A Note on Invertible Neutrosophic Square Matrices

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Abstract

The purpose of this article is to study the adjoint and inverse of neutrosophic matrices, where the inverse of a neutrosophic square matrix is defined and studied in terms of neutrosophic determinant and neutrosophic adjoint. It is shown by examples that, the converse part of the result "M is invertible if and only if $detM \neq 0$" is not true, proved by Mohammad Abobala et al. in.[2] Also some of the properties of neutrosophic adjoint are discussed.

Keywords: Neutrosophic matrix; neutrosophic adjoint; neutrosophic determinant; neutrosophic inverse.

1 Introduction

Smarandache proposed the neutrosophic logic to represent a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction, where the concept of is a new branch of philosophy. Neutrosophic concept found the way in many branch of mathematics such as graph theory,[6] number theory,[13] topology,[1] statistics,[15] algebraic equations[3,10] and Boolean algebra.[7]

In,[11] Kandasamy and Smarandache introduced the concept of neutrosophic algebraic structures. In this reference, several neutrosophic algebraic structures are introduced and studied. They introduced some neutrosophic algebraic structures like neutrosophic Fields, neutrosophic vector spaces, neutrosophic groups and neutrosophic rings.

In linear algebra, matrices are playing an important role in the theory of vector spaces and linear transformations. They were generalized to neutrosophic matrices.[8,9,12] Recently in,[2] Mohammad Abobala et al. studied the algebraic properties of neutrosophic matrices such as diagonalization problem, invertibility, determinants and he also studied algebraic representations of neutrosophic matrices by linear transformations.[4]

In this paper, the adjoint of neutrosophic matrices are introduced and the inverse of neutrosophic matrices are defined in terms of neutrosophic adjoint and neutrosophic determinant. It is shown by examples that, the converse part of the result "M is invertible if and only if $detM \neq 0$" is not true, proved by Mohammad Abobala et al. in.[2]

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2 Preliminary

In this section, we recall the notions neutrosophic numbers, neutrosophic matrices and related results.

Definition 2.1. [15] Suppose that w is a neutrosophic number, then it takes the following standard form: $w = a + bI$ where a, b are real coefficients, and I represent indeterminacy, such $0.I = 0$ and $I^n = I$ for all positive integers n.

Definition 2.2. [15] Suppose that w_1, w_2 are two neutrosophic numbers, where

$$w_1 = a_1 + b_1 I, w_2 = a_2 + b_2 I$$

To find $(a_1 + b_1 I) \div (a_2 + b_2 I)$, we can write:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} \equiv x + yI$$

where x any y are real unknowns.

$$a_1 + b_1 I \equiv (a_2 + b_2 I)(x + yI)$$
$$a_1 + b_1 \equiv a_2 x + (b_2 x + a_2 y + b_2 y)I$$

by identifying the coefficients, we get

$$a_1 = a_2 x, \quad b_1 = b_2 x + (a_2 + b_2)y.$$

One obtains unique solution, provided that

$$\begin{vmatrix} a_2 & 0 \\ b_2 & a_2 + b_2 \end{vmatrix} \neq 0 \Rightarrow a_2(a_2 + b_2) \neq 0.$$

Hence, $a_2 \neq 0$ and $a_2 \neq -b_2$ are that conditions for the division of neutrosophic real numbers exists. Then:

$$\frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} I.$$

Definition 2.3. [14] Let K be a field, the neutrosophic field $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 2.4. [13] Let $M_{m \times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$ is a neutrosophic field. We call $M_{m \times n}$ to be the neutrosophic matrix.

Definition 2.5. [2] Let $M = A + BI$ be a neutrosophic n square matrix, where A and B are two n square matrices, then M is called an invertible neutrosophic n square matrix , if and only if there exists an n square matrix $S = S_1 + S_2 I$, where S_1 and S_2 are two n square matrices such that $S \cdot M = M \cdot S = U_{n \times n}$, where $U_{n \times n}$ denotes the n square matrix.

