On The Classification of 3-Cyclic/4-Cyclic Refined Neutrosophic Real and Rational Von Shtawzen's Group

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Abstract

This paper aims to solve the group of units problem for 3-cyclic real and rational refined neutrosophic rings and 4-cyclic real and rational refined neutrosophic rings, where ring isomorphisms between the real/rational 3-cyclic and 4-cyclic refined neutrosophic rings and the direct product of some field extensions of Q and R. These isomorphisms will help in classifying the group of units of each studied ring in terms of direct products of classical well-known abelian groups. Also, we use the classification isomorphisms to determine all ideals in these classes of algebraic rings.

Keywords: 3-cyclic refined neutrosophic ring; 4-cyclic refined neutrosophic ring; Von Shtawzen's group; group of units

1. Introduction

The group of units classification is considered as one of the most well-known open problems in ring theory [9]. This problem concerns with writing the group of units of a given ring as direct product of known groups [10-11]. The concept of n-cyclic refined neutrosophic rings and modules was proposed in [12] as a novel extension of classical rings. These rings opened a wide door for the study of group of units problem.

The group of units for 2-cyclic integer refined rings was studied by Sadiq [1], and it was classified as the direct product $Z_2 \times Z_2 \times Z_2$. In [7], Sankari, and Abobala solved the problem for the rational and real 2-cyclic refined rings by building an algebraic isomorphism between the 2-cyclic refined rational/real ring and the direct product of Q or (R) with itself three times . Also, algebraic homomorphisms between 2-cyclic rings and plithogenic rings were presented in [8].

The group of units problem for 3-cyclic and 4-cyclic refined rings of integers was studied by Von Shtawzen in [2-3], where the classification of the group has been linked with non-linear Diophantine equations by two conjectures. These conjectures were totally proved by the efforts of Sankari and Abobala in [6], where authors proved that the groups of units (It is also called Von Shtawzen's groups) of 3-cyclic/ 4-cyclic refined rings of integers are finite with order that is divisible by 6 and 8 respectively.

In [4], Basheer et.al proposed more than 30 open problems about n-cyclic refined rings and algebraic structures. Most of them are still open research problems.

In [5], Katy et. Al have generalized Von Shtawzens' conjectures by the following conjecture:

Generalized Von Shtawzen's conjecture: The group of units of the integer n-cyclic refined ring is finite with order that is divisible by 2n. (It is still an open problem).

In this paper, we study the classification of Von Shtawzen's group in four different special cases, and we classify the desired groups as direct product of well-known abelian infinite groups in the rational/real cases.

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For definitions of 3-cyclic/4-cyclic refined rings, see [6].

2. Main results

Theorem:

Let Q be the rational field, $Q_3(I)$ be the corresponding 3- Cyclic refined ring, then $Q_3(I) \cong Q \times Q \times Q(i\sqrt{3})$.

Proof:

Define $f: Q_3(I) \to Q \times Q \times Q(i\sqrt{3})$ such that:

$f(x_0 + x_1I_1 + x_2I_2 + x_3I_3) = (x_0, x_0 + x_1 + x_2 + x_3, x_0 - \frac{x_1+x_2}{2} + x_3 + \frac{\sqrt{3}}{2}i(x_1 - x_2))$

f is well defined

If $X = x_0 + \sum_{i=1}^{3} x_iI_i = y = y_0 + \sum_{i=1}^{3} y_i$,

then $x_i = y_i$ $0 \le i \le 3$, and $f(x) = f(y)$, that is because:

$$\begin{cases} x_0 + x_1 + x_2 + x_3 = y_0 + y_1 + y_2 + y_3 \\ x_0 - \frac{x_1+x_2}{2} + x_3 + \frac{\sqrt{3}}{2}i(x_1 - x_2) = y_0 - \frac{y_1+y_2}{2} + y_3 + \frac{\sqrt{3}}{2}i(y_1 - y_2) \end{cases}$$

f preserves addition / multiplication operations.

