Neutrosophic Topological Vector Spaces and its Properties

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Abstract

The algebraic structures Group, Ring, Field and Vector spaces are important innovations in Mathematics. Most of the theoretical concepts of Mathematics are based on the theorems related to these algebraic structures. Initially many mathematicians developed theorems related to all these algebraic structures. In 20th century most of the researchers introduced the theorems on the algebraic structures with Fuzzy and Intuitionistic fuzzy sets. Recently in 21st century the researchers concentrated on Neutrosophic sets and introduced the algebraic structures like Neutrosophic Group, Neutrosophic Ring, Neutrosophic Field, Neutrosophic Vector spaces and Neutrosophic Linear Transformation. In the current scenario of relating the spaces with the structures, we have introduced the concepts of Neutrosophic topological vector spaces. In this article, the study of Neutrosophic Topological vector spaces has been initiated. Some basic definitions and properties of classical vector spaces are generalized in Neutrosophic environment over a Neutrosophic field with continuous functions. Neutrosophic linear transformations and their properties are also included in Neutrosophic Topological Vector spaces. This article is an extension work of fuzzy and intuitionistic fuzzy vector spaces which were introduced in fuzzy and intuitionistic fuzzy environments. Even though it is an extension work, Neutrosophic Topological Vector space will play an important role in Neural Networks, Image Processing, Machine Learning and Artificial Intelligence Algorithms.

Keywords: Topological Vector Spaces; Fuzzy, Intuitionistic Fuzz and Neutrosophic Topological Vector space; Neutrosophic continuous function; Neutrosophic proper function; Neutrosophic Linear Transformation.

1. Introduction:

A Neutrosophic topological vector space is a mathematical structure that combines elements of both Neutrosophic set theory and topological vector spaces. Neutrosophic sets, an extension of fuzzy sets, accommodate indeterminacy or uncertainty in membership values. Their applicability spans various domains, as evidenced by research in [11, 15, 16, 19, 20, 22, 27], yielding practical solutions for real-life scenarios. Researchers have successfully employed neutrosophic concepts in addressing multi-criteria decision-making challenges, as seen in studies [5, 7, 27, 28],

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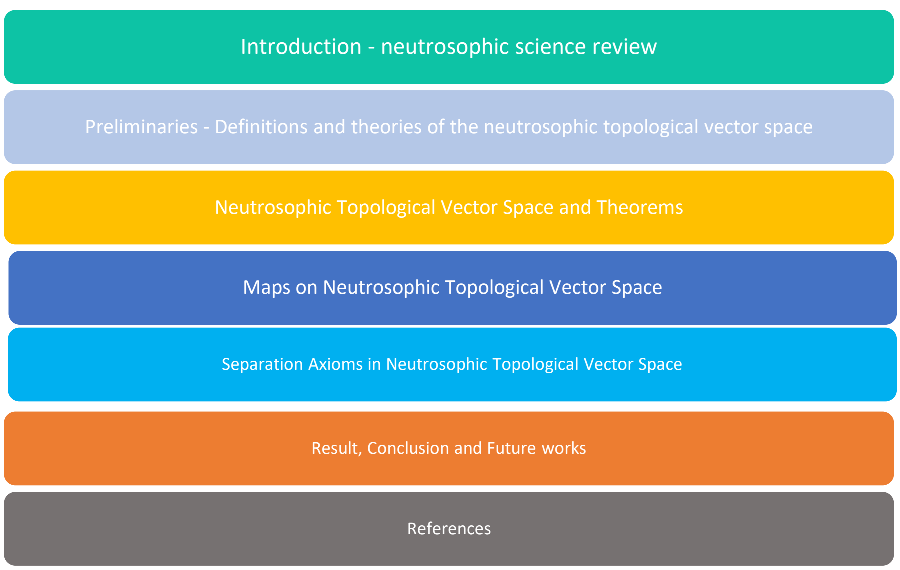
offering decisions that address diverse real-world complexities. Topological vector spaces are a type of vector space that is equipped with a topology, which provide the study of continuity and convergence of vectors. Topological vector spaces are mathematical structures that combine the algebraic properties of vector spaces with a topology, allowing for a notion of convergence. In these spaces, both vector addition and scalar multiplication are continuous operations. This framework provides a flexible setting for studying various mathematical concepts, including functional analysis and distribution theory. The topology allows for a nuanced understanding of convergence, making topological vector spaces a fundamental tool in advanced areas of mathematics and applications in physics and engineering. In a Neutrosophic topological vector space, the vectors themselves are represented as Neutrosophic sets, and the operations and topology are defined in a way which are consistent with both the Neutrosophic and topological structures. Francois Treves, introduced the concept in Topological Vector Spaces [12] in the year 1967. Buckley.J.J and Aimin Yan, explained the concept of Fuzzy topological vector space over R [8] in 1999. Fuzzy topological vector spaces merge the concepts of fuzzy sets and topological vector spaces, introducing a degree of membership for elements in the vector space. Unlike classical topological vector spaces, fuzzy topological vector spaces accommodate uncertainty, allowing elements to belong to sets to varying degrees. This framework is employed in areas such as decision-making, control theory, and artificial intelligence, providing a versatile approach to modeling imprecise or vague information within the context of vector space structures. Moumita Chiney and S. K. Samanta [21] developed IF Topological vector spaces in 2018. Intuitionistic fuzzy topological vector spaces extend the fusion of intuitionistic fuzzy sets and topological vector spaces, incorporating a more nuanced treatment of uncertainty and imprecision. In this framework, elements have membership degrees and non-membership degrees, along with a hesitation degree. This model provides a richer representation of uncertain information, making it particularly useful in decision-making and optimization problems. Intuitionistic fuzzy topological vector spaces offer a versatile mathematical structure for handling complex scenarios where degrees of belief, disbelief, and uncertainty play crucial roles. Neutrosophic vector spaces and Neutrosophic topological spaces together form the Neutrosophic topological vector spaces which is the extension work of Intuitionistic fuzzy topological vector spaces and different type of neutrosophic ideas were discussed [32-29]. This article consisting of the following sections.



2. Preliminaries:

Definition 2.1:[18] Let E is a vector space over a field K (real or complex) and a topology τ is defined on it. The set E is called a topological vector space if the maps

(i) $(x, y) \to x + y$ *from* $E \times E \to E$ and
(ii) (λ, x) ⟶ λ .x form K x E ⟶ E are continuous. It is abbreviated as TVS.