Definition 2.6. [2] Let $M = A + BI$ be a neutrosophic n square matrix. The determinant of M is defined as $detM = detA + [det(A + B) - detA]I$.

Theorem 2.7. [2] *Let $M = A + BI$ be a neutroshophic n square matrix, where A and B are two n square matrices, then $M^r = A^r + [(A + B)^r - A^r]I$.*

Theorem 2.8. [2] *Let $M = A + BI$ and $N = C + DI$ be two neutrosophic n square matrices, then*
(1) $det(M \cdot N) = detM \cdot detN$.
(2) $det(M^{-1}) = (detM)^{-1}$.
(3) $detM = 1$ *if and only if* $detA = det(A + B) = 1$

Theorem 2.9. [5] *Let $R(I)$ be any neutrosophic ring. For any $x + yI \in R(I)$, we have $(x + yI)^n = x^n + [(x + y)^n - x^n]I$.*

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3 Invertible Neutrosophic Square Matrices

We begin with the following definition.

Definition 3.1. Let $M = A + BI$ be a neutrosophic n square matrix. The adjoint matrix of M is defined as $adjM = adjA + [adj(A + B) - adjA]I$.

Example 3.2. Consider the following neutrosophic matrix:

$$M = \begin{pmatrix} 2+I & 1-I \\ -I & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} I$$

Here, $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $A + B = \begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix}$, $adjA = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $adj(A + B) = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$.
Therefore, $adjM = adjA + [adj(A + B) - adjA]I = \begin{pmatrix} 1 & -1+I \\ I & 2+I \end{pmatrix}$

Example 3.3. Consider the following neutrosophic matrix:

$$M = \begin{pmatrix} 2+I & 1+I & 3-I \\ -1+I & 3-2I & 1+3I \\ 3+2I & 4-I & 2-3I \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & 4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 3 \\ 2 & -1 & -3 \end{pmatrix} I$$

Here, $A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 3 & 1 \\ 3 & 4 & 2 \end{pmatrix}$, $A+B = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & 4 \\ 5 & 3 & -1 \end{pmatrix}$, $adjA = \begin{pmatrix} 2 & 10 & -8 \\ 5 & -5 & -5 \\ -13 & -5 & 7 \end{pmatrix}$, $adj(A+B) = \begin{pmatrix} -13 & 8 & 6 \\ 20 & -13 & -12 \\ -5 & 1 & 3 \end{pmatrix}$.
Therefore, $adjM = adjA + [adj(A + B) - adjA]I = \begin{pmatrix} 2-15I & 10-2I & -8+14I \\ 5+15I & -5-8I & -5-7I \\ -13+8I & -5+6I & 7-4I \end{pmatrix}$

Theorem 3.4. [2] *Let $M = A + BI$ be neutrosophic n square matrix, A and B are two n square matrices, then M is invertible if and only if A and $A + B$ are invertible matrices and $A^{-1} + \left[(A + B)^{-1} - A^{-1}\right] I$.*

Theorem 3.5. [2] *$M = A + BI$ is invertible matrix if and only if $detM \neq 0$*

Remark 3.6. The converse the above theorem is not true. See the following examples.

Example 3.7. Let $M = \begin{pmatrix} 2I & 4 \\ I & 3 \end{pmatrix}$. Then $detM = 2I \neq 0$. Here, $A = \begin{pmatrix} 0 & 4 \\ 0 & 3 \end{pmatrix}$ and $detA = 0$. Therefore, A is not invertible and hence M is not invertible by Theorem 3.4.

Example 3.8. Let $M = \begin{pmatrix} 1 & -2+I \\ -I & 1 \end{pmatrix}$. Then $detM = 1 - I \neq 0$. Here, $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $A + B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Here, $det(A + B) = 0$, $A + B$ is not invertible and hence M is not invertible by Theorem 3.4.