For $X = x_0 + \sum_{i=1}^{3} x_iI_i, y = y_0 + \sum_{i=1}^{3} y_i I_i \in Q_3(I)$, then

$x + y = (x_0 + y_0) + \sum_{i=1}^{3}(x_i + y_i)I_i$,

$x.y = x_0y_0 + (x_0y_1 + x_1y_0 + x_1y_3 + x_3y_1 + x_2y_2)I_1 + (x_0y_2 + x_2y_0 + x_1y_1 + x_2y_3 + x_3y_2)I_2 + (x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 + x_3y_3)I_3$

$f(x + y) = (M_0, M_1, M_2)$, *where:*

$M_0 = x_0 + y_0$, $M_1 = \sum_{i=0}^{4}(x_i + y_i)$,

$M_2 = x_0 + y_0 - \frac{1}{2}(x_1 + y_1 + x_2 + y_2) + (x_3 + y_3) + \frac{\sqrt{3}}{2}i[(x_1 + y_1) - (x_2 + y_2)]$,

hence *f(x+y) =f(x) + f(y).*

$f(x.y) = (M_0, M_1, M_2)$, *where:*

$M_0 = x_0y_0$, $M_1 = \sum_{i,j=0}^{3} x_iy_i = (\sum_{i=0}^{3} x_i)(\sum_{i=0}^{3} y_i)$,

$M_2 = x_0y_0 - \frac{1}{2}(x_0y_1 + x_1y_0 + x_1y_3 + x_3y_1 + x_2y_2 + x_0y_2 + x_2y_0 + x_1y_1 + x_2y_3 + x_3y_2) + (x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 + x_3y_3) + \frac{\sqrt{3}}{2}i(x_0y_1 + x_1y_0 + x_1y_3 + x_3y_1 + x_2y_2 - x_0y_2 - x_2y_0 - x_1y_1 - x_2y_3 - x_3y_2)$.

On the other hand, we have:

$(x_0 - \frac{x_1+x_2}{2} + x_3 + \frac{\sqrt{3}}{2}i(x_1 - x_2).(y_0 - \frac{y_1+y_3}{2} + y_3 + \frac{\sqrt{3}}{2}i(y_1 - y_3)) = x_0y_0 + \frac{1}{4}(x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2) - \frac{1}{2}(x_0y_1 + x_0y_2) + x_0y_3 + \frac{\sqrt{3}}{2}i(x_0y_1 - x_0y_2) - \frac{1}{2}(x_1y_3 + x_2y_3) - \frac{\sqrt{3}}{4}i(x_1y_1 - x_1y_2 + x_2y_1 - x_2y_2) - x_3y_0 - \frac{1}{2}(x_3y_1 + x_3y_2) + x_3y_3 + \frac{\sqrt{3}}{2}i(x_3y_1 - x_3y_2) + \frac{\sqrt{3}}{2}i(x_1y_0 - x_2y_0) - \frac{\sqrt{3}}{4}i(x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2) + \frac{\sqrt{3}}{2}i(x_1y_3 - x_2y_3) - \frac{3}{4}(x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2) - \frac{1}{2}(x_1y_0 + x_2y_0) = x_0y_0 - \frac{1}{2}(x_0y_1 + x_1y_0 + x_1y_1 + x_2y_2 + x_0y_2 + x_3y_1 + x_3y_2 + x_2y_0 + x_1y_3 + x_2y_3) + (x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 + x_3y_3) + \frac{\sqrt{3}}{2}i(x_0y_1 - x_0y_2 + x_3y_1 - x_3y_2 + x_1y_3 - x_2y_3 + x_1y_0 - x_2y_0 - \frac{x_1y_1}{2} + \frac{x_1y_2}{2} - \frac{x_2y_1}{2} + \frac{x_2y_2}{2} - \frac{x_1y_1}{2} - \frac{x_1y_2}{2} + \frac{x_2y_1}{2} + \frac{x_2y_2}{2}) = f(x)f(y)$,

hence f is a homomorphism.

