Bipolar Neutrosophic Finite Switchboard State Machines

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Abstract

The notion of bipolar neutrosophic finite switchboard state machines (BNFSSTMs), homomorphisms and strong homomorphisms of bipolar neutrosophic finite state machines (BNFSSTMs) are introduced and some related results are studied.

Keywords: neutrosophic set (NS); bipolar neutrosophic set (BNS); bipolar neutrosophic finite state machine (BNFSM); bipolar neutrosophic switchboard state machine (BNFSSTM); homomorphism and strong homomorphism.

1 Introduction

In 1965, Zadeh[1] introduced fuzzy sets (FS). Assume that $\mathcal{K}$ is a non-empty set. A mapping $\zeta : \mathcal{K} \to [0, 1]$ is a FS of $\mathcal{K}$. The FS theory has various expansions, such as intuitionistic fuzzy sets (IFSs), interval-valued fuzzy sets (IVFSs), vague sets (VSs) and so on. With, the traditional fuzzy set representation it is difficult to express the difference of the irrelevant elements from the contrary elements. Based on these observations, in 2000, Lee[2] introduced (BVFS)bipolar valued fuzzy sets, an extension of fuzzy sets whose range of the membership degree is enlarged from $[0, 1]$ to $[-1, 1]$. As a result, the fields like medical science, algebraic structures, graph theory, machine theory, decision-making, and so on have a thriving research field. The notion of neutrosophic sets was introduced by Smarandache,[3] as a generalization of FSs, IFSs and IVFSs whose elements of the universe have the degrees of truth (T), indeterminacy (I) and falsity (F) and the membership values (MSV) lies in $]0^-, 1^+[$, the non-standard unit interval (N-S).

Malik[4,5] developed the concept and approach of the fuzzy finite state machine (FFSM), fuzzy finite state submachine (FFSbM) and their decomposition. The product of the (FFSM), as well as other associated aspects, were proposed and studied by Kumbhojkar and Chaudhari in 2002.[6] The fuzzy finite switchboard state machine (FFSSM) was presented by Sato and Kuroki in 2002.[7] Jun introduced the notion of intuitionistic fuzzy finite state machines (IFFSMs) in 2005,[8] Jun suggested a commutative intuitionistic fuzzy finite switchboard state machine (IFFSSM) called the intuitionistic fuzzy finite switchboard state machine in 2006.[9] Jun and Kavikumar[10] popularized the notion of the bipolar fuzzy finite state machine (BiFFSM) in 2011. Jun and Kavikumar proposed the algebraic features of BiFFSMs and evaluated particular results in 2011. Mahmood et al.[11] introduced and researched single valued neutrosophic finite state machine homomorphism (SVNFSM), strong homomorphism, and associated features in 2018. Prabhu et al.[12] studied finite state machines via bipolar

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neutrosophic set theory, Reena[13] investigated the bipolar vague finite switchboard state machine (BVFSSM) in 2019.

Here, the notion of bipolar neutrosophic finite switchboard state machines (BNFSSTMs), homomorphism and strong homomorphism of bipolar neutrosophic finite state machines (BNFSTMs) are introduced and some related results are studied. Throughout the article, FS stands for a fuzzy set, BiNS stands for a bipolar neutrosophic set, BNFSTM stands for a bipolar neutrosophic finite state machine and BNFSSTM stands for a bipolar neutrosophic finite switchboard state machine.

2 Preliminaries

Here, we will review a few standard definitions that are relevant to this work.

Definition 2.1. [1] A mapping $\zeta : Z \to [0,1]$ is represented as a FS of a non-empty set Z.

Definition 2.2. [12] A BiNS A_N in X is defined as an object of the form

$$A_N = \{< \xi, T^+_{A_N}(\xi), I^+_{A_N}(\xi), F^+_{A_N}(\xi), T^-_{A_N}(\xi), I^-_{A_N}(\xi), F^-_{A_N}(\xi) >: \xi \in X\}.$$

The positive (+Ve) membership degree $T^+_{A_N}(\xi), I^+_{A_N}(\xi), F^+_{A_N}(\xi)$ denote the truth membership (T), indeterminate membership (I) and false membership (F) of an element of $\xi \in X$ corresponding to a BiNS A_N and the negative (-Ve) membership degree $T^-_{A_N}(\xi), I^-_{A_N}(\xi), F^-_{A_N}(\xi)$ denote the truth membership (T), indeterminate membership (I) and false membership (F) of an element of $\xi \in X$ to some implicit counter-property corresponding to a BiNS A_N.

Definition 2.3. [12] A BNFSM is a triple $B_N = (Q_N, X_N, A_N)$ where Q_N, X_N are finite non-empty sets said to be the set of states and the set of input symbols respectively and A_N is a BiNS in $Q_N \times X_N \times Q_N$. Thus $A_N = \{(\xi, T^+_{A_N}(\xi), I^+_{A_N}(\xi), F^+_{A_N}(\xi), T^-_{A_N}(\xi), I^-_{A_N}(\xi), F^-_{A_N}(\xi)) \mid \xi \in Q_N \times X_N \times Q_N\}$ where $T^+_{A_N}, I^+_{A_N}, F^+_{A_N} : Q_N \times X_N \times Q_N \to [0,1]$ and $T^-_{A_N}, I^-_{A_N}, F^-_{A_N} : Q_N \times X_N \times Q_N \to [-1,0]$ with the condition that $0 \leq T^+_{A_N} + I^+_{A_N} + F^+_{A_N} + T^-_{A_N} + I^-_{A_N} + F^-_{A_N} \leq 6$.

