_1 \\ y_0 + y_1 + y_2 = l_1 \\ x_1 = k_2 \\ y_1 = l_2 \end{cases} .\text{thus} \begin{cases} x_0 = k_0 \\ y_0 = l_0 \\ x_2 = k_1 - k_2 - k_0 \\ y_2 = l_1 - l_0 - l_2 \\ x_1 = k_2 \\ y_1 = l_2 \end{cases}$$

Example:
Consider the following NCD_R:
$(1 + 2N_1 + N_2)X + (3 + N_1 + 2N_2)Y = 7 + 12N_1 + 11N_2$.
Equation (1) is $x_0 + 3y_0 = 7$
Equation (2) is $4(x_0 + x_1 + x_2) + 6(y_0 + y_1 + y_2) = 30$
Which is equivalent to: $2(x_0 + x_1 + x_2) + 3(y_0 + y_1 + y_2) = 15$.
Take $(k_0, l_0) = (1.2)$ a solution of (1).
Take $(k_1, l_1) = (0.5)$ a solution of (2).
Equation (3) is: $x_1 + 3y_1 = 12 - 0 - (5) = 7$.
Take $(k_2, l_2) = (4{,}3)$ a solution of (3).

$$\text{Thus} \begin{cases} x_0 = 1\,.\; y_0 = 2 \\ x_1 = 1\,.y_1 = 2 \\ x_2 = -2 \\ y_2 = 1 \end{cases}$$

So that $X = 1 + N_1 - 2N_2$ $,Y = 2 + 2N_1 + N_2$, is a solution of the original equation.
The discussion of NCD_L:
$X \cdot A + Y \cdot B = C$, is equivalent to:
$(a_0x_0 + b_0y_0) + N_1[x_0a_1 + x_1a_0 + x_1a_1 + x_1a_2 + y_1b_0 + y_0b_1 + y_1b_1 + y_1b_2] + N_2[x_0a_2 + x_2a_2 + x_2a_0 + x_2a_1 + y_0b_2 + y_2b_2 + y_2b_0 + y_2b_1] = c_0 + c_1N_1 + c_2N_2$.
There for:

$$\begin{cases} a_0x_0 + b_0y_0 = c_0 \quad (1) \\ a_1x_0 + (a_0 + a_1 + a_2)x_1 + b_1y_0 + y_1(b_0 + b_1 + b_2) = c_1 \quad (2) \\ a_2x_0 + (a_0 + a_1 + a_2)x_2 + b_2y_0 + (b_0 + b_1 + b_2)y_2 = c_2 \quad (3) \end{cases}$$

Assume that (1) is solvable, i.e $\gcd(a_0, b_0)|\, c_0$, and (k_0, l_0) is a solution of (1).
Then (2) will be:
$(a_0 + a_1 + a_2)x_1 + (b_0 + b_1 + b_2)y_1 = c_1 - a_1k_0 - b_1l_0$ *it is* is solvable if and only if: gcd $(a_0 + a_1 + a_2.b_0 + b_1 + b_2)|\, c_1 - a_1k_0 - b_1l_0$
Assume that it is solvable with (k_1, l_1) as a solution.
(2) will be:
$(a_0 + a_1 + a_2)x_2 + (b_0 + b_1 + b_2)y_2 = c_2 - a_2k_0 - b_2l_0$
is solvable if and only if: gcd $(a_0 + a_1 + a_2, b_0 + b_1 + b_2)|\, c_2 - a_2k_0 - b_2l_0$
Assume that it is solvable with $(k_2.l_2)$ as a solution.

$$\text{Hence: } \begin{cases} x_0 = k_0 \;, y_0 = l_0 \\ x_1 = k_1 \;, y_1 = l_1 \\ x_2 = k_2 \,, y_2 = l_2 \end{cases}$$