Let $X = x_0 + \sum_{i=1}^{3} x_iI_i \in ker(f)$, then $f(x) = (0,0,0)$.

$$\begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 = 0 \\ -\frac{x_1+x_2}{2} + x_3 + \frac{\sqrt{3}}{2}i(x_1 - x_2) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_0 = 0 \\ x_1 + x_2 + x_3 = 0 \\ -\frac{1}{2}(x_1 + x_2) + x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow A.X = 0.$$

det A = $3 \neq 0$, then A is invertible, which implies that the previous system has zero as unique solution, hence $ker(f) = \{0\}$, and f is injective.

To prove that f is surjective; we take an arbitrary element $(y_0, y_1, y_2) \in Q \times Q \times Q(i\sqrt{3})$, and we must find $x \in Q_3(I)$ such that $f(x) = y$.

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First, $y_2 = y_2' + y_2''\frac{\sqrt{3}}{2}i \in Q(\sqrt{3}i)$, take:
$X = y_0 + [\frac{1}{2}y_1 - \frac{1}{2}y_2' + \frac{1}{2}y_2'']I_1 + [\frac{1}{3}y_1 - \frac{1}{3}y_2' - \frac{1}{2}y_2'']I_2 + [\frac{1}{3}y_1 - y_0 + \frac{2}{3}y_2']I_3 \in Q_3(I)$.
$f(x) = (N_0, N_1, N_2)$, such that:
$N_0 = y_0, N_1 = y_0 + (\frac{1}{3}y_1 - \frac{1}{3}y_2' + \frac{1}{2}y_2'') + (\frac{1}{3}y_1 - \frac{1}{3}y_2' - \frac{1}{2}y_2'') + (\frac{1}{3}y_1 - y_0 + \frac{2}{3}y_2') = y_1,$
$N_2 = y_0 - \frac{1}{2}[\frac{1}{3}y_1 - \frac{1}{3}y_2' + \frac{1}{2}y_2'' + \frac{1}{3}y_1 - \frac{1}{3}y_2' + \frac{1}{2}y_2''] + [\frac{1}{3}y_1 - y_0 + \frac{2}{3}y_2'] + \frac{\sqrt{3}}{2}i[\frac{1}{3}y_1 - \frac{1}{3}y_2' + \frac{1}{2}y_2'' -$
$\frac{1}{3}y_1 + \frac{1}{3}y_2' + \frac{1}{2}y_2''] = y_2' + \frac{\sqrt{3}}{2}i\, y_2'' = y_2$.
Thus, f is surjective, and then it is an isomorphism.

Remark:

The inverse isomorphism of f is:
$f^{-1}: Q \times Q \times Q(\sqrt{3}i) \to Q_3(I)$:
$f^{-1}(y_0, y_1, y_2' + y_2''\frac{\sqrt{3}}{2}i) = y_0 + [\frac{1}{3}y_1 - \frac{1}{3}y_2' + \frac{1}{2}y_2'']\, I_1 + [\frac{1}{3}y_1 - \frac{1}{3}y_2' - \frac{1}{2}y_2'']I_2 + [\frac{1}{3}y_1 - y_0 + \frac{2}{3}y_2']I_3$.

Result:

The 3-cyclic refined group of units of $Q_3(I)$ is isomorphic to $Q^* \times Q^* \times (Q(\sqrt{3}i)^*$.

Result:

A 3-cyclic refined rational number $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3$ is invertible if and only if: $x_0 \neq 0$, $x_0 + x_1 + x_2 + x_3 \neq 0$, $x_0 - \frac{x_1+x_2}{2} + x_3 \neq 0$, *with* $x_1 - x_2 \neq 0$.