Definition 2.2:[14] A fuzzy Topological vector space Equipped with a fuzzy topology such that the two maps

(i) $\phi : E \times E \to$ E defined by $(x, y) \to x + y$
(ii) $\psi : K \times E \to E$ defined by $(\lambda, x) \to \lambda x$

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are continuous when K has the usual topology and $E \times E, K \times E$ are given the product fuzzy topologies.

Definition 2.3:[18] Let τ be an IFTS on X, the pair (X, τ) is called an IFTVS if the following two operations of IFS on X

(i) $I^{+}: X \times X \rightarrow X, (x, y) = x + y$
(ii) $I \circ: F \times X \rightarrow X, (k, x) = kx$

are IF continuous, when F has the usual intuitionistic fuzzy topology and $X \times X$ and $F \times X$ the corresponding product intuitionistic fuzzy topologies.

Definition 2.4:[26] Let $\tau \subseteq \mathcal{N}(X)$, then τ is called a Neutrosophic topology on X if

(i) $\bar{X}$ and $\bar{\phi}$ belong to τ.
(ii) The union of any number of Neutrosophic sets in τ belongs to τ
(iii) The intersection of any two Neutrosophic sets in τ belongs to τ.

The pair (X, τ) is called a Neutrosophic topological space over X. Moreover, the members of τ are said to be Neutrosophic open sets in X. If $A^c \in \tau$, then $A \in \mathcal{N}(X)$ is said to be Neutrosophic closed set in X.

Definition 2.5:[2] Let $(V, +, \cdot)$ be any vector space over a field K and let $V(I) = < V \cup I >$ be a neutrosophic set generated by V and I.

(i) If $V(I)$ is a neutrosophic vector space over a neutrosophic field K, then triple $(V(I), +, \cdot)$ is called a weak neutrosophic vector space over a field K.

(ii) If $V(I)$ is a neutrosophic vector space over a neutrosophic field K(I), then $(V(I), +, \cdot)$ is called a strong neutrosophic vector space over a field K(I).

The elements of V(I) are called neutrosophic vectors and the elements of K(I) are called neutrosophic scalars.

Definition 2.6:[3] Let $V(I)$ and $W(I)$ be strong neutrosophic vector spaces over a neutrosophic field $K(I)$ and let $\phi: V(I) \rightarrow W(I)$ be a mapping of $V(I)$ into $W(I)$. Then ϕ is called a neutrosophic vector space homomorphism if the following conditions holds:

(i) ϕ is a vector space homomorphism.

(ii) $\phi(I) = I$.

If ϕ is a bijective neutrosophic vector space homomorphism, then ϕ is called a neutrosophic vector space isomorphism and we write $V(I) \cong W(I)$.

3.Neutrosophic Topological Vector Space:

Neutrosophic Topological Vector Spaces represent an extension of traditional topological vector spaces to accommodate neutrosophic sets, which capture the notion of indeterminacy, ambiguity, and unknown information. This chapter explores the intersection of neutrosophic set theory and topological vector spaces, offering a comprehensive understanding of their properties and applications. We delve into the mathematical foundations, defining neutrosophic topological vector spaces and exploring their unique characteristics. The integration of neutrosophic logic into this context provides a powerful tool for handling uncertain and imprecise information, making Neutrosophic Topological Vector Spaces a valuable framework for various real-world applications.

Definition 3.1: Let X and Y be two non-empty sets and let N, M are the neutrosophic sets. A neutrosophic subset $\mathcal{F}$ of $X \times Y$ is said to be a neutrosophic proper function from the neutrosophic set N to the neutrosophic M if

(i) $\mathcal{F}(x, y) \leq N(x) \cap M(y)$, for each $(x, y) \in X \times Y$.

(ii) For each $x \in X$, there exists a unique $y_0 \in Y$ such that $\mathcal{F}(x, y_0) = N(x)$ and $\mathcal{F}(x, y) = (0,1) if\ y \neq y_0$.

Henceforth $\mathcal{F}: N \rightarrow M$ implies $\mathcal{F}$ is a neutrosophic proper function from $N \in \mathcal{N}^X$ into $M \in \mathcal{N}^Y$, where $\mathcal{N}^X$ and $\mathcal{N}^Y$ are the collection of all neutrosophic subsets of X and Y respectively.

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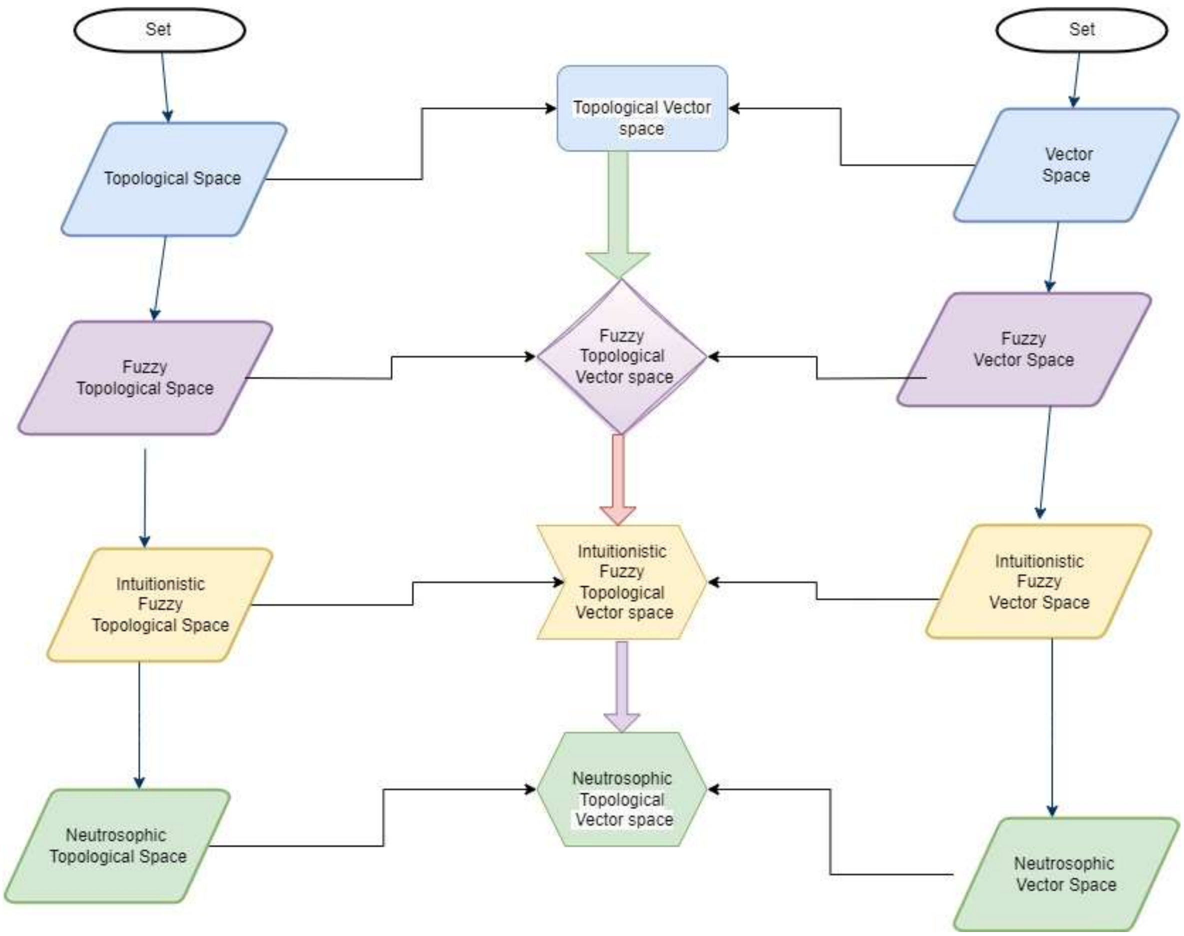
Definition 3.2: A Neutrosophic set $N = (\mu_N, \mathrm{I}_N, \vartheta_N)$ of a vector space X over the field K is said to be Neutrosophic vector space over X if