Example 2.4. Let $Q_N = \{\kappa, \delta\}$ and $X_N = \{a\}$ be finite nonempty sets of states and input symbols respectively. Let $Q_N \times X_N \times Q_N = \{(\kappa, a, \kappa), (\kappa, a, \delta), (\delta, a, \kappa), (\delta, a, \delta)\}$. Thus A_N is a BiNS in $Q_N \times X_N \times Q_N$, i.e., $A_N = \{< (\kappa, a, \kappa), 0.5, 0.3, 0.1, -0.06, -0.04, -0.01 >,$
$< (\kappa, a, \delta), 0.8, 0.5, 0.4, -0.01, -0.05, -0.06 >,$
$< (\delta, a, \kappa), 0.3, 0.2, 0.7, -0.02, -0.003, -0.05 >,$
$< (\delta, a, \delta), 0.3, 0.4, 0.5, -0.04, -0.05, -0.01 >\}.$
Thus, A_N is a BNFSM.

Definition 2.5. [12] Suppose $B_N = (Q_N, X_N, A_N)$ be a BNFSM. The BiNS A_N can be extended to other BiNS

$$A^*_N = \{(\nu, T^+_{A^*_N}(\nu), I^+_{A^*_N}(\nu), F^+_{A^*_N}(\nu), T^-_{A^*_N}(\nu), I^-_{A^*_N}(\nu), F^-_{A^*_N}(\nu)) \mid \nu \in Q_N \times X^*_N \times Q_N\}$$

where $T^+_{A^*_N}, I^+_{A^*_N}, F^+_{A^*_N} : Q_N \times X^*_N \times Q_N \to [0,1]$ and $T^-_{A^*_N}, I^-_{A^*_N}, F^-_{A^*_N} : Q_N \times X^*_N \times Q_N \to [-1,0]$ and determined by
$T^+_{A^*_N}(\kappa, \lambda, \delta) = \{1, if \kappa = \delta; 0, if \kappa \neq \delta \}$
$I^+_{A^*_N}(\kappa, \lambda, \delta) = \{1, if \kappa = \delta; 0, if \kappa \neq \delta\}$
$F^+_{A^*_N}(\kappa, \lambda, \delta) = \{0, if \kappa = \delta; 1, if \kappa \neq \delta \}$
$T^-_{A^*_N}(\kappa, \lambda, \delta) = \{-1, if \kappa = \delta; 0, if \kappa \neq \delta \}$
$I^-_{A^*_N}(\kappa, \lambda, \delta) = \{-1, if \kappa = \delta; 0, if \kappa \neq \delta\}$
$F^-_{A^*_N}(\kappa, \lambda, \delta) = \{0, if \kappa = \delta; -1, if \kappa \neq \delta \}.$
Also,
i) $T^+_{A^*_N}(\kappa, \nu a, \delta) = \vee_{r \in Q_N}[T^+_{A^*_N}(\kappa, \nu, r) \wedge T^+_{A_N}(r, a, \delta)]$
$ii) I^+_{A^*_N}(\kappa, \nu a, \delta) = \vee_{r \in Q_N}[I^+_{A^*_N}(\kappa, \nu, r) \wedge I^+_{A_N}(r, a, \delta)]$
$iii) F^+_{A^*_N}(\kappa, \nu a, \delta) = \wedge_{r \in Q_N}[F^+_{A^*_N}(\kappa, \nu, r) \vee F^+_{A_N}(r, a, \delta)]$

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$iv)T^{-}_{A^*_N}(\kappa, \nu a, \delta) = \wedge_{r\in Q_N}[T^{-}_{A^*_N}(\kappa, \nu, r) \vee T^{-}_{A_N}(r, a, \delta)]$
$v)I^{-}_{A^*_N}(\kappa, \nu a, \delta) = \wedge_{r\in Q_N}[I^{-}_{A^*_N}(\kappa, \nu, r) \vee I^{-}_{A_N}(r, a, \delta)]$
$vi)F^{-}_{A^*_N}(\kappa, \nu a, \delta) = \vee_{r\in Q_N}[F^{-}_{A^*_N}(\kappa, \nu, r) \wedge F^{-}_{A_N}(r, a, \delta)]$,
$\forall \kappa, \delta \in Q_N, \nu \in X^*_N$ and $a \in X_N$.

Theorem 2.6. [12] *Let $B_N = (Q_N, X_N, A_N)$ be a BNFSM. Then*
i) $T^{+}_{A^*_N}(\kappa, \nu y, \delta) = \vee_{r\in Q_N}[T^{+}_{A^*_N}(\kappa, \nu, r) \wedge T^{+}_{A^*_N}(r, y, \delta)]$
$ii)I^{+}_{A^*_N}(\kappa, \nu y, \delta) = \vee_{r\in Q_N}[I^{+}_{A^*_N}(\kappa, \nu, r) \wedge I^{+}_{A^*_N}(r, y, \delta)]$
$iii)F^{+}_{A^*_N}(\kappa, \nu y, \delta) = \wedge_{r\in Q_N}[F^{+}_{A^*_N}(\kappa, \nu, r) \vee F^{+}_{A^*_N}(r, y, \delta)]$
$iv)T^{-}_{A^*_N}(\kappa, \nu y, \delta) = \wedge_{r\in Q_N}[T^{-}_{A^*_N}(\kappa, \nu, r) \vee T^{-}_{A^*_N}(r, y, \delta)]$
$v)I^{-}_{A^*_N}(\kappa, \nu y, \delta) = \wedge_{r\in Q_N}[I^{-}_{A^*_N}(\kappa, \nu, r) \vee I^{-}_{A^*_N}(r, y, \delta)]$
$vi)F^{-}_{A^*_N}(\kappa, \nu y, \delta) = \vee_{r\in Q_N}[F^{-}_{A^*_N}(\kappa, \nu, r) \wedge F^{-}_{A^*_N}(r, y, \delta)]$,
$\forall \kappa, \delta \in Q_N$ *and* $\nu, y \in X^*_N$.

3 Bipolar neutrosophic finite switchboard state machines

Let X^*_N mean the set of all words of elements of X of finite length. Let λ signify the empty words in X^*_N and $|\nu|$ indicate the length of ν for each $\nu \in X^*_N$.