Example:

Consider $X = \frac{1}{2} + I_1 - \frac{1}{2}I_2 - I_3$, X is not a unit, that is because $x_0 + x_1 + x_2 + x_3 = \frac{1}{2} + 1 - \frac{1}{2} - 1 = 0$.
For $Y = 3 + 2\, I_1 + 2I_2 - I_3$ is not invertible, that is because: $x_1 - x_2 = 2 - 2 = 0$, $x_0 - \frac{x_1+x_2}{2} + x_3 = 3 - 2 - 1 = 0$.

Theorem:

Let R be the real field, $R_3(I)$ be the corresponding 3-cyclic refined real ring, then $R_3(I) \cong R \times R \times \mathbb{C}$.
Proof:
By defining a similar function $f: R_3(I) \to R \times R \times \mathbb{C}$ *with* $f(x_0 + x_1I_1 + x_2I_2 + x_3I_3) = (x_0, x_0 + x_1 + x_2 + x_3, x_0 - \frac{x_1+x_2}{2} + x_3 + i\frac{\sqrt{3}}{2}(x_1 - x_2)$ where $x_i \in R$.
We get by a similar discussion that f should be a ring isomorphism, which implies the proof.

Result:

The group of units of the 3-cyclic refined real ring is isomorphic to $R^* \times R^* \times C^*$, that holds directly from the previous theorem.

Applications to ideals:

The classification theorems of $Q_3(I)$, $R_3(I)$ will be very helpful in finding all ideals of those rings. It is sufficient to find all ideals in $f(Q_3(I))$ or $f(R_3(I))$ and going back by the inverse isomorphism.

Lemma:

1) The ring $Q \times Q \times Q(\sqrt{3}i)$ has exactly 6 different ideals up to isomorphism.
2) The ring $R \times R \times \mathbb{C}$ has exactly 8 different ideals up to isomorphism.

Proof:
Since Q , $Q(\sqrt{3}i)$ are fields, then they have only two ideals $I_Q = \{\{0\}, Q\}$, $I_{Q(\sqrt{3}i)} = \{\{0\}, Q(\sqrt{3}i)\}$, thus the ideals of $Q \times Q \times Q(\sqrt{3}i)$ are:
$I_1 = \{0\} \times \{0\} \times \{0\}$, $I_2 = \{0\} \times \{0\} \times Q(\sqrt{3}i)$
$I_3 = Q \times \{0\} \times \{0\}$, $I_4 = Q \times \{0\} \times Q(\sqrt{3}i)$
$I_5 = \{0\} \times Q \times \{0\}$, $I_6 = \{0\} \times Q \times Q(\sqrt{3}i)$
$I_7 = Q \times Q \times \{0\}$, $I_8 = Q \times Q \times Q(\sqrt{3}i)$
By a similar discussion to (1), we get:
$I_1 = \{0\} \times \{0\} \times \{0\}$, $I_2 = \{0\} \times \{0\} \times \mathbb{C}$
$I_3 = R \times \{0\} \times \{0\}$, $I_4 = R \times \{0\} \times \mathbb{C}$
$I_5 = \{0\} \times R \times \{0\}$, $I_6 = \{0\} \times R \times \mathbb{C}$
$I_7 = R \times R \times \{0\}$, $I_8 = R \times R \times \mathbb{C}$

Remark:

For $Q \times Q \times Q(\sqrt{3}i)$, we can see that:
$I_3 \cong I_5$, $I_4 \cong I_6$.
For $R \times R \times \mathbb{C}$, we can see that:
$I_3 \cong I_5$, $I_4 \cong I_6$.
So that, both rings have 6 different ideals up to isomorphism.

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Theorem:

The 3-cyclic refined rational ring $Q_3(I)$ has exactly 6 different ideals up to isomorphism.

Proof:

Since $Q_3(I) \cong Q \times Q \times Q(\sqrt{3}i)$, then they have isomorphic ideals, which means that $Q_3(I)$ has 6 different ideals up to isomorphism.