(i) $N + N + N \subseteq N$

(ii) $\alpha N \subseteq N$, for all scalar α.

This definition is equivalent to the definition defined in 2.5.

The following flowcharts explains the generalization of the Neutrosophic Topological Vectors spaces from the various authors contribution .



Definition 3.3: Let $\mathcal{N}(X)$ is a Neutrosophic vector space over a field K (real or complex) and a topology τ is defined on it. The set $\mathcal{N}(X)$ is called a Neutrosophic topological vector space if the maps

(i) $N^{\oplus}: (V \times V, \tau \times \tau) \rightarrow (V, \tau)$
(ii) $N^{\odot}: (V \times V, \tau \times \tau) \rightarrow (V, \tau)$

are Neutrosophic continuous. The pair $(\mathcal{N}(\mathcal{X}), \tau)$ is denoted as Neutrosophic Topological vector space. Moreover, the elements of τ are called Neutrosophic open sets.

Definition 3.4: Neutrosophic Proper Function: Let X be a vector space over the field K with θ as the null vector. Let V be an Neutrosophic vector space over X, a ∈ X and k ∈ be fixed. Let us define the Neutrosophic proper functions.

$\mathcal{N}^{\oplus}: V \times V \times V \rightarrow V$ by $\mathcal{N}^{\oplus}\big((x, y, z), t\big) = \begin{cases} (V \times V \times V)(x, y, z) & if\ x + y + z = t \\ (0,1) & if\ x + y + z \neq t \end{cases}$

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$$\mathcal{N}^{\odot}: K \times V \to V \text{ by } \mathcal{N}^{\odot}((k,x),t) = \begin{Bmatrix} K \times (V)(k,x) & if\ kx = y, k \neq 0 \\ sup_{x\in X} V(x) & if\ kx = y, k = 0 \\ (0,1) & if\ kx \neq y \end{Bmatrix}$$

$$\mathcal{N}^{a}: V \to V \text{ by } \mathcal{N}^{a}((k,x),t) = \begin{Bmatrix} V(x) & if\ y = k_0 x, k \neq 0 \\ (0,1) & if\ kx \neq y \end{Bmatrix}$$

$$\mathcal{N}^{k_0}: V \to V \text{ by } \mathcal{N}^{k_0}((x,y)) = \begin{Bmatrix} K \times V(k,x) & if\ y = k_0 x, k_0 \neq 0 \\ sup_{x\in X} V(x) & if\ k_0 x = y, k_0 = 0 \\ (0,1) & if\ k_0 x \neq y \end{Bmatrix}$$

$$\mathcal{N}_V^{L_{k,m,n}}((x,y,z),t) = \begin{Bmatrix} (V \times V \times V)(x,y,z), if\ kx + my + nz = t, k, m, n \neq 0 \\ V(x), \quad if\ kx = t, k \neq 0, m, n = 0 \\ V(y), \quad if\ my = t, m \neq 0, k, m = 0 \\ V(z), \quad if\ nz = t, n \neq 0, k, m = 0 \\ sup_{x\in X} V(s) \quad if\ kx + my + nz = t, k, m, n = 0 \\ (0,1) \quad if\ kx + my + nz \neq t \end{Bmatrix}$$

$for\ all\ x, y, z \in X, k, m, n \in K.$

Definition 3.5: A Neutrosophic topology τ on V is called a Neutrosophic vector topology if the Neutrosophic proper functions $\mathcal{N}^{\oplus}: (V \times V \times V \to V, \tau \times \tau \times \tau) \to (V, \tau)$ and $\mathcal{N}^{\odot}: (K \times V, \tau \times x) \to (V, \tau)$ are Neutrosophic continuous. The pair (V, τ) is said to be a Neutrosophic topological vector space if τ is a Neutrosophic vector topology on V.

Definition 3.6: Weak Neutrosophic Topological Vector Space: Let V be a Neutrosophic vector space over the field K. Let τ be a Neutrosophic topology on V then (V, τ) is called weak Neutrosophic topological vector space.

Definition 3.7: Strong Neutrosophic Topological Vector Space: Let V be a Neutrosophic vector space over the Neutrosophic field K. Let τ be a Neutrosophic topology on V then (V, τ) is called strong Neutrosophic topological vector space.

Example:3.8 R(I) is a weak Neutrosophic vector topological vector space over a field Q and it is a strong Neutrosophic topological vector space over a Neutrosophic field Q(I).

Example:3.9 $R^n(I)$ is a weak Neutrosophic vector topological vector space over a field R and it is a strong Neutrosophic topological vector space.

Example:3.10 $M_{m\times n}(I) = \{[a_{ij}]: a_{ij} \in Q(I)\}$ is a weak Neutrosophic vector topological vector space over a field Q and it is a strong Neutrosophic topological vector space over a Neutrosophic field Q(I).