Definition 3.1. A BNFSM $B_N = (Q_N, X_N, A_N)$ is said to be switching if it satisfies
$T^{+}_{A^*_N}(\delta, h, \zeta) = T^{+}_{A^*_N}(\zeta, h, \delta)$
$I^{+}_{A^*_N}(\delta, h, \zeta) = I^{+}_{A^*_N}(\zeta, h, \delta)$
$F^{+}_{A^*_N}(\delta, h, \zeta) = F^{+}_{A^*_N}(\zeta, h, \delta)$
$T^{-}_{A^*_N}(\delta, h, \zeta) = T^{-}_{A^*_N}(\zeta, h, \delta)$
$I^{-}_{A^*_N}(\delta, h, \zeta) = I^{-}_{A^*_N}(\zeta, h, \delta)$
$F^{-}_{A^*_N}(\delta, h, \zeta) = F^{-}_{A^*_N}(\zeta, h, \delta)$,
$\forall \zeta, \delta \in Q_N$ and $h \in X_N$.

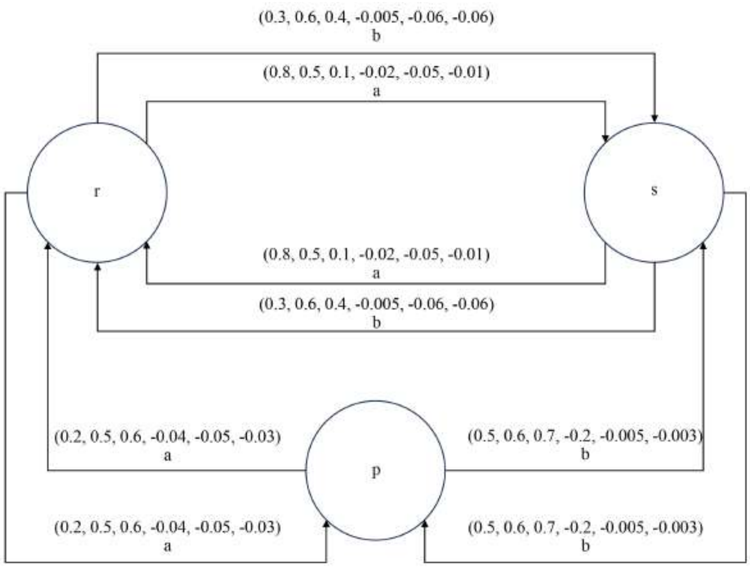

Figure 1: BNFSM-Switching

Definition 3.2. A BNFSM $B_N = (Q_N, X_N, A_N)$ is supposed to be commutative if it fulfills
$T^{+}_{A^*_N}(\delta, ab, \zeta) = T^{+}_{A^*_N}(\delta, ba, \zeta)$
$I^{+}_{A^*_N}(\delta, ab, \zeta) = I^{+}_{A^*_N}(\delta, ba, \zeta)$
$F^{+}_{A^*_N}(\delta, ab, \zeta) = F^{+}_{A^*_N}(\delta, ba, \zeta)$

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$T^{-}_{A^*_N}(\delta, ab, \zeta) = T^{-}_{A^*_N}(\delta, ba, \zeta)$
$I^{-}_{A^*_N}(\delta, ab, \zeta) = I^{-}_{A^*_N}(\delta, ba, \zeta)$
$F^{-}_{A^*_N}(\delta, ab, \zeta) = F^{-}_{A^*_N}(\delta, ba, \zeta)$
$\forall \zeta, \delta \in Q_N$ and $a, b \in X_N$.

Example 3.3. The BNFSM in Figure 1 is not commutative.

Definition 3.4. If a BNFSM $B_N = (Q_N, X_N, A_N)$ is both switching and commutative, then is a BNFSSM.

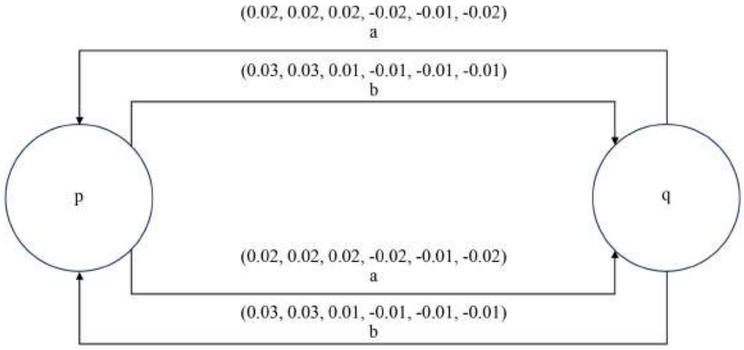

Figure 2: BNFSSM

Theorem 3.5. *If $B_N = (Q_N, X_N, A_N)$ is a commutative BNFSSM, then*
$T^{+}_{A^*_N}(\delta, \nu a, \zeta) = T^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$I^{+}_{A^*_N}(\delta, \nu a, \zeta) = I^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$F^{+}_{A^*_N}(\delta, \nu a, \zeta) = F^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$T^{-}_{A^*_N}(\delta, \nu a, \zeta) = T^{-}_{A^*_N}(\delta, a\nu, \zeta)$
$I^{-}_{A^*_N}(\delta, \nu a, \zeta) = I^{-}_{A^*_N}(\delta, a\nu, \zeta)$
$F^{-}_{A^*_N}(\delta, \nu a, \zeta) = F^{-}_{A^*_N}(\delta, a\nu, \zeta)$,
$\forall \zeta, \delta \in Q_N, a \in X_N$ *and* $\nu \in X^*_N$.