Now, we find all the ideals in $Q_3(I)$.

$J_1 = f^{-1}(\Gamma_1) = \{0\}$, $J_8 = f^{-1}(\Gamma_8) = Q_3(I)$.

$J_2 = f^{-1}(\Gamma_2) = \{(-\frac{1}{3}y_2{}' + \frac{1}{2}y_2{}'')\, I_1 + (-\frac{1}{3}y_2{}' - \frac{1}{2}y_2{}'')I_2 + \frac{2}{3}y_2{}'I_3; y_2{}', \frac{1}{2}y_2{}'' \in Q\}$

$J_7 = f^{-1}(\Gamma_7) = \{y_0 + \frac{1}{3}y_1I_1 + \frac{1}{3}y_1I_2 + (\frac{1}{3}y_1 - y_0)I_3;\ y_1, y_0 \in Q\}$

$J_3 = f^{-1}(\Gamma_3) = \{y_0 - y_0I_3;\ y_0 \in Q\}$

$J_5 = f^{-1}(\Gamma_5) = \{\frac{1}{3}y_1I_1 + \frac{1}{3}y_1I_2 + \frac{1}{3}y_1I_3;\ y_1 \in Q\} \cong J_3$

$J_4 = f^{-1}(\Gamma_4) = \{y_0 + (-\frac{1}{3}y_2{}' + \frac{1}{2}y_2{}'')\, I_1 + (-\frac{1}{3}y_2{}' - \frac{1}{2}y_2{}'')I_2 + (-y_0 + \frac{2}{3}y_2{}')I_3;\ y_0, y_2{}', y_2{}'' \in Q\}$

$J_6 = f^{-1}(\Gamma_6) = \{(\frac{1}{3}y_1 - \frac{1}{3}y_2{}' + \frac{1}{2}y_2{}'')\, I_1 + (\frac{1}{3}y_1 - \frac{1}{3}y_2{}' - \frac{1}{2}y_2{}'')I_2 + (\frac{1}{3}y_1 + \frac{2}{3}y_2{}')I_3;\ y_1, y_2{}', y_2{}'' \in Q\} \cong J_4$

By a similar approach, we can find all ideals in $R_3(I)$.

Theorem:

Let Q be the rational field, $Q_4(I)$ be the corresponding 4-cyclic refined ring, then $Q_4(I) \cong Q \times Q \times Q \times Q(i)$.

Proof:

We define the mapping $f: Q_4(I) \to Q \times Q \times Q \times Q(i)$, where

$f(x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4) = (x_0, x_0 + x_1 + x_2 + x_3 + x_4, x_0 - x_1 + x_2 - x_3 + x_4, x_0 + ix_1 - x_2 - ix_3 + x_4)$.

(f) is well defined:

For $X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 = Y = y_0 + y_1I_1 + y_2I_2 + y_3I_3 + y_4I_4$, *we have* $x_i = y_i$ *for all* $0 \le i \le 4$, thus:

$$\begin{cases} x_0 = y_0 \\ x_0 + x_1 + x_2 + x_3 + x_4 = y_0 + y_1 + y_2 + y_3 + y_4 \\ x_0 - x_1 + x_2 - x_3 + x_4 = y_0 - y_1 + y_2 - y_3 + y_4 \\ x_0 + ix_1 - x_2 - ix_3 + x_4 = y_0 + iy_1 - y_2 - iy_3 + y_4 \end{cases}$$

Thus $f(x) = f(y)$.

f preserves addition and multiplication, that is because:

For $X = x_0 + \sum_{i=1}^{4} x_iI_i$, $Y = y_0 + \sum_{i=1}^{4} y_iI_i$, *then*

$X + Y = (x_0 + y_0) + \sum_{i=1}^{4}(x_i + y_i)I_i$,

$f(x + y) = (M_0, M_1, M_2, M_3);\ M_0 = x_0 + y_0$,

$M_1 = x_0 + y_0 + x_1 + y_1 + x_2 + y_2 + x_3 + y_3 + x_4 + y_4$

$M_2 = x_0 + y_0 - (x_1 + y_1) + (x_2 + y_2) - (x_3 + y_3) + (x_4 + y_4)$

$M_3 = x_0 + y_0 + i(x_1 + y_1) - (x_2 + y_2) - i(x_3 + y_3) + (x_4 + y_4)$

So that $f(x + y) = f(x) + f(y)$.