Theorem 3.11 Every strong Neutrosophic topological vector space is a weak Neutrosophic topological vector space.

Proof: Suppose that τ is a topology on V and V(I), V is weak Neutrosophic vector space V(I) is strong Neutrosophic vector space over a Neutrosophic field K(I). Since K ⊆ K(I) for every field K, it follows that (V, τ) is a weak Neutrosophic topological vector space.

Theorem 3.12 Every strong (weak) Neutrosophic topological vector space is a topological vector space.

Proof: Suppose that V(I) is a strong Neutrosophic topological vector space over a Neutrosophic field K(I). Obviously, $(V(I), +, .)$ is an abelian group.

Let $u = a + bI, v = c + dI \in V(I),$ and $\alpha = k + mI, \beta = p + nI \in K(I)$ where $a, b, c, d \in V$ and $k, m, n, p \in K$, then

1). $a\ (u + v) = (k + mI)(a + bI + c + dI)$

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$= ka + kc + [kb + kd + ma + mb + mc + md]I$

$= (k + mI)(a + bI) + (k + mI)(c + dI)$

$= a\,u + a\,v.$

2. $(a + b)u = (k + mI + p + nI)(a + bI)$

$= ka + pa + [kb + pb + ma + na + mb + nb]I$

$= (k + mI)(a + bI) + (p + nI)(a + bI)$

$= a\,u + b\,u$

3. $(ab)u = ((k + mI)(p + nI))(a + bI)$

$= kpa + [kpb + kna + mpa + mna + knb + mpb + mnb]I$

$= (k + mI)((p + nI)(a + bI))$

$= a\,(b\,u)$

4. $For\ 1 + 1 + 0I\ \hat{\text{I}}\ K(I),\quad we\ have$

$1u = (1 + 0I)(a + bI)$

$= a(b + 0 + 0)I$

$= a + bI.$

Hence V(I) is a vector space. Since τ is a topology on V over a field, (V, τ) is topological vector space.

Theorem 3.13 A Neutrosophic topology τ on V is a Neutrosophic topological vector space if and only if the Neutrosophic functions $(V \times V \times V, \tau \times \tau \times \tau) \to (V, \tau)$ is Neutrosophic continuous function.

Proof: Let τ be a Neutrosophic vector topology τ on V and k, m,n ∈ K. Since k ∈ K is normal element of K with respect to V, then the Neutrosophic proper function

$\mathcal{N}_V^{L_{k,m,n}}: (V \times V \times V, \tau \times \tau \times \tau) \to (V, \tau)$ defined by

$$\mathcal{N}((x,y,z),t) = \begin{cases} V(x,y,z) & if\ x+y+z=t \\ (0,1) & if\ x+y+z \neq t \end{cases}$$ is Neutrosophic continuous.

Also, by definition of Neutrosophic vector topology, $N^{\odot}: (\mathbb{K} \times V, v \times \tau) \to (V, \tau)$ is Neutrosophic continuous.

Then $\mathcal{N}^{\odot} \circ \mathcal{N}_k : (V, \tau) \to (V, \tau)$ is defined by

$$\mathcal{N}^{\odot} \circ \mathcal{N}_k(x,y) = \begin{cases} V(x) & if\ y = kx, k \neq 0 \\ sup_{s\in X} V(s) & if\ y = kx, k = 0 \\ (0,1) & otherwise \end{cases}$$ is Neutrosophic continuous.

Similarly, $\mathcal{N}^{\odot} \circ \mathcal{N}_m : (V, \tau) \to (V, \tau)$ defined by

$$\mathcal{N}^{\odot} \circ \mathcal{N}_m(z,t) = \begin{cases} V(zx) & if\ t = mx, m \neq 0 \\ sup_{s\in X} V(s) & if\ y = mz, k = 0 \\ (0,1) & otherwise \end{cases}$$ is Neutrosophic continuous.

Thus by the result,

$(\mathcal{N}^{\odot} \circ \mathcal{N}_k) \times (\mathcal{N}^{\odot} \circ \mathcal{N}_m): (V \times V \times V, \tau \times \tau \times \tau) \to (V \times V \times V, \tau \times \tau \times \tau)$ defined by

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$$\left(\mathcal{N}^{\odot} \circ \mathcal{N}_k\right) \times \left(\mathcal{N}^{\odot} \circ \mathcal{N}_m\right)\left((x,z),(y,t)\right) = \begin{cases} (V \times V)(x,z) & if\ (x,z) = (y,t) \\ (0,1) & if\ (x,z) \neq (y,t) \end{cases}$$

is Neutrosophic continuous. Therefore $\mathcal{N}^{\oplus} \circ \left[\left(\mathcal{N}^{\odot} \circ \mathcal{N}_k\right) \times \left(\mathcal{N}^{\odot} \circ \mathcal{N}_m\right)\right] = \mathcal{N}_V^{L_{k,m,n}}$ is Neutrosophic continuous.

Conversely, let $\mathcal{N}_V^{L_{k,m,n}}$ is Neutrosophic continuous function for all $k, m, n \in K$.

We know that the projection mapping $p_I: (K \times V, v \times \tau) \to (V, \tau)$ defined by

$p_I\left((k,x),z\right) = \begin{cases} (K \times V)(k,x) & if\ z = x \\ (0,1) & otherwise \end{cases}$ and since θ is normal of V with respect to V, then $\mathcal{N}_\theta: (V,\tau) \to (V \times V \times V, \tau \times \tau \times \tau)$ defined by

$$\mathcal{N}_\theta\left(x,(x_1,y_1)\right) = \begin{cases} V(x) & if\ (x_1,y_1) = (x.\theta) \\ (0,1) & if\ (x_1,y_1) \neq (x.\theta) \end{cases}$$

are Neutrosophic continuous functions.

$\mathcal{N}_\theta. \circ p_I : (K \times V, v \times \tau) \to (V,\tau)$ defined by

$\mathcal{N}_\theta \circ p_I\left((k,x),(x_1,y_1)\right) = \begin{cases} (K \times V)(k,x) & if\ (x_1,y_1) = (x.\theta) \\ (0,1) & if\ (x_1,y_1) \neq (x.\theta) \end{cases}$ is Neutrosophic continuous.