Proof. Given $B_N = (Q_N, X_N, A_N)$ is a commutative BNFSSM. Let $\zeta, \delta \in Q_N$ and a $\in X_N$ and $\nu \in X^*_N$. We claim the outcome by doing induction on $|\nu| = n$. Assume that $n = 0$. Then $\nu = \lambda$. Thus
$T^{+}_{A^*_N}(\delta, \nu a, \zeta) = T^{+}_{A^*_N}(\delta, \lambda a, \zeta) = T^{+}_{A^*_N}(\delta, a, \zeta) = T^{+}_{A^*_N}(\delta, a\lambda, \zeta)$
$= T^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$I^{+}_{A^*_N}(\delta, \nu a, \zeta) = I^{+}_{A^*_N}(\delta, \lambda a, \zeta) = I^{+}_{A^*_N}(\delta, a, \zeta) = I^{+}_{A^*_N}(\delta, a\lambda, \zeta)$
$= I^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$F^{+}_{A^*_N}(\delta, \nu a, \zeta) = F^{+}_{A^*_N}(\delta, \lambda a, \zeta) = F^{+}_{A^*_N}(\delta, a, \zeta) = F^{+}_{A^*_N}(\delta, a\lambda, \zeta)$
$= F^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$T^{-}_{A^*_N}(\delta, \nu a, \zeta) = T^{-}_{A^*_N}(\delta, \lambda a, \zeta) = T^{-}_{A^*_N}(\delta, a, \zeta) = T^{-}_{A^*_N}(\delta, a\lambda, \zeta)$
$= T^{-}_{A^*_N}(\delta, a\nu, \zeta)$
$I^{-}_{A^*_N}(\delta, \nu a, \zeta) = I^{-}_{A^*_N}(\delta, \lambda a, \zeta) = I^{-}_{A^*_N}(\delta, a, \zeta) = I^{-}_{A^*_N}(\delta, a\lambda, \zeta)$
$= I^{-}_{A^*_N}(\delta, a\nu, \zeta)$
$F^{-}_{A^*_N}(\delta, \nu a, \zeta) = F^{-}_{A^*_N}(\delta, \lambda a, \zeta) = F^{-}_{A^*_N}(\delta, a, \zeta) = F^{-}_{A^*_N}(\delta, a\lambda, \zeta)$
$= F^{-}_{A^*_N}(\delta, a\nu, \zeta)$.
Assume the outcome is valid $\forall u \in X^*_N$ for $|u| = n - 1$, where $n > 0$. Suppose $b \in X_N$ be such that $\nu = ub$. Thus
$T^{+}_{A^*_N}(\delta, \nu a, \zeta) = T^{+}_{A^*_N}(\delta, uba, \zeta)$
$= \sup_{\wp \in Q_N}(T^{+}_{A^*_N}(\delta, u, \wp) \wedge T^{+}_{A^*_N}(\wp, ba, \zeta))$
$= \sup_{\wp \in Q_N}(T^{+}_{A^*_N}(\delta, u, \wp) \wedge T^{+}_{A^*_N}(\wp, ab, \zeta))$
$= T^{+}_{A^*_N}(\delta, uab, \zeta)$
$= \sup_{\wp \in Q_N}(T^{+}_{A^*_N}(\delta, ua, \wp) \wedge T^{+}_{A_N}(\wp, b, \zeta))$

$= \sup_{\wp \in Q_N} (T^{+}_{A^*_N}(\delta, au, \wp) \wedge T^{+}_{A_N}(\wp, b, \zeta))$
$= T^{+}_{A^*_N}(\delta, aub, \zeta)$
$= T^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$I^{+}_{A^*_N}(\delta, \nu a, \zeta) = I^{+}_{A^*_N}(\delta, uba, \zeta)$
$= \sup_{\wp \in Q_N} (I^{+}_{A^*_N}(\delta, u, \wp) \wedge I^{+}_{A^*_N}(\wp, ba, \zeta))$
$= \sup_{\wp \in Q_N} (I^{+}_{A^*_N}(\delta, u, \wp) \wedge I^{+}_{A^*_N}(\wp, ab, \zeta))$
$= I^{+}_{A^*_N}(\delta, uab, \zeta)$
$= \sup_{\wp \in Q_N} (I^{+}_{A^*_N}(\delta, ua, \wp) \wedge I^{+}_{A_N}(\wp, b, \zeta))$
$= \sup_{\wp \in Q_N} (I^{+}_{A^*_N}(\delta, au, \wp) \wedge I^{+}_{A_N}(\wp, b, \zeta))$
$= I^{+}_{A^*_N}(\delta, aub, \zeta)$
$= I^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$F^{+}_{A^*_N}(\delta, \nu a, \zeta) = F^{+}_{A^*_N}(\delta, uba, \zeta)$
$= \inf_{\wp \in Q_N} (F^{+}_{A^*_N}(\delta, u, \wp) \vee F^{+}_{A^*_N}(\wp, ba, \zeta))$
$= \inf_{\wp \in Q_N} (F^{+}_{A^*_N}(\delta, u, \wp) \vee F^{+}_{A^*_N}(\wp, ab, \zeta))$
$= F^{+}_{A^*_N}(\delta, uab, \zeta)$
$= \inf_{\wp \in Q_N} (F^{+}_{A^*_N}(\delta, ua, \wp) \vee F^{+}_{A_N}(\wp, b, \zeta))$
$= \inf_{\wp \in Q_N} (F^{+}_{A^*_N}(\delta, au, \wp) \vee F^{+}_{A_N}(\wp, b, \zeta))$
$= F^{+}_{A^*_N}(\delta, aub, \zeta)$
$= F^{+}_{A^*_N}(\delta, a\nu, \zeta)$
$T^{-}_{A^*_N}(\delta, \nu a, \zeta) = T^{-}_{A^*_N}(\delta, uba, \zeta)$
$= \inf_{\wp \in Q_N} (T^{-}_{A^*_N}(\delta, u, \wp) \vee T^{-}_{A^*_N}(\wp, ba, \zeta))$
$= \inf_{\wp \in Q_N} (T^{-}_{A^*_N}(\delta, u, \wp) \vee T^{-}_{A^*_N}(\wp, ab, \zeta))$
$= T^{-}_{A^*_N}(\delta, uab, \zeta)$
$= \inf_{\wp \in Q_N} (T^{-}_{A^*_N}(\delta, ua, \wp) \vee T^{-}_{A_N}(\wp, b, \zeta))$
$= \inf_{\wp \in Q_N} (T^{-}_{A^*_N}(\delta, au, \wp) \vee T^{-}_{A_N}(\wp, b, \zeta))$
$= T^{-}_{A^*_N}(\delta, aub, \zeta)$
$= T^{-}_{A^*_N}(\delta, a\nu, \zeta)$
$I^{-}_{A^*_N}(\delta, \nu a, \zeta) = I^{-}_{A^*_N}(\delta, uba, \zeta)$
$= \inf_{\wp \in Q_N} (I^{-}_{A^*_N}(\delta, u, \wp) \vee I^{-}_{A^*_N}(\wp, ba, \zeta))$
$= \inf_{\wp \in Q_N} (I^{-}_{A^*_N}(\delta, u, \wp) \vee I^{-}_{A^*_N}(\wp, ab, \zeta))$
$= I^{-}_{A^*_N}(\delta, uab, \zeta)$
$= \inf_{\wp \in Q_N} (I^{-}_{A^*_N}(\delta, ua, \wp) \vee I^{-}_{A_N}(\wp, b, \zeta))$
$= \inf_{\wp \in Q_N} (I^{-}_{A^*_N}(\delta, au, \wp) \vee I^{-}_{A_N}(\wp, b, \zeta))$
$= I^{-}_{A^*_N}(\delta, aub, \zeta)$
$= I^{-}_{A^*_N}(\delta, a\nu, \zeta)$
$F^{-}_{A^*_N}(\delta, \nu a, \zeta) = F^{-}_{A^*_N}(\delta, uba, \zeta)$
$= \sup_{\wp \in Q_N} (F^{-}_{A^*_N}(\delta, u, \wp) \wedge F^{-}_{A^*_N}(\wp, ba, \zeta))$
$= \sup_{\wp \in Q_N} (F^{-}_{A^*_N}(\delta, u, \wp) \wedge F^{-}_{A^*_N}(\wp, ab, \zeta))$
$= F^{-}_{A^*_N}(\delta, uab, \zeta)$
$= \sup_{\wp \in Q_N} (F^{-}_{A^*_N}(\delta, ua, \wp) \wedge F^{-}_{A_N}(\wp, b, \zeta))$
$= \sup_{\wp \in Q_N} (F^{-}_{A^*_N}(\delta, au, \wp) \wedge F^{-}_{A_N}(\wp, b, \zeta))$
$= F^{-}_{A^*_N}(\delta, aub, \zeta)$
$= F^{-}_{A^*_N}(\delta, a\nu, \zeta)$.
Hence the outcome is true for $|\nu| = n$. □