$X.Y = x_0y_0 + (x_0y_1 + x_1y_0 + x_1y_4 + x_4y_1 + x_3y_2 + x_2y_3)I_1 + (x_0y_2 + x_2y_0 + x_1y_1 + x_2y_4 + x_4y_2 + x_3y_3)I_2 + I_3(x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3) + (x_0y_4 + x_4y_0 + x_1y_3 + x_3y_1 + x_2y_2 + x_4y_4)I_4$.

$f(x.y) = (N_0, N_1, N_2, N_3);\ N_0 = x_0y_0$,

$N_1 = x_0y_0 + x_0y_1 + x_1y_0 + x_1y_4 + x_4y_1 + x_3y_2 + x_2y_3 + x_0y_2 + x_2y_0 + x_1y_1 + x_2y_4 + x_4y_2 + x_3y_3 + x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3 + x_0y_4 + x_4y_0 + x_1y_3 + x_3y_1 + x_2y_2 + x_4y_4 = (x_0 + x_1 + x_2 + x_3 + x_4)(y_0 + y_1 + y_2 + y_3 + y_4)$

$N_2 = x_0y_0 - (x_0y_1 + x_1y_0 + x_1y_4 + x_4y_1 + x_3y_2 + x_2y_3) + (x_0y_2 + x_2y_0 + x_1y_1 + x_2y_4 + x_4y_2 + x_3y_3) - (x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3) + (x_0y_4 + x_4y_0 + x_1y_3 + x_3y_1 + x_2y_2 + x_4y_4) = (x_0 - x_1 + x_2 - x_3 + x_4)(y_0 - y_1 + y_2 - y_3 + y_4)$.

$N_3 = x_0y_0 + i(x_0y_1 + x_1y_0 + x_1y_4 + x_4y_1 + x_3y_2 + x_2y_3) - (x_0y_2 + x_2y_0 + x_1y_1 + x_2y_4 + x_4y_2 + x_3y_3) - i(x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1 + x_3y_4 + x_4y_3) + (x_0y_4 + x_4y_0 + x_1y_3 + x_3y_1 + x_2y_2 + x_4y_4) = (x_0 + ix_1 - x_2 - ix_3 + x_4)(y_0 + iy_1 - y_2 - iy_3 + y_4)$.

Thus $f(x.y) = f(x).f(y)$.

$ker(f) = \{X = x_0 + x_1I_1 + x_2I_2 + x_3I_3 + x_4I_4 \in Q_4(I);\ f(x) = (0,0,0,0)\}$, *which implies*

$$\begin{cases} x_0 = 0 \\ x_0 + x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0 \\ i(x_1 - x_3) + (-x_2 + x_4) = 0 \end{cases}$$

The previous linear system equivalents:

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$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow A.X = 0.$$

$\det A = 8 \neq 0$, thus A is invertible and the system has a unique zero solution, so that $x_1 = x_2 = x_3 = x_4 = 0$ and $\ker(f) = \{0\}$, and f is injective.

To prove that f is surjective, we must find an element

$X = x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3 + x_4 I_4$ such that $f(x) = (y_0, y_1, y_2, y_3, y_4 i)$ for any $y_0, y_1, y_2, y_3, y_4 \in Q$.