Therefore $\mathcal{N}^{\odot} = (\mathcal{N}_V^{L_{k,m,n}} \circ \mathcal{N}_\theta \circ p_I):\ (K \times V, v \times \tau) \to (V,\tau)$, where

$(\mathcal{N}_V^{L_{k,m,n}} \circ \mathcal{N}_\theta \circ p_I)((k,x),z) = \begin{cases} (K \times V)(k,x) & if\ z = kx, k \neq 0 \\ sup_{s \in X}\ V(s) & if\ z = kx, k = 0 \\ (0,1) & if\ z \neq kx \end{cases}$ is Neutrosophic continuous.

Since $\mathcal{N}_V^{L_{k,m,n}}$ is Neutrosophic continuous for all k, m∈ K, taking k = 1, m = 1. We have

$\mathcal{N}^{\oplus}: (V \times V \times V \to V, \tau \times \tau \times \tau) \to (V,\tau)$ is Neutrosophic continuous. Hence Proved.

Theorem 3.13:

Definition 3.13: A Neutrosophic proper function $\mathcal{N}: V \to W$ is said to be a neutrosophic linear transformation if

(i) If $\mathcal{N}(\theta,\theta',\theta'') = sup_{(x,y,z)\in(X\times Y\times Z)}\ \mathcal{N}(x,y,z)$,

(ii) $\mathcal{N}(kx,ky,kz) = \begin{cases} \mathcal{N}(x,y,z) & if\ k \neq 0 \\ sup_{(x,y,z)\in(X\times Y\times Z)}\mathcal{N}(x,y,z) & if\ k = 0 \end{cases}$

(iii) If $\mathcal{N}(kx,ky,kz) = V(kx)$ and $\mathcal{N}(ma,mb,mc) = V(ma)$ imply $\mathcal{N}(kx,ky,kz) + \mathcal{N}(ma,mb,mc) = V(kx + mz)$ for all $a, x \in X, b, y \in Y, c, z \in Z$ and $k, m \in K$.

Theorem 3.14: Let $\mathcal{N}(\mathcal{X})$ be an Neutrosophic set in vector space over $\mathcal{X}$. Then the following are equivalent:

1. $\mathcal{N}(\mathcal{X})$ is an Neutrosophic vector space over $\mathcal{X}$.
2. For all scalars α, β then $\alpha x + \beta x \subseteq \mathcal{N}(\mathcal{X})\ \forall\ x \in \mathcal{N}(\mathcal{X})$
3. For all scalars α, β and for all $x, y \in \mathcal{N}(\mathcal{X})$, then $\mu_N(\alpha x + \beta y) \geq \mu_N(x) \wedge \mu_N(y)$, $\nu_N(\alpha x + \beta y) \leq \nu_N(x) \vee \nu_N(y)$ and $\sigma(\alpha x + \beta y) \leq \sigma_N(x) + \sigma_N(y)$.

Proof: Clearly (1) ⟹ (2) and (2) ⟹ (3) by the definition of Neutrosophic vector space.

To prove (2) ⟹ (1) : $\mathcal{N}(\mathcal{X}) + \mathcal{N}(\mathcal{X}) = 1.\mathcal{N}(\mathcal{X}) + 1.\mathcal{N}(\mathcal{X}) \subseteq \mathcal{N}(\mathcal{X})$,

$\alpha\mathcal{N}(\mathcal{X}) = \alpha\mathcal{N}(\mathcal{X}) + 0\mathcal{N}(\mathcal{X}) \subseteq \mathcal{N}(\mathcal{X})$ this proves the equivalent conditions.

Theorem 3.15: If (V, τ) is an Neutrosophic topological vector space, then $\mathbb{N}^k$ is an Neutrosophic homeomorphism of (V, τ) onto itself, for all $k(\neq 0) \in K$.

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Proof: Let (V, τ) is an Neutrosophic topological vector space and $k, m, n \in K$. Since $k \in K$ is normal element of K with respect to V, the Neutrosophic proper function

$N^{\odot}: (\mathbb{K} \times V, \upsilon \times \tau) \to (V, \tau)$ defined by $\mathcal{N}_k\big(x, (x_1, k_1)\big) = \begin{Bmatrix} V(x) & if\ (x_1, k_1) = (x, k) \\ (0,1) & if\ (x_1, k_1) \neq (x, k) \end{Bmatrix}$ is Neutrosophic continuous.

Hence for $k \neq 0$, $N^{\odot} \circ \mathcal{N}_k = \mathbb{N}^{k}: (V, \tau) \to (V, \tau)$ defined by $\mathbb{N}^k(x, y) = \begin{Bmatrix} V(x) & if\ y = kx, \\ (0,1) & if\ y \neq kx \end{Bmatrix}$ is

Neutrosophic continuous.

Similarly, for $k \neq 0$, the Neutrosophic proper function $(\mathbb{N}^k)^{-1}: (V, \tau) \to (V, \tau)$ defined by $(\mathbb{N}^k)^{-1}(x, y) = \begin{Bmatrix} V(x) & if\ y = \frac{1}{k}x, \\ (0,1) & if\ y \neq \frac{1}{k}x \end{Bmatrix}$ is Neutrosophic continuous.

Also $\mathbb{N}^k \circ (\mathbb{N}^k)^{-1} = I_V = (\mathbb{N}^k)^{-1} \circ \mathbb{N}^k$. Hence $\mathbb{N}^k$ is an Neutrosophic homeomorphism of (V, τ) onto itself.

4. Maps on Neutrosophic Topological Vector Spaces:

Neutrosophic topological vector spaces are a fascinating extension of classical topological vector spaces, incorporating the concept of neutrosophy to handle indeterminacy, vagueness, and uncertainty. Here are a few theorems along with their brief proofs in the context of neutrosophic topological vector spaces:

Theorem 4.1: (**Neutrosophic Sum of Vectors)** Let *V* be a neutrosophic topological vector space, and let *x*, *y* be neutrosophic vectors in *V*. Then, the neutrosophic sum *x* + *y* is also a neutrosophic vector in *V*.

Proof: For the neutrosophic sum to be well-defined, the classical vector sum *x* + *y* must be in *V*, and the neutrosophic uncertainties of *x* and *y* must be propagated to the neutrosophic sum. This can be established by applying the neutrosophic extension of the classical vector addition and ensuring the neutrosophic operations preserve the topology of *V*.

Theorem 4.2: (**Continuity of Neutrosophic Scalar Multiplication)** Let *V* be a neutrosophic topological vector space, and let *α* be a neutrosophic scalar. Then, the neutrosophic scalar multiplication *αx* is a continuous mapping from *V* to itself.

