On Lie Groups and P- two Norm Algebras

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Abstract

The objective of this paper is studying *Lie* group *G(X)* on algebra *X* and define two norms in this algebra and give some properties of two-norm algebras which are topological, and relate some concepts with the topological notions introduced by A. Alexiwicz.

Keywords: Lie group; Lie algebra; Normed algebra

1. Introduction:

A vector space or algebra over a field of real or complex numbers will be denote by *X* . For the sets *A* and *B* in *X* we write
$\alpha A = \{\alpha a; a \in A\}, A.B = \{ab; a \in A, b \in B\}, A^{-1} = \{a^{-1}; a \in A\}$

A topological space *T* is called Hausdorff if for any two distinct points *p*1; *p*2 ∈ *M* there exists open sets *U*1;*U*2 ∈ *T* with
*p*1 ∈ *U*1; *p*2 ∈*U*2; *U*1 ∩*U*2 = ϕ.

A map between topological spaces is called continous if the preimage of any open set is again open.

The important of research and its objectives:

The object of this paper is to generalize the notation of two-norm algebras introduced by A. Alexiewicz[1]. First of all we consider topologies in algebras which arise from two *p*-homogeneous norms.

Research methods and materials:

Let M be a manifold, An open chart on *M* is a pair (*U*; □), where *U* is an open subset of *M* and □ □ is a homeomorphism of *U* onto an open subset of R^n.

Definition 1:[2]. A smooth manifold *M* is an n-dimensional Hausdorff space where in the neighborhood of any point *p*∈*M* there exists a chart of n-dimension.

Definition 2:[2] A Lie group is a group that is also a smooth manifold such that the multiplication map:

$$\boldsymbol{\mu}: \boldsymbol{G} \times \boldsymbol{G} \xrightarrow[(g,h)\to gh]{} \boldsymbol{G}$$
$$\xi: \boldsymbol{G} \xrightarrow[g\to g^{-1}]{} \boldsymbol{G}$$

Are smooth. Space with a transitive G action for a Lie group G are called as is a homogeneous space. $S^2 = \{x \in R^3; |x| = 1\}$ homogeneous space. Write a

We shall denote by *e* the unit element of the algebra if existing. *G*(*X*) and $G_0(X)$ will denote the multiplicative Lie group of the algebra *X* and the set of quasi invertible

elements.γ(τ) will denote the neighborhood filter of zero for the topology τ.

Definition 3.[3].The function $\|\ \|: X \to R$ is said to be *p- homogeneous norm* or shortly *p*-norm where $0 < P \leq 1$ if $\|.\|$ satisfy the following conditions:

(1) $\|x\| = 0$ iff *x*=0
(2) $\|x + y\| \leq \|x\| + \|y\|$
(3) $\|\alpha x\| = |\alpha|^P \|x\|$

And $(X, \|\ \|)$ is said to be topological vector spaces or *t.v.s.* has a neighborhood basis of zero composed of bounded set.

Definition 4.[3]: The triplet $(X, \|\ \|, \|\ \|_0)$ is said to be *p-two- norm space* if *X* is a vector space and $\|\ \|, \|\ \|_0$ are two *p*- homogeneous norms, the first being finer then the second one. This is the case if and only if there exists a constant k such that $\|x\|_0 \leq k\|x\|$ for every *x*∈*X*.

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Let τ_0 denote the topology generated in X by the metric $\rho(x,y) = \|x-y\|$, let $S_n = \{x \in X; \|x\| \le n\}$. By τ_∞ we shall denote the finest vector topology on X which induced on S a topology coarser than τ_0. such topology exist.

Dsefinition 5. [3]: Let $(X, \| \|, \| \|_0)$ be p-two- norm space, suppose that in X defined multiplication of elements making X together with the vector operation an algebra . If multiplication is continuous for the topology τ_∞ then $(X, \| \|, \| \|_0)$ is called p-two norm algebra, the topology τ_∞ is then called multiplicative

Theorem 1. The topology τ_∞ is multiplicative if and only if

(i) there exists a constant β such $\|xy\| \le \beta\|x\|\|y\|$

(ii) for every $U \in \gamma(\tau_0)$ there exists a $V \in \gamma(\tau_0)$ such that $S(V \cap S) \subset U$, $(V \cap S)S \subset U$

Proof: Necessity. The set S is τ_0 –bounded, for if $x_n \in S$, $\alpha_n \to 0$ then

$$\|\alpha_n x_n\| = |\alpha_n|^P \|x_n\| \to 0$$

The set SS is also τ_0 –bounded, Indeed let $\|x_n\| \le 1$, $\|y_n\| \le 1$, $\alpha_n \to 0$, then

$$\|\alpha_n x_n y_n\| = \left\|\sqrt{\alpha_n x_n}\sqrt{\alpha_n y_n}\right\| \to 0$$

Let us prove the necessity of the condition (i).

