A Discussion of FGF Conjecture in Some Quasi-Frobenus Rings

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Abstract

A ring R is called right *FGF* ring, if every finitely generated right R-module embeds in a free right R-module. It is well known that a quasi-Frobenus ring R is right *FGF* ring, but the converse is still an open question. In this note we give some equivalent additional conditions that convert the right *FGF* ring to the QF ring. Other known results that characterize the class of quasi-Frobenius rings are fond. In the process, some new results that characterize the class of IF-rings are provided.

Keywords: QF- ring; FGF-ring; IF-ring; FP-injective module; Flat module.

1. Introduction.

Throughout this paper, R is an associative ring with identity; all modules are unitary. It is well known that a ring R is *quasi-Frobenius (briefly, QF)* when *every right R-module embeds in a free right R-module.* This is one of many characterizations for the class of QF rings. For detailed information about QF rings (see [1] – [3]). If the embedding is restricted to finitely generated right R-modules, i.e. if every finitely generated right R-module embeds in a free right R-module, then R is called right *FGF* ring, and the question of whether any right *FGF* ring is QF is known as the *FGF* conjecture (or Faith's Problem). This conjecture goes back to the early eighties of the last century, but is still an open question. From that time until now there are many affirmative answers to this question with different additional conditions. Among them, when R is left perfect, left self-injective left or right noetherian , when the injective hull of R as a right R-module is a projective R-module or when every finitely generated right R-module essentially embeds in a free right R-module. (see [3]–[7] for a discussion and first results and [8]–[12] for more recent results.

In this note we are concerned with the *FGF* conjecture with additional conditions. By the first theorem we give some affirmative answers to this question by finding some equivalent additional conditions that characterize the class of QF rings. The main result of this paper is a characterization of a QF ring R as a right *FGF* with one of the conditions, *every flat submodule of any flat right R-module is a pure, every factor module of a flat right R-module by a flat submodule is flat, every flat right R-module is FP-injective.* We also get some known partial affirmative answers for that question, namely we find the additional conditions, when R is *left IF-ring* and when R is *right perfect*. Thus shedding a few light on an old open question in ring theory.

In the process, we obtain some results that characterize the class of IF rings. By the second theorem that basically results by proving the first theorem, we give some equivalent conditions that characterize the class of IF rings. Finally we note that the question of whether *any right FGF ring R with every flat right R-module is f-injective is QF* remains open.

Now we recall some known notions and facts needed in the sequel.

A short exact sequence of right R-modules

$0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$

is said to be *pure* if the sequence

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$0 \to \mathrm{Hom_R}(M, A) \to \mathrm{Hom_R}(M, B) \to \mathrm{Hom_R}(M, C) \to 0$

is exact, for every finitely presented left R-module M.

A submodule N of a right R-module M is said to be *pure submodule*

if the canonical short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

is pure.

A right R-module M is said to be *FP-injective* (or *absolutely pure*) if it is pure in every module that contains it. Detailed information for the concepts of pure submodules and FP-injective modules can be found in [13] and [14].

A ring R is called *right IF-ring* if every injective right R-module is flat. If R is right and left *IF-ring* then R is called *IF-ring* [15]. For all ring-theoretical terminology not defined here, the reader is referred to [1] and [2].

Lemma 1. *For any ring R, the following statements are equivalent.*

(1) *R is right IF ring.*
(2) *Every right R-module embeds in a flat right R-module.*
(3) *Every finitely generated right R-module embeds in a flat right R- module.*
(4) *Every FP-injective right R-module is flat.*

Proof. (1) ⇒ (2) since the injective hull of every right R-module is flat.

(2) ⇒ (3) is trivial.

(3) ⇒ (1) follows from the fact that : if each finitely generated

submodule of an injective right R-module M can be embedded in a flat

right R-module then M is flat [16, Lemma2].

(1) ⇒ (4) follows from the fact that: every FP injective submodule of a flat module is flat [17, Corollary1.17].

(4) ⇒ (1) is trivial. ■

Lemma 2. [see 18**, Lemma2**] Let

$$0 \longrightarrow N \longrightarrow P \longrightarrow F \longrightarrow 0$$

be a short exact sequence of right R-modules with P flat. The sequence is pure exact if and only if F is flat.

Lemma 3. [14, Propositions 2.3.1 and 2.3.2] *A right R-module M is FP-injective (or absolutely pure) if and only if M is a pure submodule of an injective right R-module.*

Lemma 4. [19, p. 280] *For every direct system* (M_i , f_{ij}) *of right R-modules, the canonical epimorphism* $\bigoplus_{i \in I} M_i \to \lim_{\rightarrow} M_i \to 0$ *is pure.*

Lemma 5. *If R is a right IF-ring and every flat right R-module is FP-injective, then every factor module of an FP-injective right R-module by a pure submodule is FP-injective.*

Proof. Assume that R is a right IF-ring and every flat right R-module is FP-injective, M is an FP-injective right R-module, and N its pure submodule, then M is flat by Lemma1(4), and M/N is flat by Lemma

2. Hence M/N is FP-injective by assumption. ■

A ring R is called *right coherent*, if every finitely generated right ideal is finitely presented or equivalently every finitely generated submodule of every finitely generated right R-module is finitely presented.

