A Short Contribution to Split-Complex Linear Diophantine Equations in Two Variables

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Abstract

In this work, we study the split-complex integer solutions for the split-complex linear Diophantine equation in two variables $AX + BY = C$ where A, B, C, X, Y are split-complex integers. An algorithm for generating all solutions will be obtained by transforming the split-complex equation to a classical equivalent system of linear Diophantine equations in four variables.

Keywords: split-complex integer; split-complex Diophantine equation; general solution.

1. Introduction and preliminaries

Split-complex numbers are considered as a generalization of real numbers. The commutative ring of split-complex numbers is defined as follows:

$S = \{a + bJ; a, b \in R, J^2 = 1\}$.

This algebraic class of numbers was studied by many authors from many different sides such as geometry, physics, and linear spaces [1,3,4]. It is good to for the reader to check another similar structure (weal fuzzy complex numbers) [2,5].

The set of split-complex integers makes a commutative subring of S, where it is defined as:

$S - I = \{a + bJ; a, b \in Z, J^2 = 1\}$. [6]

The split-complex integers and number theory were studied in [6], where many concepts such as division, congruencies, and Euler's function were handled in the split-complex integer version.

These concepts have helped with generalizing RSA algorithm into a novel version depending on split-complex integers [6].

In this paper, we study the split-complex linear Diophantine equation in two variables, where a general algorithm for its solutions and solvability will be provided with an example.

This work lies under the section of number theory (non-classical number theory), and it can be considered as the first try with Diophantine equations defined over split-complex integer ring.

Main discussion.

Definition:

Let $A = a_1 + a_2J, B = b_1 + b_2J, C = c_1 + c_2J$ are three split-complex integers, the split-complex linear Diophantine equation in two variables is defined as follows:

$AX + BY = C; X = x_1 + x_2J, Y = y_1 + y_2J$ are split-complex integers.

Theorem.

Let $AX + BY = C$ be a split-complex linear Diophantine equation in two variables, then it is solvable if and only if:

$$\begin{cases} gcd(a_1 + a_2, b_1 + b_2) \setminus c_1 + c_2 \\ gcd(a_1 - a_2, b_1 - b_2) \setminus c_1 - c_2 \end{cases}$$

Proof.

The equation $AX + BY = C$ is equivalent to:

$(a_1 + a_2J)(x_1 + x_2J) + (b_1 + b_2J)(y_1 + y_2J) = c_1 + c_2J$

Doi: https://doi.org/10.54216/GJMSA.060204
Received: January 25, 2022 Revised: March 14, 2023 Accepted: May 09, 2023

$$\Rightarrow \begin{cases} a_1x_1 + b_1y_1 + a_2x_2 + b_2y_2 = c_1 & (1) \\ a_2x_1 + a_1x_2 + b_1y_2 + b_2y_1 = c_2 & (2) \end{cases}$$

Assume that $AX + BY = C$ is solvable, then $(1), (2)$ are solvable.

By adding (1) to (2) and subtracting (2) from (1), we get:

$$\begin{cases} (a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2 \\ (a_1 - a_2)(x_1 - x_2) + (b_1 - b_2)(y_1 - y_2) = c_1 + c_2 \end{cases}$$

Thus, $gcd(a_1 + a_2, b_1 + b_2) \setminus c_1 + c_2$ and $gcd(a_1 - a_2, b_1 - b_2) \setminus c_1 - c_2$.

For the converse, suppose that $gcd(a_1 + a_2, b_1 + b_2) \setminus c_1 + c_2, gcd(a_1 - a_2, b_1 - b_2) \setminus c_1 - c_2$, then the following two Diophantine equations are solvable:

$$\begin{cases} (a_1 + a_2)(x_1 + x_2) + (b_1 + b_2)(y_1 + y_2) = c_1 + c_2 \\ (a_1 - a_2)(x_1 - x_2) + (b_1 - b_2)(y_1 - y_2) = c_1 + c_2 \end{cases}$$

Thus, the Diophantine equation are solvable:

$$\begin{cases} a_1x_1 + a_1x_2 + a_2x_1 + a_2x_2 + b_1y_1 + b_1y_2 + b_2y_1 + b_2y_2 = c_1 + c_2 & (1) \\ a_1x_1 + a_2x_2 - a_1x_2 - a_2x_1 + b_1y_1 - b_1y_2 - b_2y_1 + b_2y_2 = c_1 - c_2 & (2) \end{cases}$$

Adding (1) to (2), and subtracting (2) from (1) gives:

$$\begin{cases} a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2 = c_1 \\ a_2x_1 + a_1x_2 + b_1y_2 + b_2y_1 = c_2 \end{cases}$$

This implies:

$(a_1 + a_2J)(x_1 + x_2J) + (b_1 + b_2J)(y_1 + y_2J) = c_1 + c_2J$ is solvable.

Hence, our proof is complete.

Algorithm for the solution.

To solve the split-complex linear Diophantine equation in two variables $AX + BY = C$, follow these steps:

Step (1).