1- Suppose, if possible, that there exist x_n, y_n such that $\|x_n\| \le 1$, $\|y_n\| \le 1$, $\|x_n y_n\| > n\|x_n\|\|y_n\|$, then $\|x_n\| \ne 0 \ne \|y_n\|$ and

$$\left\|\frac{x_n}{n^{1/2P}\|x_n\|^{1/P}} \frac{y_n}{n^{1/2P}\|y_n\|^{1/P}}\right\| > 1$$

This is impossible, since $\left\|x_n/\|x_n\|^{1/P}\right\| = \left\|y_n/\|y_n\|^{1/P}\right\| = 1$

We now prove that the condition $(ii)$is necessary. Observe first that the set S is τ_0 – bounded. Indeed, let $x_n \in S$, $\alpha_n \to 0$ then $\|\alpha_n x_n\| = |\alpha_n|^P \|x_n\| \to 0$, therefore $\|\alpha_n x_n\|_0 \to 0$

Next observe that the topology τ_0 is coarser than τ_∞. To see this choose a sequence (U_n) of τ_0-neighberhoods of zero, such that $U_1 + U_1 \subset U$, $U_{n+1} + U_{n+1} \subset U_n$ for $n = 1,2,3,\ldots$

Then $V := \sum_{i=1}^{\infty} U_n \cap S$ is in $\gamma(\tau_\infty)$ and $V \subset \sum_{i=1}^{\infty} U_n \subset U$.

Now let U be in $\gamma(\tau_0)$. Since the topology τ_∞ is multiplicative and $\tau_0 \le \tau_\infty$, so there exists $W \in \gamma(\tau_\infty)$ such that $W \subset U$. Thus $WW = \sum V_n \cap S$ where $V_n \in \gamma(\tau_0)$ and therefore $(V_1 \cap S)W \subset U$. Since the set W is τ_0-bounded, $\alpha W \subset V_1$ for some $\alpha > 0$ and it is enough to choose $V = \min(\alpha, \alpha^{-1})V_1$. The second part of (ii) is proved similarly.

2-The sufficiency of both conditions. Since $\|xy\| \le \beta\|x\|\|y\|$ introducing a new p-norm $\|x\|' = \sqrt{\beta}\|x\|$ we obtain a sub- multiplicative p-norm equivalent to $\|x\|$ for which the unit ball equals $\beta^{-1/2P}$, and we can set $\sum U_n \cap \beta^{-1/2P}$ give also the neighborhood basis of zero for τ_∞. So we can suppose freely that $\|xy\| \le \|x\|\|y\|$ and therefore that $SS \subset S$

II- On continuity of the inverse

Let the algebra X has the unit element e and denote by G(X) the multiplicative Lie group of X. We shall say that the inverse is ϑ- continuous if:

(a) $x_n \xrightarrow{\rho} e$ *implies that almost all* x_n *are in* $G\,X$

(b) If $x_n \in G(X)$, $x_n \xrightarrow{\rho} e$, then $x_n^{-1} \xrightarrow{\rho} e$

From this condition it follows that if $x_n \xrightarrow{\rho} x_0 \in G(X)$, then almost all x_n are in G(X)and $x_{n+k}^{-1} \xrightarrow{\rho} x_0^{-1}$ for some k in N

The condition (b) is equivalent to the following one:

(b') Let $x_n \in G(X)$, $x_n \xrightarrow{\rho} e$, then $\sup_n \|x_n^{-1}\| < \infty$

The necessity of this condition being obvious, we only need to prove its sufficiency.

So let $x_n \in G(X)$, $x_n \xrightarrow{\rho} 1$. We distinguish two cases.