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Lemma 6. [13, Theorem 3.2] *A ring R is right coherent if and only if every direct limit of FP-injective right R-modules is FP-injective.*

Lemma 7. [21, Theorem18] *A ring R is right noetherian if and only if every cyclic right R-module is finitely presented.*

Lemma 8. [17, Corollary1.10] *A ring R is right noetherian if and only if every right FP-injective R-module is injective.*

A ring R is called *right perfect*, if each right R-module has a projective cover. From [22, Lemma 2] by considering R as a small preadditive category with single object whose endomorphism ring is R, we can get the following lemma.

Lemma 9. *If R is right noetherian and R is left IF-ring, then R is right perfect.*

Now we are in position to state the following partial answer to the FGF conjecture.

Theorem 1. *Let R be a right FGF ring, then the following assertions are equivalent.*

(1) *Every flat submodule of any flat right R-module is pure.*
(2) *Every factor module of a flat right R-module by a flat submodule is flat.*
(3) *Every flat right R-module is FP-injective.*
(4) *R is left IF-ring.*
(5) *R is right perfect.*
(6) *R is QF-ring.*

Proof. Since R is right FGF ring, it is clear that R is right IF ring (See Lemma 1).

(1) ⇒ (2) is clear by Lemma 2 and the fact that, a submodule N of a right R-module M is pure if and only if the canonical short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

is pure.

(2) ⇒ (3). Let M be a flat right R-module, E its injective hull. Then E is flat right R-module and E/M is flat by (2), and the canonical short exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow E/M \longrightarrow 0$$

is pure by Lemma 2, so M is a pure submodule of the injective right R-module E and M is FP-injective by Lemma 3.

(2) ⇒ (4). Assume that every flat right R-module is FP-injective. To prove that R is left IF-ring, it is sufficient to prove that R is right coherent and right self FP-injective [15, Theorem1]. We have by assumption R is right self FP-injective. To show that R is right coherent, it is sufficient to prove that every direct limit of FP-injective right R-modules is FP-injective [see Lemma 6]. Let $(M_i, f_{ij})_I$ be a direct system of FP-injective right R-modules. Then $\bigoplus_{i \in I} M_i$ is FP-injective right R-module [20, p.156]

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and the canonical epimorphism $\bigoplus_{i \in I} M_i \to \varinjlim M_i \to 0$ is pure [see Lemma 4]. So $\varinjlim M_i$ is FP-injective right R-module [see Lemma 5]. Thus R is left IF-ring.

(4) ⇒ (5). Let R be a left IF-ring. Then R is IF-ring, so R is right coherent and hence every finitely generated submodule of any finitely generated right R-module is finitely presented. Let M be a cyclic right R-module. Since R is right FGF ring, there is a finitely generated free right R-module contains M, then M is finitely presented, and R is right neotherian by Lemma 7. So R is right perfect by Lemma 9.

(5) ⇒ (6). Let R be a right perfect ring. Then every flat right R-module is projective [23, Theorem P]. Since R is right IF-ring, then every injective right R-module is projective. Hence R is QF-ring by [24, Theorem 5.3].

(6) ⇒ (1). Let N be a flat submodule of a flat right R-module M. Since R is QF-ring, then both M and N are injective, so N is a direct summand of M. But every direct summand is a pure submodule [18, Lemma1]. This completes the proof. ■

Remark 1. In the proof of the Theorem 1 we used the fact that every finitely generated right R-module embeds in a projective right R-module to prove the conditions (4) - (6) that is R is right perfect and then QF. But to prove the conditions (1) - (4), we only used the fact that R is right IF-ring. This suggests the following characterizations for IF-rings.

Theorem 2. For any ring R the following assertions are equivalent.

(1) R is right IF-ring and every flat submodule of any flat right R-module is pure.
(2) R is right IF-ring and every factor module of a flat right R-module by a flat submodule is flat.
(3) Every flat right R-module is FP-injective and every FP injective right R-module is flat.
(4) R is IF-ring.

Proof. The proof of the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) is similar to the one of the same in the proof of the Theorem1. It remains to prove the implication (4) ⇒ (1). Let R be an IF-ring, then by [15, Theorem 2] R is coherent (right and left). Let M be a flat right R-module, then the character module $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is injective left R-module [see 25, Chap. 5, sec. 5.3]. Since R is left IF-ring, M^* is flat left R-module, so the character module of is M^{**} of M^* is injective right R-module. By [26, Theorem 1] M is FP-injective right R-module. So we have the fact that, every flat right R-module is FP-injective. Now assume that N is flat submodule of the flat right R-module M, then N is FP-injective (absolutely pure), so N is pure submodule of M. Hence (1) holds. ■

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