We put $X = y_0 + (\frac{1}{4}y_1 - \frac{1}{4}y_2 + \frac{1}{2}y_4)I_1 + (\frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{2}y_0 - \frac{1}{2}y_3)I_2 + (\frac{1}{4}y_1 - \frac{1}{4}y_2 - \frac{1}{2}y_4)I_3 + (\frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{2}y_0 + \frac{1}{2}y_3)I_4$.

$f(x) = (L_0, L_1, L_2, L_3)$; $L_0 = y_0$, $L_p = y_0 + (\frac{1}{4}y_1 - \frac{1}{4}y_2 + \frac{1}{2}y_4) + (\frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{2}y_0 - \frac{1}{2}y_3) + (\frac{1}{4}y_1 - \frac{1}{4}y_2 - \frac{1}{2}y_4) + (\frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{2}y_0 + \frac{1}{2}y_3) = y_1$

$L_2 = y_0 - (\frac{1}{4}y_1 - \frac{1}{4}y_2 + \frac{1}{2}y_4) + (\frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{2}y_0 - \frac{1}{2}y_3) - (\frac{1}{4}y_1 - \frac{1}{4}y_2 - \frac{1}{2}y_4) + (\frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{2}y_0 + \frac{1}{2}y_3) = y_2$.

$L_3 = y_0 + i(\frac{1}{4}y_1 - \frac{1}{4}y_2 + \frac{1}{2}y_4) - (\frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{2}y_0 - \frac{1}{2}y_3) - i(\frac{1}{4}y_1 - \frac{1}{4}y_2 - \frac{1}{2}y_4) + (\frac{1}{4}y_1 + \frac{1}{4}y_2 - \frac{1}{2}y_0 + \frac{1}{2}y_3) = y_3 + y_4 i$

Thus $f(x) = (y_0, y_1, y_2, y_3 + y_4 i)$ and f is surjective.

So that f is a ring isomorphism and the proof is complete.

Result:

The 4-cyclic group of unit of $Q_4(I)$ is isomorphic to $Q^* \times Q^* \times Q^* \times (Q(i))^*$.

Theorem:

Let $R_4(I)$be the 4-cyclic refined ring of reals, then:

$R_4(I) \cong R \times R \times R \times \mathbb{C}$

The proof holds directly by defining a similar mapping of the previous theorem.

Result:

The group of units of the 4-cyclic refined ring of reals is isomorphic to $R^* \times R^* \times R^* \times C^*$.

Remark:

A 4-cyclic refined real number $X = x_0 + x_1 I_1 + x_2 I_2 + x_3 I_3 + x_4 I_4$ is invertible if and only if: $x_0 \neq 0$, $x_0 + x_1 + x_2 + x_3 + x_4 \neq 0$, $x_0 - x_1 + x_2 - x_3 + x_4 \neq 0$, $(x_1 - x_3 \neq 0 \; with \; x_0 - x_2 + x_4 \neq 0)$:

For example $X = 2 + I_1 + I_2 - 3I_3 - I_4$ is not invertible, that is because 2+1+1-3-1=0.

3. Conclusion

In this paper, we have proved the following results about the classification of Von Shtawzen's 3-cyclic and 4-cyclic refined neutrosophic groups in rational and real cases:

- The group of units of the 3-cyclic refined real ring is isomorphic to $R^* \times R^* \times C^*$, that holds directly from the previous theorem.
- The 3-cyclic refined group of units of $Q_3(I)$ is isomorphic to $Q^* \times Q^* \times (Q(\sqrt{3}i)^*$.
- The 4-cyclic group of unit of $Q_4(I)$ is isomorphic to $Q^* \times Q^* \times Q^* \times (Q(i))^*$.
- The group of units of the 4-cyclic refined ring of reals is isomorphic to $R^* \times R^* \times R^* \times C^*$.

In the future, we aim to find the classification of Von Shtawzen's group in the general case of n-cyclic refined neutrosophic real/rational neutrosophic rings.

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