Proof: To prove the continuity of neutrosophic scalar multiplication, we need to demonstrate that for any neutrosophic vector *x* and a neutrosophic open set *U* in *V*, the pre-image of *U* under the neutrosophic scalar multiplication is an open set. This involves applying the neutrosophic extension of the classical scalar multiplication and showing that the neutrosophic topology is preserved.

Theorem 4.3: (**Neutrosophic Subspace Topology)** Let *V* be a neutrosophic topological vector space, and let *W* be a neutrosophic subspace of *V*. Then, the topology of *W* induced by *V* is a neutrosophic topological vector space.

Proof: To establish the neutrosophic subspace topology, we need to verify that the neutrosophic operations defined on *W* are well-defined and that the neutrosophic topology induced by *V* is preserved in *W*. This involves showing that neutrosophic open sets in *W* are the intersections of neutrosophic open sets in *V* with *W*, and neutrosophic vector operations respect the subspace structure.

Definition 4.4: Let V, W are the neutrosophic topological vector space. A neutrosophic proper function $N_f: V \to W$ is said to be Neutrosophic linear transformation if

(i) $N_f(\theta, \theta') = sup_{(x,y)\in(X\times Y)} N_f(x, y)$,

(ii) $N_f(kx, ky) = \begin{cases} N_f(x, y) & if\ k \neq 0 \\ sup_{(x,y)\in(X\times Y)} N_f(x, y) & if\ k = 0 \end{cases}$

(iii) $N_f(kx, ky) = V(kx)$ and $N_f(mz, mw) = V(mz)$ imply $N_f(kx + mz, ky + mw) = V(kx + mz)$, for all $x, z \in X, y, w \in Y$ and $k, m \in K$.

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Theorem 4.5: Let $N_f : V \to W$ be a neutrosophic linear transformation. Then

(i) $N_f^{-1}(W)$ is a Neutrosophic vector space over X

(ii) $N_f(V)$ is a Neutrosophic vector space over Y

Proof: (i) For any $x \in X$,

$[N_f^{-1}(W) + N_f^{-1}(W) + N_f^{-1}(W)](x) = \cup_{w=x+y+z}\{[N_f^{-1}(W)(x)] \cap [N_f^{-1}(W)(y)] \cap N_f^{-1}(W)(z)]\}$

$= \cup_{w=x+y+z}\{V(x) \cap V(y) \cap W(t_y) \cap W(t_z)\}$, for $t_z, t_y, t_z \in Y$ such that $N_f(x, t_x) = V(x)$, $N_f(y, t_y) = V(y)$ and $N(z, t_z) = V(z)$

$= \cup_{w=x+y+z}\{ (\mathrm{V(x)} \cap \mathrm{V(y)} \cap \mathrm{V(z)}) \cap (\mathrm{W}(\mathrm{t_y}) \cap \mathrm{W}(\mathrm{t_z})) \}$

$\leq \cup_{w=x+y+z} \{(\mathrm{V}(\mathrm{x}+\mathrm{y}+\mathrm{z}) \cap (\mathrm{W}(t_x + \mathrm{t_y} + \mathrm{t_z})\}$ [as V, W are neutrosophic vector spaces]

$= \cup_{w=x+y+z}\{N(x+y+z, t_x + t_y + t_z) \cap W(t_x + t_y + t_z)\}$ [since N is a linear mapping]

$= \cup_{w=x+y+z}\{N(x, t_y + t_z) \cap W(t_y + t_z)\}$

$= \begin{cases} V(x) \cap W(t_x), for\ t_x \in Y\ unique, such\ that\ N(x, t_x) = V(x) \\ (0,1), \quad otherwise \end{cases}$

$= N^{-1}(W)(w)$.

Hence $N^{-1}(W) + N^{-1}(W) + N^{-1}(W) \subseteq N^{-1}(W)$.

For $k \neq 0$, $[kN^{-1}(W)](x) = V\left(\frac{x}{y}\right) \cap W\left(y_{\frac{x}{k}}\right), for\ y_{\frac{x}{k}} \in Y$ unique such that $N\left(\frac{x}{k}, y_{\frac{x}{k}}\right) = V(\frac{x}{y})$

$= V\left(x, ky_{\frac{x}{k}}\right), since\ N\ is\ linear\ N\left(x, ky_{\frac{x}{k}}\right) = N(\frac{x}{k}, y_{\frac{x}{k}})$

$= [N^{-1}(W)](x), for\ all\ x \in X.$

For $k = 0$, $x \neq \theta$, $[kN_f^{-1}(W)](x) = (0,1) \leq [N_f^{-1}(W)](x)$.

Again for $k = 0$, $x = \theta$, $[kN_f^{-1}(W)](\theta) = \cup_{a \in X}[N_f^{-1}(W)(a)]$

$= \cup_{a \in X}\{\cup_{b \in Y} N(a,b) \cap W(b)\}$

$\leq [\cup_{(a,b)\in(X\times Y)} N(a,b)] \cap [\cup_{b \in Y} W(b)]$

$= N(\theta, \theta') \cap \{\cup_{b\in Y} W(b)\}$ [since N is linear]

$= N(\theta, \theta') \cap W(\theta')$.

Again $[N_f^{-1}(W)](\theta) = \cup_{b\in Y}\{N(\theta, t) \cap W(b)\} = N(\theta, \theta') \cap W(\theta')$ [since N is linear].

Therefore $kN^{-1}(W) \subseteq N^{-1}(W)$, for all $k \in K$.

Hence (i) is proved.

(ii) For any $z \in Y$,

$[N_f(V) + N_f(V) + N_f(V)] = \cup_{w=x+y+z}\{ [N_f(W)(x)] \cap [N_f(W)(y)] \cap N_f(W)(z)]\}$

$= \cup_{w=x+y+z}\{\{[\cup_{b\in Y} F(b,x) \cap V(b)]\} \cap [\cup_{a\in X}\{F(a,y)\} \cap V(a)]\}$

$= \cup_{w=x+y+z}\{\cup\{V(b') \cap V(a') : a', b' \in X\ such\ that\ N_f(b', x) = V(b'), N_f(a', y) = V(a')\}\}$

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$\leq \cup_{w=x+y+z}\{\cup\{V(b'+a'): b', a' \in X \text{ such that } N_f(b',x) = V(b'), N_f(a',y) = V(a')\}\}$

$= \cup_{w=x+y+z}\{\cup\{V(b'+a'): a', b' \in X\} \text{ such that } N_f(b'+a', x+y+z) = N_f(b'+a', w) = V(b'+a')\}$ since N_f is linear.