Theorem 3.6. *If $B_N = (Q_N, X_N, A_N)$ is a BNFSSM, then*
$T^{+}_{A^*_N}(\delta, h, \zeta) = T^{+}_{A^*_N}(\zeta, h, \delta)$
$I^{+}_{A^*_N}(\delta, h, \zeta) = I^{+}_{A^*_N}(\zeta, h, \delta)$

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$F^{+}_{A^*_N}(\delta,h,\zeta)=\mathcal{F}^{+}_{A^*_N}(\zeta,h,\delta)$
$T^{-}_{A^*_N}(\delta,h,\zeta)=T^{-}_{A^*_N}(\zeta,h,\delta)$
$I^{-}_{A^*_N}(\delta,h,\zeta)=I^{-}_{A^*_N}(\zeta,h,\delta)$
$F^{-}_{A^*_N}(\delta,h,\zeta)=F^{-}_{A^*_N}(\zeta,h,\delta),$
$\forall\zeta,\delta\in Q_N$ *and* $h\in X^*_N$.

Proof. Given $B_N=(Q_N,X_N,A_N)$ is a BNFSSM. Let $\zeta,\delta\in Q_N$ and $h\in X^*_N$. We claim the outcome by induction for $|h|=n$ if $n=0$, then $h=\lambda$. Thus

$$T^{+}_{A^*_N}(\delta,h,\zeta)=T^{+}_{A^*_N}(\delta,\lambda,\zeta)=T^{+}_{A^*_N}(\zeta,\lambda,\delta)=T^{+}_{A^*_N}(\zeta,h,\delta)$$
$$I^{+}_{A^*_N}(\delta,h,\zeta)=I^{+}_{A^*_N}(\delta,\lambda,\zeta)=I^{+}_{A^*_N}(\zeta,\lambda,\delta)=I^{+}_{A^*_N}(\zeta,h,\delta)$$
$$F^{+}_{A^*_N}(\delta,h,\zeta)=F^{+}_{A^*_N}(\delta,\lambda,\zeta)=F^{+}_{A^*_N}(\zeta,\lambda,\delta)=F^{+}_{A^*_N}(\zeta,h,\delta)$$
$$T^{-}_{A^*_N}(\delta,h,\zeta)=T^{-}_{A^*_N}(\delta,\lambda,\zeta)=T^{-}_{A^*_N}(\zeta,\lambda,\delta)=T^{-}_{A^*_N}(\zeta,h,\delta)$$
$$I^{-}_{A^*_N}(\delta,h,\zeta)=I^{-}_{A^*_N}(\delta,\lambda,\zeta)=I^{-}_{A^*_N}(\zeta,\lambda,\delta)=I^{-}_{A^*_N}(\zeta,h,\delta)$$
$$F^{-}_{A^*_N}(\delta,h,\zeta)=F^{-}_{A^*_N}(\delta,\lambda,\zeta)=F^{-}_{A^*_N}(\zeta,\lambda,\delta)=F^{-}_{A^*_N}(\zeta,h,\delta).$$