Write the equivalent system:

$$\begin{cases} (a_1 + a_2)Z_1 + (b_1 + b_2)T_1 = c_1 + c_2 & (1) \\ (a_1 - a_2)Z_2 + (b_1 - b_2)T_2 = c_1 - c_2 & (2) \end{cases}; Z_1 = x_1 + x_2, Z_2 = x_1 - x_2, T_1 = y_1 + y_2, T_2 = y_1 - y_2$$

Step (2).

Let $(Z_1{}^*, T_1{}^*)$ be a solution for (1), and $(Z_2{}^*, T_2{}^*)$ be a solution for (2), then:

$d_1 = gcd(a_1 + a_2, b_1 + b_2), d_2 = gcd(a_1 - a_2, b_1 - b_2)$

$$\begin{cases} Z_1 = Z_1{}^* + k_1 \dfrac{b_1 + b_2}{d_1} \\ T_1 = T_1{}^* - k_1 \dfrac{a_1 + a_2}{d_1} \end{cases}; \ k_1 \in Z$$

Also,

$$\begin{cases} Z_2 = Z_2{}^* + k_2 \dfrac{b_1 - b_2}{d_2} \\ T_2 = T_2{}^* - k_2 \dfrac{a_1 - a_2}{d_2} \end{cases}; \ k_2 \in Z$$

Step (3).

The solution of $AX + BY = C$ is:

$$X = \frac{1}{2}\left[Z_1{}^* + Z_2{}^* + k_1\frac{b_1 + b_2}{d_1} + k_2\frac{b_1 - b_2}{d_2}\right] + \frac{1}{2}J\left[T_1{}^* + T_2{}^* - k_1\frac{a_1 + a_2}{d_1} - k_2\frac{a_1 - a_2}{d_1}\right]$$

$$Y = \frac{1}{2}\left[Z_1{}^* - Z_2{}^* + k_1\frac{b_1 + b_2}{d_1} - k_2\frac{b_1 - b_2}{d_2}\right] + \frac{1}{2}J\left[T_1{}^* - T_2{}^* - k_1\frac{a_1 + a_2}{d_1} + k_2\frac{a_1 - a_2}{d_1}\right]$$

With $k_1, k_2 \in Z$.

Example.

Consider the following split-complex linear Diophantine equation:

$(2 + J)X + (5 - J)Y = 14 + 23J$, we have:

$$\begin{cases} a_1 = 2, a_2 = 1 \\ b_1 = 5, b_2 = -1 \\ c_1 = 14, c_2 = 23 \end{cases}$$

Also,

$$\begin{cases} a_1 + a_2 = 3, a_1 - a_2 = 1 \\ b_1 + b_2 = 4, b_1 - b_2 = 6 \\ c_1 + c_2 = 37, c_1 - c_2 = -9 \end{cases}$$

$gcd(a_1 + a_2, b_1 + b_2) = gcd(3,4) = 1 \setminus c_1 + c_2$

$gcd(a_1 - a_2, b_1 - b_2) = gcd(1,6) = 1 \setminus c_1 - c_2$

Thus, it is solvable.

Doi: https://doi.org/10.54216/GJMSA.060204
Received: January 25, 2022 Revised: March 14, 2023 Accepted: May 09, 2023

The equivalent system is:

$$\begin{cases}3Z_1 + 4T_1 = 37\\ Z_2 + 6T_2 = -9\end{cases}$$

We have: $Z_1{}^* = 3, T_1{}^* = 7, Z_2{}^* = 3, T_2{}^* = -2$.

$$\begin{cases}Z_1 = 3 + k_1\frac{4}{1} = 3 + 4k_1\\ T_1 = 7 - k_1\frac{3}{1} = 7 - 3k_1\end{cases};\ k_1 \in Z$$

$$\begin{cases}Z_2 = 3 + k_2\frac{6}{1} = 3 + 6k_2\\ T_1 = -2 - k_1\frac{1}{1} = -2 - k_2\end{cases};\ k_2 \in Z$$

Thus:

$X = \frac{1}{2}[6 + 4k_1 + 6k_2] + \frac{1}{2}J[4k_1 - 6k_2]$

$Y = \frac{1}{2}[5 - 3k_1 - k_2] + \frac{1}{2}J[9 - 3k_1 + k_2]; k_1, k_2 \in Z$ and $5 - 3k_1 - k_2 \equiv 0(mod2), 9 - 3k_1 + k_2 \equiv 0(mod2)$.

for example, we can generate a solution for:

$k_1 = 2, k_2 = 1$.

$X = \frac{1}{2}[6 + 8 + 6] + \frac{1}{2}J[8 - 6] = 10 + J$.

$Y = \frac{1}{2}[-2] + \frac{1}{2}J[9 - 6 + 1] = -1 + 2J$.

The all solution of the original split-complex Diophantine equation can be obtained by taking k_1, k_2 such that $k_1 + k_2 \equiv 1(mod2)$.

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Doi: https://doi.org/10.54216/GJMSA.060204
Received: January 25, 2022 Revised: March 14, 2023 Accepted: May 09, 2023