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Theorem 3.9. *Let $M = A + BI$ be neutrosophic n square matrix, A and B are two n square matrices, then M is invertible if and only if $detA \neq 0$ and $det(A + B) \neq 0$ and*

$$M^{-1} = \frac{1}{detM}(adjM)$$

.

Proof. By definition 2.2,

$$\frac{1}{detM} = \frac{1}{detA + [det(A + B) - detA]I}$$

exists only if

$$detA \neq 0 \text{ and } detA + det(A + B) - detA \neq 0,$$
$$\text{i.e., } detA \neq 0 \text{ and } det(A + B) \neq 0.$$

Also,

$$\begin{aligned}
M^{-1} &= \frac{1}{detM}(adjM) \\
&= \left(\frac{1}{detA + [det(A + B) - detA]\, I}\right)(adjA + [adj(A + B) - adjA]I) \\
&= \left(\frac{1}{detA} - \left(\frac{det(A + B) - detA}{detA\, det(A + B)}\right) I\right)(adjA + [adj(A + B) - adjA]I) \\
&= \left(\frac{1}{detA} - \left(\frac{1}{detA} - \frac{1}{det(A + B)}\right) I\right)(adjA + [adj(A + B) - adjA]I) \\
&= \frac{adjA}{detA} - \frac{adjA}{detA}I + \frac{adj(A + B)}{det(A + B)}I \\
&= A^{-1} - A^{-1}I + (A + B)^{-1}I \\
&= A^{-1} + \left[(A + B)^{-1} - A^{-1}\right] I
\end{aligned}$$

Hence the result holds by Theorem 3.4. □

Example 3.10. Consider the neutrosophic matrix $M = \begin{pmatrix} 1 & -1 + I \\ I & 2 + I \end{pmatrix}$, where, $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $A + B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$.

Here, $detA = 2$, $det(A+B) = 3$, hence M is invertible and $detM = 2+I$. Also, $adjM = \begin{pmatrix} 2 + I & 1 - I \\ -I & 1 \end{pmatrix}$.

Thus,

$$\begin{aligned}
M^{-1} &= \frac{1}{detM}(adjM) \\
&= \frac{1}{2 + I}\begin{pmatrix} 2 + I & 1 - I \\ -I & 1 \end{pmatrix} \\
&= \left(\frac{1}{2} - \frac{1}{6}I\right)\begin{pmatrix} 2 + I & 1 - I \\ -I & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2}I \\ -\frac{1}{3}I & \frac{1}{2} - \frac{1}{6}I \end{pmatrix}
\end{aligned}$$

Remark 3.11. If M is a invertible neutrosophic matrix and M^{-1} is its inverse, then $adjM = detM \cdot M^{-1}$.

Theorem 3.12. *Let $M = A + BI$ and $N = C + DI$ be neutrosophic invertible n square matrices. Then MN is also invertible and $(MN)^{-1} = N^{-1}M^{-1}$.*

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Proof. If M is invertible then $det(A) \neq 0$, $det(A+B) \neq 0$. Similarly, if N is invertible then $det(C) \neq 0$, $det(C+D) \neq 0$. This implies that, $det(AC) = detA\ detC \neq 0$ and $det[(A+C)(C+D)] = det(A+B)\ det(C+D) \neq 0$. Hence, $MN = AC + [BC+BD+AD] = AC + [(A+B)(C+D) - AC]$ is invertible. Also by associativity of matrix multiplication, we have

$$(MN)(N^{-1}M^{-1}) = M(NN^{-1})M^{-1} = MM^{-1} = U_{n\times n}$$
$$(N^{-1}M^{-1})(MN) = N^{-1}(M^{-1}M)N = N^{-}N = U_{n\times n}.$$

Thus,$(MN)^{-1} = N^{-1}M^{-1}$. □

Theorem 3.13. *Let $M = A + BI$ and $N = C + DI$ be two neutrosophic n square matrices. Then the following properties holds.*

(1) $det(adjM) = (detM)^{n-1}$.