Example:
Consider the following NCD_L:
$X \cdot (1 + 2N_1 + N_2) + Y \cdot (3 - N_1 + N_2) = 13 + 2N_1 + 2N_2$.
Equation (1) is: $x_0 + 3y_0 = 13$, it is solvable, we can take $(k_0, l_0) = (1.4)$.
Equation (I) is: $4x_1 + 3y_1 = 2 - (2)(1) + (1)(4) = 4$, it is solvable, we can take $(k_1, l_1) = (1.0)$.
Equation (II) is: $4x_2 + 3y_2 = 2 - (1)(1) - (1)(4) = -3$, it is solvable, we can take $(k_2, l_2) = (0.-1)$.
Thus: $(X, Y) = (1 + N_1.4 - N_2)$ is a solution of the original equation.
The discussion of $NCD_{(R.L)}$:
$A \cdot X + Y \cdot B = C$, is equivalent to:
$(a_0x_0 + b_0y_0) + N_1[a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + y_0b_1 + y_1b_0 + y_1b_1 + y_1b_2] + +N_2[a_0x_2 + a_2x_0 + a_2x_2 + a_2x_1 + y_0b_2 + y_2b_0 + y_2b_2 + y_2b_1] = c_0 + c_1N_1 + c_2N_2$.
It is equivalent to:

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$$\begin{cases} a_0x_0 + b_0y_0 = c_0 \quad (1) \\ a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + y_0b_1 + y_1(b_0 + b_1 + b_2) = c_1 \quad (2) \\ a_0x_2 + a_2x_0 + a_2x_2 + a_2x_1 + y_0b_2 + y_2(b_0 + b_1 + b_2) = c_2 \quad (3) \end{cases}$$

Equation (1) is solvable if and only if $\gcd(a_0, b_0) | c_0$.

Assume that it is solvable with (k_0, l_0) as a solution.

By adding (1),(2),(3), we get:

$(a_0 + a_1 + a_2)(x_0 + x_1 + x_2) + (b_0 + b_1 + b_2)(y_0 + y_1 + y_2) = c_0 + c_1 + c_2$

Assume that it is solvable, i.e gcd $(a_0 + a_1 + a_2, b_0 + b_1 + b_2) | c_0 + c_1 + c_2$ with (k_1, l_1) as a solution.

From equation (2), we can write:

$$a_0x_1 + (b_0 + b_1 + b_2)y_1 = c_1 - b_1l_0 - a_1k_1$$

Assume that it is solvable, i.e. gcd $(a_0 . b_0 + b_1 + b_2) | c_1 - b_1l_0 - a_1k_1$, with (k_2, l_2) as a solution.

There for, we get:

$$\begin{cases} x_0 = k_0 \ , y_0 = l_0 \\ k_2 = x_1 \ , l_2 = y_1 \\ x_0 + x_1 + x_2 = k_1 \ , y_0 + y_1 + y_2 = l_1 \end{cases}$$

Hence: $\begin{cases} x_0 = k_0 \ , y_0 = l_0 \\ x_1 = k_2 \ , y_1 = l_2 \\ x_2 = k_1 - k_0 - k_2 \\ y_2 = l_1 - l_0 - l_2 \end{cases}$

Example:

Consider the following $NCD_{(R.L)}$:

$(1 + N_1 + N_2)X + Y \cdot (3 + N_1 + N_2) = 10 + 6N_1,$

According to our discussion, we can see that:

Equation (1) is: $x_0 + 3y_0 = 10$, take the solution: $(k_0, l_0) = (1,3)$.

Equation (I) is: $3(x_0 + x_1 + x_2) + 5(y_0 + y_1 + y_2) = 16$ take the solution: $(k_1, l_1) = (2,2)$.

Equation (II) is: $x_1 + 5y_1 = 6 - (1)(3) - (1)(2) = 1$, take the solution: $(k_2, l_2) = (1,0)$.

$x_0 = 1 \ . y_0 = 3$

Thus: $x_1 = 1 \ . y_1 = 0$

$x_2 = 0 \ . y_2 = -1$

And $(X, Y) = (1 + N_1, 3 - N_2)$ is a solution of the original equation.

4. Conclusion

In this paper, we used some logical algebraic elements to extend any ring into a non-commutative ring contains the original ring with many generalized substructures and special elements. On the other hand, we studied the substructures of non-commutative logical rings such as AH-homomorphisms and AH-ideals with many examples that explain their algebraic validity. Also, we discussed the possibility of solving linear Diophantine equation with two variables in the non-commutative logical ring of integers, where we present an easy algorithm to solve this kind of generalized Diophantine equations.

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