Accordingly the outcome is valid for $n=0$. The outcome is valid all $b\in X^*_N$ with $|b|=n-1, n>0$, we have
$T^{+}_{A^*_N}(\delta,b,\zeta)=T^{+}_{A^*_N}(\zeta,b,\delta), I^{+}_{A^*_N}(\delta,b,\zeta)=I^{+}_{A^*_N}(\zeta,b,\delta), F^{+}_{A^*_N}(\delta,b,\zeta)=F^{+}_{A^*_N}(\zeta,b,\delta),$
$T^{-}_{A^*_N}(\delta,b,\zeta)=T^{-}_{A^*_N}(\zeta,b,\delta), I^{-}_{A^*_N}(\delta,b,\zeta)=I^{-}_{A^*_N}(\zeta,b,\delta), F^{-}_{A^*_N}(\delta,b,\zeta)=F^{-}_{A^*_N}(\zeta,b,\delta).$
Let $\nu\in X_N$ and $h\in X^*_N$ be such that $h=b\nu$. Then
$T^{+}_{A^*_N}(\delta,h,\zeta)=T^{+}_{A^*_N}(\delta,b\nu,\zeta)=\sup_{\wp\in Q_N}[T^{+}_{A^*_N}(\delta,b,\wp)\wedge T^{+}_{A_N}(\wp,\nu,\zeta)]$
$=\sup_{\wp\in Q_N}[T^{+}_{A^*_N}(\wp,b,\delta)\wedge T^{+}_{A_N}(\zeta,\nu,\wp)]$
$=\sup_{\wp\in Q_N}[T^{+}_{A^*_N}(\wp,b,\delta)\wedge T^{+}_{A^*_N}(\zeta,\nu,\wp)]$
$=\sup_{\wp\in Q_N}[T^{+}_{A^*_N}(\zeta,\nu,\wp)\wedge T^{+}_{A^*_N}(\wp,b,\delta)]$
$=T^{+}_{A^*_N}(\zeta,\nu b,\delta)=T^{+}_{A^*_N}(\zeta,b\nu,\delta)=T^{+}_{A^*_N}(\zeta,h,\delta)$
$I^{+}_{A^*_N}(\delta,h,\zeta)=I^{+}_{A^*_N}(\delta,b\nu,\zeta)=\sup_{\wp\in Q_N}[I^{+}_{A^*_N}(\delta,b,\wp)\wedge I^{+}_{A_N}(\wp,\nu,\zeta)]$
$=\sup_{\wp\in Q_N}[I^{+}_{A^*_N}(\wp,b,\delta)\wedge I^{+}_{A_N}(\zeta,\nu,\wp)]$
$=\sup_{\wp\in Q_N}[I^{+}_{A^*_N}(\wp,b,\delta)\wedge I^{+}_{A^*_N}(\zeta,\nu,\wp)]$
$=\sup_{\wp\in Q_N}[I^{+}_{A^*_N}(\zeta,\nu,\wp)\wedge I^{+}_{A^*_N}(\wp,b,\delta)]$
$=I^{+}_{A^*_N}(\zeta,\nu b,\delta)=I^{+}_{A^*_N}(\zeta,b\nu,\delta)=I^{+}_{A^*_N}(\zeta,a,\delta)$
$F^{+}_{A^*_N}(\delta,h,\zeta)=F^{+}_{A^*_N}(\delta,b\nu,\zeta)=\inf_{\wp\in Q_N}[F^{+}_{A^*_N}(\delta,b,\wp)\vee F^{+}_{A_N}(\wp,\nu,\zeta)]$
$=\inf_{\wp\in Q_N}[F^{+}_{A^*_N}(\wp,b,\delta)\vee F^{+}_{A_N}(\zeta,\nu,\wp)]$
$=\inf_{\wp\in Q_N}[F^{+}_{A^*_N}(\wp,b,\delta)\vee F^{+}_{A^*_N}(\zeta,\nu,\wp)]$
$=\inf_{\wp\in Q_N}[F^{+}_{A^*_N}(\zeta,\nu,\wp)\vee F^{+}_{A^*_N}(\wp,b,\delta)]$
$=F^{+}_{A^*_N}(\zeta,\nu b,\delta)=F^{+}_{A^*_N}(\zeta,b\nu,\delta)=F^{+}_{A^*_N}(\zeta,a,\delta)$
$T^{-}_{A^*_N}(\delta,h,\zeta)=T^{-}_{A^*_N}(\delta,b\nu,\zeta)=\inf_{\wp\in Q_N}[T^{-}_{A^*_N}(\delta,b,\wp)\vee T^{-}_{A_N}(\wp,\nu,\zeta)]$
$=\inf_{\wp\in Q_N}[T^{-}_{A^*_N}(\wp,b,\delta)\vee T^{-}_{A_N}(\zeta,\nu,\wp)]$
$=\inf_{\wp\in Q_N}[T^{-}_{A^*_N}(\wp,b,\delta)\vee T^{-}_{A^*_N}(\zeta,\nu,\wp)]$
$=\inf_{\wp\in Q_N}[T^{-}_{A^*_N}(\zeta,\nu,\wp)\vee T^{-}_{A^*_N}(\wp,b,\delta)]$
$=T^{-}_{A^*_N}(\zeta,\nu b,\delta)=T^{-}_{A^*_N}(\zeta,b\nu,\delta)=T^{-}_{A^*_N}(\zeta,a,\delta)$
$I^{-}_{A^*_N}(\delta,h,\zeta)=I^{-}_{A^*_N}(\delta,b\nu,\zeta)=\inf_{\wp\in Q_N}[I^{-}_{A^*_N}(\delta,b,\wp)\vee I^{-}_{A_N}(\wp,\nu,\zeta)]$
$=\inf_{\wp\in Q_N}[I^{-}_{A^*_N}(\wp,b,\delta)\vee I^{-}_{A_N}(\zeta,\nu,\wp)]$
$=\inf_{\wp\in Q_N}[I^{-}_{A^*_N}(\wp,b,\delta)\vee I^{-}_{A^*_N}(\zeta,\nu,\wp)]$
$=\inf_{\wp\in Q_N}[I^{-}_{A^*_N}(\zeta,\nu,t)\vee I^{-}_{A^*_N}(\wp,b,\delta)]$
$=I^{-}_{A^*_N}(\zeta,\nu b,\delta)=I^{-}_{A^*_N}(\zeta,b\nu,\delta)=I^{-}_{A^*_N}(\zeta,a,\delta)$
$F^{-}_{A^*_N}(\delta,h,\zeta)=F^{-}_{A^*_N}(\delta,b\nu,\zeta)=\sup_{\wp\in Q_N}[F^{-}_{A^*_N}(\delta,b,\wp)\wedge F^{-}_{A_N}(\wp,\nu,\zeta)]$
$=\sup_{\wp\in Q_N}[F^{-}_{A^*_N}(\wp,b,\delta)\wedge F^{-}_{A_N}(\zeta,\nu,\wp)]$