(2) $adj(MN) = adjM\ adjN$.

(3) $adj(M^m) = (adjM)^m$ *for any positive integer* m.

(4) $adj(M^T) = (adjM)^T$.

(5) $adj(kM) = k^{n-1}adjM$ *for any neutrosophic number* k.

(6) $adj(adjM) = (detM)^{n-2}M$

Proof. (1).

$$\begin{aligned} det(adjM) &= det(adjA + [adj(A+B) - adjA]I) \\ &= det(adjA) + [det(adjA + adj(A+B) - adjA) - det(adjA)]\, I \\ &= det(adjA) + [det(adj(A+B)) - det(adjA)]\, I \\ &= (detA)^{n-1} + \left[(det(A+B))^{n-1} - (detA)^{n-1}\right] I \\ &= (detA)^{n-1} + \left[(detA + det(A+B) - detA)^{n-1} - (detA)^{n-1}\right] I \\ &= (detA + [det(A+B) - detA]\, I)^{n-1} \\ &= (detM)^{n-1}. \end{aligned}$$

(2). $MN = AC + [BC+BD+AD] = AC + [(A+B)(C+D) - AC]$.

$$\begin{aligned} adj(MN) &= adj(AC) - [adj(A+B)(C+D) - adj(AC)]I \\ &= adjA\ adjC + [adj(A+B)adj(C+D) - adjA\ adjD]I \\ &= (adjA + [adj(A+B) - adjA]I)\ (adjC + [adj(C+D) - adjC]I \\ &= adjM\ adjN. \end{aligned}$$

(3). We can prove this easily by using property (2).

(4).

$$\begin{aligned} adj(M^T) &= adj(A^T + B^T I) \\ &= adjA^T + \left[adj(A^T + B^T) - adjA^T\right] I \\ &= (adjA)^T + \left[adj(A+B)^T - (adjA)^T\right] I \\ &= (adjA)^T + \left[(adj(A+B))^T - (adjA)^T\right] I \\ &= (adjA)^T + [adj(A+B) - adjA]^T\, I \\ &= (adjA + [adj(A+B) - adjA]\, I)^T \\ &= (adjM)^T. \end{aligned}$$

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(5).

$$\begin{aligned} adj(kM) &= adj(kA + kBI) \\ &= adj(kA) + [adj(kA + kB) - adj(kA)]\, I \\ &= adj(kA) + [adjk(A + B) - adj(kA)]\, I \\ &= k^{n-1} adjA + \left[k^{n-1} adj(A + B) - k^{n-1} adjA\right] I \\ &= k^{n-1} \left(adjA + [adj(A + B) - adjA]\, I\right) \\ &= k^{n-1} adjM. \end{aligned}$$

(6).

$$\begin{aligned} adj(adjM) &= adj\left(adjA + [adj(A + B) - adjA]I\right) \\ &= adj(adjA) + [adj(adj(A + B)) - adj(adjA)]I \\ &= (detA)^{n-2} \cdot A + [(det(A + B))^{n-2} \cdot (A + B) - (detA)^{n-2} \cdot A]I \\ &= \left((detA)^{n-2} + [(det(A + B))^{n-2} - (detA)^{n-2}]I\right) \cdot (A + [A + B - A]I) \\ &= (detA + [det(A + B) - detA]I)^{n-2} \cdot (A + BI) \\ &= (detM)^{n-2} M. \end{aligned}$$

□

4 Conclusion

In this article, the adjoint of neutrosophic square matrices was defined and the inverse of invertible neutrosophic square matrices was studied in terms of neutrosophic adjoint and neutrosophic determinant. It was shown that, the necessary and sufficient condition for the invertibility of a neutrosophic square matrix $M = A + BI$ is $detA \neq 0$ and $det(A + B) \neq 0$(which is not equivalent to $detM \neq 0$).

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