Now, $N_f(V)(w) = \cup_{b\in X}\{N_f(t,w) \cap V(b)\}$

$= \cup_{b\in X}\{V(b_1): b_1 \in X, N_f(b_1,w) = V(b_1)\}$.

Hence $N_f(V) + N_f(V) + N_f(V) \subseteq N_f(V)$.

For any scalar $k \neq 0, w \in Y$,

$[kN_f(V)](z) = N_f(V)\left(\frac{z}{k}\right)$

$= \cup_{b\in X}\left\{N_f\left(b, \frac{w}{k}\right) \cap V(b)\right\}$

$= \cup\left\{V(b): b \in X \text{ such that } N_f\left(b, \frac{w}{k}\right) = V(b)\right\}$

$= \cup\{V(kb): b \in X \text{ such that } N_f(kb, w) = V(kb)\}$ [since V is a Neutrosophic vector space and N_f is linear.

$\leq \cup_{a\in X}\{F(a,w) \cap V(a)\} = N_f(V)(w)$

If $k = 0, w \neq \theta'$ then $[kN_f(V)](w) = (0,1) \leq N_f(V)(w)$.

Again if If $k = 0, w = \theta'$ then $[kN_f(V)](\theta') = \cup_{b\in Y}\{F(V)(b)\}$

$= \cup_{b\in Y}\{\cup_{a\in X} N_f(a,b) \cap V(a)\}$

$\leq [\cup_{(a,b)\in X\times Y} N_f(a,b)] \cap [\cup_{a\in X} V(a)] = N_f(\theta,\theta') \cap V(\theta)$, since N_f is linear.

Now $[kN_f(V)](\theta') = \cup_{a\in X} N_f(a,\theta') \cap V(a) = N_f(\theta,\theta') \cap V(\theta)$, since N_f is linear.

Therefore $kN_f(V) \subseteq N_f$ for all $k \in K$. Hence (ii) proved.

Theorem 4.6: Let $N_f: V \to W$ be an neutrosophic linear transformation. If σ be an neutrosophic vector topology on W, then $\tau = \{N_f^{-1}(W_1): W_1 \in \sigma\}$ is an neutrosophic vector topology on V.

Proof: Obviously τ is a neutrosophic topology on V. Let $V_1 \in \tau$, then there exists $W_1 \in \sigma$ such that $V_1 = N_f^{-1}(W_1)$.

Since $N_F: (V,\tau) \to (W,\sigma)$ is neutrosophic continuous, $N_f \times N_f \times N_f: (V \times V \times V, \tau \times \tau \times \tau) \to (W \times W \times W, \sigma \times \sigma \times \sigma)$ is also neutrosophic continuous.

Again since (W,σ) is a neutrosophic topological vector space $\mathcal{N}_W^{L_{k,m,n}}: \to (W \times W \times W, \sigma \times \sigma \times \sigma) \to (W,\sigma)$ is neutrosophic continuous, hence $\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(W_1) \in \sigma \times \sigma \times \sigma$ then

$(N_f \times N_f \times N_f)^{-1}\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(W_1) \in \tau \times \tau \times \tau$.

Now, $(N_f \times N_f \times N_f)^{-1}\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(W_1)(x_1,x_2,x_3)$

$= (V \times V \times V)(x_1,x_2,x_3) \cap \left[\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(W_1)\right](y_1,y_2,y_3)$, where $N_f(x_i,y_i) = V(x_i)$, for $i = 1,2,3$.

$= (V \times V \times V)(x_1,x_2,x_3) \cap (W \times W \times W)(y_1,y_2,y_3) \cap W_1(ky_1 + my_2 + ny_3)$

$= (V \times V \times V)(x_1,x_2,x_3) \cap W_1(ky_1 + my_2 + ny_3)$ [since $V(x_i) \leq W(y_i)$, for $i = 1,2,3$] ……….(I)

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Again $\left[\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(V_1)(x_1,x_2,x_3)\right]$

$= (V \times V \times V)(x_1,x_2,x_3) \cap (V_1)(kx_1 + mx_2 + nx_3)$

$= (V \times V \times V)(x_1,x_2,x_3) \cap \left[N_f^{-1}(W_1)\right](kx_1 + mx_2 + nx_3)$

$= (V \times V \times V)(x_1,x_2,x_3) \cap [V(kx_1 + mx_2 + nx_3) \cap W_1(ky_1 + my_2 + ny_3)]$, [since F is linear]

$= (V \times V \times V)(x_1,x_2,x_3) \cap W_1(ky_1 + my_2 + ny_3)$, [since V *is a neutrosophic vector space*].....(II).

From (I) and (II) we have, $\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(V_1) = \left(N_f \times N_f \times N_f\right)^{-1}\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(W_1) \in \tau \times \tau \times \tau.$

Therefore (V, τ) is a Neutrosophic topological vector space.

Theorem 4.7: Let $N_f: V \to W$ be a injective neutrosophic linear transformation. If τ is a neutrosophic vector topology on V, $\sigma = \{W' \subseteq W: N_f^{-1}(W') \in \tau\}$ is an neutrosophic vector topology on $N_f(V)$. If further N_f is surjective, then σ is an neutrosophic vector topology on W.

Proof: Since N_f is injective for all $V_1 \subseteq V$, $N_f^{-1}\left(N_f(V_1)\right) = V_1$.

It is easy to verify that σ is a neutrosophic vector topology on the vector space $N_f(V) = W_1(say)$. Since N_f is injective, $N_f: (V, \tau) \to (W, \sigma)$ is neutrosophic open.

Let $W' \in \sigma$, then $N_f^{-1}(W') \in \tau$. Since τ is a neutrosophic vector topology on V, $\left(N_f \times N_f \times N_f\right)^{-1}\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(W_1) \in \tau \times \tau \times \tau$

Since , $N_f \times N_f \times N_f: (V \times V \times V, \tau \times \tau \times \tau) \to (W \times W \times W, \sigma \times \sigma \times \sigma)$ is neutrosophic open,

$\left(N_f \times N_f \times N_f\right)\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}\left(N_f^{-1}(W')\right) \in \sigma \times \sigma \times \sigma$.