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$= \sup_{\wp \in Q_N} [F^-_{A^*_N}(\wp, b, \delta) \wedge F^-_{A^*_N}(\zeta, \nu, \wp)]$
$= \sup_{\wp \in Q_N} [F^-_{A^*_N}(\zeta, \nu, \wp) \wedge F^-_{A^*_N}(\wp, b, \delta)]$
$= F^-_{A^*_N}(\zeta, \nu b, \delta) = F^-_{A^*_N}(\zeta, b\nu, \delta) = F^-_{A^*_N}(\zeta, a, \delta).$
Thus the result is true for $|b| = n$. □

Theorem 3.7. *If $B_N = (Q_N, X_N, A_N)$ is a BNFSSM, then*
$T^+_{A^*_N}(\delta, ab, \zeta) = T^+_{A^*_N}(\delta, ba, \zeta)$
$I^+_{A^*_N}(\delta, ab, \zeta) = I^+_{A^*_N}(\delta, ba, \zeta)$
$F^+_{A^*_N}(\delta, ab, \zeta) = F^+_{A^*_N}(\delta, ba, \zeta)$
$T^-_{A^*_N}(\delta, ab, \zeta) = T^-_{A^*_N}(\delta, ba, \zeta)$
$I^-_{A^*_N}(\delta, ab, \zeta) = I^-_{A^*_N}(\delta, ba, \zeta)$
$F^-_{A^*_N}(\delta, ab, \zeta) = F^-_{A^*_N}(\delta, ba, \zeta),$
$\forall \zeta, \delta \in Q_N$ *and* $a, b \in X^*_N$.

Proof. Given $B_N = (Q_N, X_N, A_N)$ is a BNFSSM. Let $\zeta, \delta \in Q_N$ and $a, b \in X^*_N$. We claim by induction on $|b| = n$ if $n = 0$, then $b = \lambda$. Thus
$T^+_{A^*_N}(\delta, ab, \zeta) = T^+_{A^*_N}(\delta, a\lambda, \zeta) = T^+_{A^*_N}(\delta, a, \zeta) = T^+_{A^*_N}(\delta, \lambda a, \zeta)$
$= T^+_{A^*_N}(\delta, ba, \zeta)$
$I^+_{A^*_N}(\delta, ab, \zeta) = I^+_{A^*_N}(\delta, a\lambda, \zeta) = I^+_{A^*_N}(\delta, a, \zeta) = I^+_{A^*_N}(\delta, \lambda a, \zeta)$
$= I^+_{A^*_N}(\delta, ba, \zeta)$
$F^+_{A^*_N}(\delta, ab, \zeta) = F^+_{A^*_N}(\delta, a\lambda, \zeta) = F^+_{A^*_N}(\delta, a, \zeta) = F^+_{A^*_N}(\delta, \lambda a, \zeta)$
$= F^+_{A^*_N}(\delta, ba, \zeta)$
$T^-_{A^*_N}(\delta, ab, \zeta) = T^-_{A^*_N}(\delta, a\lambda, \zeta) = T^-_{A^*_N}(\delta, a, \zeta) = T^-_{A^*_N}(\delta, \lambda a, \zeta)$
$= T^-_{A^*_N}(\delta, ba, \zeta)$
$I^-_{A^*_N}(\delta, ab, \zeta) = I^-_{A^*_N}(\delta, a\lambda, \zeta) = I^-_{A^*_N}(\delta, a, \zeta) = I^-_{A^*_N}(\delta, \lambda a, \zeta)$
$= I^-_{A^*_N}(\delta, ba, \zeta)$
$F^-_{A^*_N}(\delta, ab, \zeta) = F^-_{A^*_N}(\delta, a\lambda, \zeta) = F^-_{A^*_N}(\delta, a, \zeta) = F^-_{A^*_N}(\delta, \lambda a, \zeta)$
$= F^-_{A^*_N}(\delta, ba, \zeta)$.
Thus the outcome is valid for $n = 0$. Assume the result is true for $|b| = n - 1$, i.e., $\forall c \in X^*_N$ with $|c| = n - 1, n > 0$. Let $b \in \zeta$ be such that $b = cp$. Thus
$T^+_{A^*_N}(\delta, ab, \zeta)$
$= T^+_{A^*_N}(\delta, acp, \zeta)$
$= \sup_{r \in Q_N} [T^+_{A^*_N}(\delta, ac, r) \wedge T^+_{A_N}(r, p, \zeta)]$
$= \sup_{r \in Q_N} [T^+_{A^*_N}(\delta, ca, r) \wedge T^+_{A_N}(r, p, \zeta)]$
$= \sup_{r \in Q_N} [T^+_{A^*_N}(r, ca, \delta) \wedge T^+_{A_N}(\zeta, p, r)]$
$= \sup_{r \in Q_N} [T^+_{A_N}(\zeta, p, r) \wedge T^+_{A^*_N}(r, ca, \delta)]$
$= T^+_{A^*_N}(\zeta, pca, \delta)$
$= \sup_{r \in Q_N} [T^+_{A^*_N}(\zeta, pc, r) \wedge T^+_{A^*_N}(r, a, \delta)]$
$= \sup_{r \in Q_N} [T^+_{A^*_N}(\zeta, cp, r) \wedge T^+_{A^*_N}(r, a, \delta)]$
$= T^+_{A^*_N}(\zeta, cpa, \delta)$
$= T^+_{A^*_N}(\delta, cpa, \zeta)$
$= T^+_{A^*_N}(\delta, ba, \zeta)$
$I^+_{A^*_N}(\delta, ab, \zeta)$
$= I^+_{A^*_N}(\delta, acp, \zeta)$
$= \sup_{r \in Q_N} [I^+_{A^*_N}(\delta, ac, r) \wedge I^+_{A_N}(r, p, \zeta)]$
$= \sup_{r \in Q_N} [I^+_{A^*_N}(\delta, ca, r) \wedge I^+_{A_N}(r, p, \zeta)]$
$= \sup_{r \in Q_N} [I^+_{A^*_N}(r, ca, \delta) \wedge I^+_{A_N}(\zeta, p, r)]$
$= \sup_{r \in Q_N} [I^+_{A_N}(\zeta, p, r) \wedge I^+_{A^*_N}(r, ca, \delta)]$
$= I^+_{A^*_N}(\zeta, pca, \delta)$