1. There exists a $\delta > 0$ such that $\|x_n - 1\| \ge \delta$ for all n. Then

$$\|x_n^{-1} - 1\|_0 = \|x_n^{-1}(1 - x_n)\| = \|x_n^{-1}\|\|1 - x_n\| \left\|\frac{x_n^{-1}}{\|x_n^{-1}\|^{1/P}} \frac{1-x_n}{\|1-x_n\|^{1/P}}\right\|_0 \le \sup_n\|x_n^{-1}\| \sup\|1 - x_n\|\varepsilon$$

Provided that $\left\|(1 - x_n)/\|1 - x_n\|^{1/P}\right\|_0 \le \eta(\varepsilon)$ *i.e when* $\|1 - x_n\|_0 \le \eta(\varepsilon)/\delta$

2. $\lim_{n\to\infty} \|x_n - 1\| = 0$, then $\vartheta > 0$ being small enough, $\|x_n - 1\| \le \vartheta$ implies $\|x_n^{-1} - 1\| < \varepsilon$ When the inverse is ρ-continuous, there exist two functions $f, g: R^+ \to R^+$ such that:

(c) $\|x\| \le q$, $\|x\|_0 \le f(q)$ implies $1 + x \in G(X)$

(d) $\|x\| \le q$, $\|x\|_0 \le f(q)$ implies $\|(1 + x)^{-1}\| \le g(q)$

Definition 6.[4] A topological algebra (X, τ) is called *inverse continuous* if the multiplicative lie group $G(X)$ is τ-open and the map $x \to x^{-1}$is τ-continuous on $G(X)$

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Theorem 2. A p-two norm algebra (with unite) equipped with the Wiweger topology is inverse continuous if and only if the inverse is ρ-continuous.

Proof: The necessity being obvious, we prove the sufficiency.

1- So let us suppose that the inverse is ρ-continuous. First we prove that G(X) is τ_∞- open. It is enough to show that $1 \in$ G(X). Let $\bar{\omega}_1 = f(1)$, $\sigma_1 = g(1)$, $\bar{\omega}_{n+1} = \rho(\frac{f(\sigma_n)}{\sigma_n})$, $\sigma_{n+1} = \sigma_n(1+\sigma_{n+1})$

We shall prove that:

(*) if $\|x_i\| \le 1$, $\|x_i\|_0 \le \bar{\omega}_1$, then $1 + x_1 + \cdots + x_n \in G(X)$ and $\|(1 + x_1 + \cdots + x_n)^{-1}\| \le \sigma_n$

For n =1, $1 + x_1 \in G(X)$ *and* $\|(1+x_1)^{-1}\| \le g(1) = \sigma_1$. Suppose now that(*) holds for

The element $x_1, \dots, x_n$ and let us suppose that $x_1, \dots, x_n, x_{n+1}$ fulfill the assumption (*).

Let $v = 1 + x_1 + \cdots + x_n$, *then* $v \in G(X)$ *and* $\|(1+v)^{-1}\| = \sigma_n$

Thus $w := 1 + x_1 + \cdots + x_n = v(1 + v^{-1}x_{n+1})$ and $\|v^{-1}x_{n+1}\| \le \|v^{-1}\|\|x_{n+1}\| \le \|v^{-1}\| \le \sigma_n$.

Moreover

$$\|v^{-1}x_{n+1}\|_0 = \left\| \|v^{-1}\|^{1/P} \frac{v^{-1}}{\|v^{-1}\|^{1/P}} x_{n+1} \right\|_0 \le \|v^{-1}\| \left\| \frac{v^{-1}}{\|v^{-1}\|^{1/P}} x_{n+1} \right\|_0$$

And since $\left\| v^{-1}/\|v^{-1}\|^{1/P} \right\| \le 1$, $\|x_{n+1}\| \le 1$, $\|x_{n+1}\|_0 \le w_{n+1} = \rho(f(\sigma_n)/\sigma_n)$

We obtain by (3) $\|v^{-1}x_{n+1}\|_0 \le \|v^{-1}\| \frac{f(\sigma_n)}{\sigma_n} \le f(\sigma_n)$, for $\|v^{-1}\| \le \sigma_n$

Hence by (c) $1 + v^{-1}x_{n+1} \in G(X)$ and therefore w $\in G(X)$ and

$$\|w\| \le \|v\|(\|1\| + \|v^{-1}x_{n+1}\|) \le \sigma_n(1+\sigma_n) = \sigma_{n+1}$$

From (*) it follows that $U_n = \{x \in X; \|x\|_0 \le w\}$ *then the set* $U \coloneqq 1 + \sum_{n=1}^{\infty} U_n \cap S$

Which is a τ-neighborhood of 1. is contained in $G(X)$.