Now $\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(W')(y_1,y_2,y_3)$

$= \cup_{y_4 \in Y}\left\{\left[(\mathcal{N}_W^{L_{k,m,n}})\left((y_1,y_2,y_3),y_4\right)\right] \cap W'(y_4)\right\}$

$= (W_1 \times W_1 \times W_1)(y_1,y_2,y_3) \cap W'(ky_1 + my_2 + ny_3)$

$= \left[N_f(V) \times N_f(V) \times N_f(V)\right](y_1,y_2,y_3) \cap N_f(V_1)'(ky_1 + my_2 + ny_3)$ [since N_f is injective, there is $V_1 \subseteq V$ *such that* $N_f(V_1) = W'$].

Again, $N_f(V_1)(ky_1 + my_2 + ny_3) = \cup_{b \in X} N_f\left(b,(ky_1 + my_2 + ny_3)\right) \cap V_1(b)$

$= V(b) \cap V_1(b)$, where $b \in X$ with $N_f\left(b,(ky_1 + my_2 + ny_3)\right) = V(b)$;

$= v_1(b)$, where $b \in X$ with$N_f\left(b,(ky_1 + my_2 + ny_3)\right) = V(b)$;

$= V_1(kx_1 + mx_2 + nx_3)$, for $(x_1,x_2,x_3) \in X \times X \times X$ such that $N_f(x_i,y_i) = V(x_i)$ for $i = 1,2,3$ as N_f is linear.

Therefore $\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1}(W')(y_1,y_2,y_3)$

$= (V \times V \times V)(x_1,x_2,x_3) \cap V_1(kx_1 + mx_2 + nx_3)$ where $N_f(x_i,y_i) = V(x_i)$ for $i = 1,2,3$ …………(III)

Again $\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1} N_f^{-1}(W')(x_1,x_2,x_3)$

$= \cup_{b \in X} \mathcal{N}_V^{L_{k,m,n}}\left((x_1,x_2,x_3),b\right) \cap N_f^{-1}(W')(kx_1 + mx_2 + nx_3)$

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$= (V \times V \times V)(x_1, x_2, x_3)N_f^{-1}(W')(kx_1 + mx_2 + nx_3)$, since N_f is injective, $N_f^{-1}(W') = N_f^{-1}(N_f(V_1) = V_1$.

Hence $[N_f(V) \times N_f(V) \times N_f(V)]\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1} N_f^{-1}(W')(y_1, y_2, y_3)$

$= \bigcup_{(b,a)\in X\times X}\left\{(N_f \times N_f \times N_f)(b,a), (y_1, y_2, y_3) \cap \left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1} N_f^{-1}(W')(b,a)\right\}$

$= (V \times V \times V)(x_1, x_2, x_3) \cap V_1(kx_1 + mx_2 + nx_3)$, where $N_f(x_i, y_i) = V(x_i)$ for $i = 1,2,3$ …………(IV)

Therefore from (III) and (IV), we have $\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1} N_f^{-1}(W') = [N_f(V) \times N_f(V) \times N_f(V)]\left(\mathcal{N}_W^{L_{k,m,n}}\right)^{-1} N_f^{-1}(W') \in \sigma \times \sigma \times \sigma$ and hence σ is a neutrosophic vector topology on $N_f(V)$.

If further N_f is injective, then $N_f(V) = W$. Hence Proved.

These theorems offer a indication into the rich mathematical landscape of neutrosophic topological vector spaces. Each theorem provides a building block for understanding the structure and properties of these spaces, paving the way for further exploration and applications in various fields. The proofs involve extending classical concepts to the neutrosophic framework while ensuring the preservation of key properties.

5. Separation Axioms in Neutrosophic Topological Vector Space:

Separation axioms are important properties in topology that describe the extent to which points and sets can be separated by open sets. Extending these axioms to neutrosophic topological vector spaces involves considering indeterminacy and uncertainty. Here are a few separation axioms for neutrosophic topological vector spaces along with their proofs:

Theorem 5.1: (**Neutrosophic Hausdorff Property)** A neutrosophic topological vector space V is said to satisfy the neutrosophic Hausdorff property if for any distinct neutrosophic vectors x, y in V, there exist disjoint neutrosophic open sets U and V containing x and y respectively.

Proof:

Let x and y be distinct neutrosophic vectors in V. By definition, there are three components in each neutrosophic vector, denoted as $x = x_t, x_f, x_i$ and $y = y_t, y_f, y_i$. Define the following open sets:

$$U = \{v \in V : d(v,x) < (1 - x_i)\}$$

$$V = \{v \in V : d(v,y) < (1 - y_i)\}$$

where $d(v,w)$ is the distance between neutrosophic vectors v and w. It can be shown that U and V are disjoint neutrosophic open sets that contain x and y respectively. This demonstrates the neutrosophic Hausdorff property.

Theorem 5.2: (Neutrosophic Regular Space) A neutrosophic topological vector space V is said to be neutrosophically regular if for every neutrosophic closed set F and every neutrosophic vector x not in F, there exist disjoint neutrosophic open sets U and V such that $x \in U$ and $F \subset V$.

Proof:

Let F be a neutrosophic closed set and x be a neutrosophic vector not in F. Using a similar argument as in the proof of the neutrosophic Hausdorff property, we can define disjoint neutrosophic open sets U and V as follows:

$U = v \in V : d(v,x) < (1 - x_i)$

$V = v \in V : d(v,F) < (1 - min(F_t, F_i)$

It can be shown that U and V are disjoint neutrosophic open sets that satisfy the conditions of the neutrosophic regular space.

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These theorems establish important separation axioms in the context of neutrosophic topological vector spaces. They reflect the interplay between the neutrosophic properties of vectors and the topology of the space, extending classical separation axioms to accommodate uncertainty and indeterminacy.

6. Conclusion:

This paper introduces the novel concept of Neutrosophic Topological Vector Spaces (NTVS) and elucidates Neutrosophic proper functions to fortify this theoretical framework. The presented theorems contribute to the advancement of these concepts, marking only the initial phase of this innovative idea. Future extensions are envisioned in metric spaces, exploring aspects such as boundedness, connectedness, and compactness, with further application in finite-dimensional spaces. Additionally, the exploration of Neutrosophic linear transformations expands the scope to include Neutrosophic matrix representation. The study of maps on Neutrosophic Topological Vector Spaces serves as a foundation for extending theorems on Neutrosophic continuity, homeomorphism, and separation axioms—crucial elements for understanding the product of connected and compact spaces. This groundwork establishes the groundwork for applying Neutrosophic Topological Vector Spaces in diverse fields such as image processing, machine learning, and decision-making, showcasing the broad potential impact of this innovative framework in practical applications.

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