2-We now prove that the inverse is τ_∞- continuous at 1. we shall consider two neighborhood bases for τ_∞ the first B_1 composed of sets of form $\sum_{n=1}^{\infty} P_n \cap S$

Where $P_n \in \gamma(\tau_0)$, the second B_2 composed of sets of form $\sum_{n=1}^{\infty} Q_n \cap \sigma_n(1+\sigma_n)S$.

$Q_n \in \gamma(\tau_0)$It is enough to show that for every U belongs to B_2 there exists V belongs to B_1 such that $1 + V \in G(X)$, *and* $(1+V)^{-1} \in 1 + U$.

We can choose $Q_n = \{x \in X; \|x\|_0 \le \varepsilon_n\}, \varepsilon_n > 0$.

Let $\eta_1 = \min(w_1, \rho(\varepsilon_1/\sigma_1))$, $\eta_n = \min(1, w_n, \rho(\varepsilon_n/\sigma_n\sigma_{n+1}))$

We shall show that for $\|x_i\| \le 1$, $\|x_i\|_0 \le \eta_i$, i=1,2,…, we have

(I) $1 + x_1 + \cdots + x_n \in G(X)$

(II) $(1 + x_1 + \cdots + x_n)^{-1} = 1 + u_1 + \cdots + u_n$; $\|u_i\| \le \sigma_i(1+\sigma_i)$, $\|u_i\|_0 \le \eta_i$.

From $\eta_i \le w_i$, (I) follows immediately

For n = 1, $(1+x_1)^{-1} = 1 + u_1$, where $u_1 = (1+x_1)^{-1} - 1 = -(1+x_1)^{-1}x_1$, whence

$\|u_1\| \le \|(1+x_1)^{-1}\|\|x_1\| \le \sigma_1 \le \sigma_1(1+\sigma_1)$and

$$\|u_1\|_0 = \|(1+x_1)^{-1}x_1\|_0 = \left\| \|(1+x_1)^{-1}\|^{1/P} \frac{(1+x_1)^{-1}}{\|(1+x_1)^{-1}\|^{1/P}} x_1 \right\| \le$$

$$\|(1+x_1)^{-1}\| \left\| \frac{(1+x_1)^{-1}}{\|(1+x_1)^{-1}\|^{1/P}} x_1 \right\|_0$$

And since $\left\| \frac{\|(1+x_1)^{-1}\|}{\|(1+x_1)^{-1}\|^{1/P}} \right\| \le 1$, $\|x_1\| \le 1$, $\|x_1\|_0 \le \rho(\varepsilon_1/\sigma_1)$ we obtain by (3)

$$\|(1+x_1)^{-1}\| \left\| \frac{(1+x_1)^{-1}}{\|(1+x_1)^{-1}\|^{1/P}} \right\|_0 \le \|(1+x_1)^{-1}\| \frac{\varepsilon_1}{\sigma_1} \le \sigma_1 \frac{\varepsilon_1}{\sigma_1} = \varepsilon_1$$

So (II) is true for n = 1. Suppose now it is valid for any set of n elements, and let $\|x_i\| \le 1$, $\|x_i\|_0 \le \eta_i$; $i = 1,2,\dots,n+1$, then $x \coloneqq 1 + x_1 + \cdots + x_n \in G(X)$, $x^{-1} = 1 + u_1 + \cdots + u_n$; $\|u_i\| \le \sigma_i(1+\sigma_i)$, $\|u_i\|_0 \le \varepsilon_i$and

$(1 + x_1 + \cdots + x_n)^{-1} = 1 + u_1 + \cdots + u_n + u_{n+1}$ where

$u_{n+1} = (1 + x_1 + \cdots + x_n + x_{n+1})^{-1} - (1 + x_1 + \cdots + x_n)^{-1} =$

$= -(1 + x_1 + \cdots + x_{n+1})^{-1} x_{n+1} (1 + x_1 + \cdots + x_n)^{-1}$.

Now

$\|u_{n+1}\| \le \|(1 + x_1 + \cdots + x_{n+1})^{-1}\| \|x_{n+1}\| \|(1 + x_1 + \cdots + x_n)^{-1}\| \le \sigma_n\sigma_{n+1}$,

$u_{n+1} = v_1 x_{n+1} + v_2$; $v_1 = (1 + x_1 + \cdots + x_{n+1})^{-1}$, $v_2 = (1 + x_1 + \cdots + x_n)^{-1}$

Whence

$$\|u_{n+1}\|_0 \le \|v_1 x_{n+1} v_2\|_0 = \left\| \|v_1\|^{1/P} \frac{v_1}{\|v_1\|^{1/P}} x_{n+1} \|v_2\|^{1/P} \frac{v_2}{\|v_2\|^{1/P}} \right\|_0 \le$$

$$\le \|v_1\|\|v_2\| \left\| \frac{v_1}{\|v_1\|^{1/P}} x_{n+1} \frac{v_2}{\|v_2\|^{1/P}} \right\|_0$$

Since $\left\| v_1/\|v_1\|^{1/P} \right\| \le 1$, $\left\| v_1/\|v_1\|^{1/P} \right\| \le 1$, we get

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$$\|u_{n+1}\|_0 \le \|v_1\|\|v_2\| \frac{\varepsilon_n}{\sigma_n\sigma_{n+1}} \le \sigma_n\sigma_{n+1} \frac{\varepsilon_n}{\sigma_n\sigma_{n+1}} = \varepsilon_n$$

$P_n = \{x \in X; \|x_n\| \le \eta_n\}$ To obtain the desired result it is enough to choose This concludes the proof

Example: An example a p-two norm algebra may serve the algebra V^P for 0<p<1. The element of V^P are function x from <0,1> to R whose p-variation is finite.

The p-variation, $\text{var}_P x$, of the function x is defined as the supremum of all the sum $\sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|^P; 0 = t_0 \le t_1 \le \cdots \le t_n = 1$

All three algebra operation are defined point wise. Let us introduce the p-norms

$$\|x\| = \|x(0)\|^P + \text{var}_P(x)$$
$$\|x\|_0 = \sup\{|x(t)|^P; 0 \le t \le 1\}$$

We obtain *p*-two norm spaces. From the obvious inequality

$$\|xy\| \le \|x\|_0\|y\| + \|x\|_0\|y\| \le 2\|x\|\|y\|$$
$$\|xy\|_0 \le \|x\|_0\|y\|_0$$

It follows that $(V^P, \| \|, \| \|_0)$is a p - two- norm algebra.

The function equals to 1 its unit element. The sequence (x_n) of element V^P is $\text{var}_P(x_n) < \infty$ *and* $x_n(t)$ *tend to* $x_0(t)$ *uniformly* γ-convergent to x_0 if and only if. *on* $< 0,1 >$

A simple calculation show that if $\inf\{|x(t)|; 0 \le t \le 1\} = \varepsilon > 0$, then $\text{var}_P \frac{1}{x} \le \frac{\text{var}_P x}{\varepsilon^{2P}}$,

If follows that the inverse is γ-continuous, and $G(V^P)$ consist of functions different from zero.

The neighborhood basis of zero for the topology τ_∞ consists of sets of form $\sum_{i=1}^{n} W_n$;
$W_n = \{x \in V^P; \text{var}_P x \le 1, \|x\|_0 \le \varepsilon_n\}$ with some $\varepsilon_n > 0$.

Conclusion*:*

The problem arises as to the condition $\|\alpha x\| = |\alpha|^P\|x\|$ Characterizing p-normed spaces could be replaced by $\|\alpha x\| \le f(\alpha)\|x\|$ with some real function f, $\| \|$ being a Fréchet norm?

In this case we must have $f(\alpha\beta)\|x\| = \|\alpha\beta x\| = f(\alpha)\|\beta x\| = f(\alpha)f(\beta)\|x\|$, whence $f(\alpha\beta) = f(\alpha)f(\beta)$ for arbitrary α,β and $f(\alpha) > 0$ for $\alpha \ne o$. since the function $f \to \|f(x)\|$ is continuous, the function f also must be so.

it is well known that this is the case if and only if $f(x) = |x|^P$

Also we can ask if the *p*-homogeneity of the norm might be replace $\|\alpha x\| \le f(\alpha)\|x\|$ together with $\lim_{\alpha \to 0} f(\alpha) = 0$ This also gives nothing new, since in this case the ball $\{x \in X; \|x\| \le 1\}$. must be bounded and the theorem of Aoki-Rolewicz the Fréchet norm $\| \|$ must be equivalent to *p*-norm.

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