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$= \sup_{r\in Q_N}[I^{+}_{A^*_N}(\zeta, pc, r) \wedge I^{+}_{A^*_N}(r, a, \delta)]$
$= \sup_{r\in Q_N}[I^{+}_{A^*_N}(\zeta, cp, r) \wedge I^{+}_{A^*_N}(r, a, \delta)]$
$= I^{+}_{A^*_N}(\zeta, cda, \delta)$
$= I^{+}_{A^*_N}(\zeta, cpa, \delta)$
$= I^{+}_{A^*_N}(\delta, ba, \zeta)$
$F^{+}_{A^*_N}(\delta, ab, \zeta)$
$= F^{+}_{A^*_N}(\delta, acp, \zeta)$
$= \inf_{r\in Q_N}[F^{+}_{A^*_N}(\delta, ac, r) \vee F^{+}_{A_N}(r, p, \zeta)]$
$= \inf_{r\in Q_N}[F^{+}_{A^*_N}(\delta, ca, r) \vee F^{+}_{A_N}(r, p, \zeta)]$
$= \inf_{r\in Q_N}[F^{+}_{A^*_N}(r, ca, \delta) \vee F^{+}_{A_N}(\zeta, p, r)]$
$= \inf_{r\in Q_N}[F^{+}_{A_N}(\zeta, p, r) \vee F^{+}_{A^*_N}(r, ca, \delta)]$
$= F^{+}_{A^*_N}(\zeta, pca, \delta)$
$= \inf_{r\in Q_N}[F^{+}_{A^*_N}(\zeta, pc, r) \vee F^{+}_{A^*_N}(r, a, \delta)]$
$= \inf_{r\in Q_N}[F^{+}_{A^*_N}(\zeta, cp, r) \vee F^{+}_{A^*_N}(r, a, \delta)]$
$= F^{+}_{A^*_N}(\zeta, cpa, \delta)$
$= F^{+}_{A^*_N}(\zeta, cpa, \delta)$
$= F^{+}_{A^*_N}(\delta, ba, \zeta)$
$T^{-}_{A^*_N}(\delta, ab, \zeta)$
$= T^{-}_{A^*_N}(\delta, acp, \zeta)$
$= \inf_{r\in Q_N}[T^{-}_{A^*_N}(\delta, ac, r) \vee T^{-}_{A_N}(r, p, \zeta)]$
$= \inf_{r\in Q_N}[T^{-}_{A^*_N}(\delta, ca, r) \vee T^{-}_{A_N}(r, p, \zeta)]$
$= \inf_{r\in Q_N}[T^{-}_{A^*_N}(r, ca, \delta) \vee T^{-}_{A_N}(\zeta, p, r)]$
$= \inf_{r\in Q_N}[T^{-}_{A_N}(\zeta, p, r) \vee T^{-}_{A^*_N}(r, ca, \delta)]$
$= T^{-}_{A^*_N}(\zeta, pca, \delta)$
$= \inf_{r\in Q_N}[T^{-}_{A^*_N}(\zeta, pc, r) \vee T^{-}_{A^*_N}(r, a, \delta)]$
$= \inf_{r\in Q_N}[T^{-}_{A^*_N}(\zeta, cp, r) \vee T^{-}_{A^*_N}(r, a, \delta)]$
$= T^{-}_{A^*_N}(\zeta, cpa, \delta)$
$= T^{-}_{A^*_N}(\zeta, cpa, \delta)$
$= T^{-}_{A^*_N}(\delta, ba, \zeta)$
$I^{-}_{A^*_N}(\delta, ab, \zeta)$
$= I^{-}_{A^*_N}(\delta, acp, \zeta)$
$= \inf_{r\in Q_N}[I^{-}_{A^*_N}(\delta, ac, r) \vee I^{-}_{A_N}(r, p, \zeta)]$
$= \inf_{r\in Q_N}[I^{-}_{A^*_N}(\delta, ca, r) \vee I^{-}_{A_N}(r, p, \zeta)]$
$= \inf_{r\in Q_N}[I^{-}_{A^*_N}(r, ca, \delta) \vee I^{-}_{A_N}(\zeta, p, r)]$
$= \inf_{r\in Q_N}[I^{-}_{A_N}(\zeta, p, r) \vee I^{-}_{A^*_N}(r, ca, \delta)]$
$= I^{+}_{A^*_N}(\zeta, pca, \delta)$
$= \inf_{r\in Q_N}[I^{-}_{A^*_N}(\zeta, pc, r) \vee I^{-}_{A^*_N}(r, a, \delta)]$
$= \inf_{r\in Q_N}[I^{-}_{A^*_N}(\zeta, cp, r) \vee I^{-}_{A^*_N}(r, a, \delta)]$
$= I^{-}_{A^*_N}(\zeta, cpa, \delta)$
$= I^{-}_{A^*_N}(\zeta, ba, \delta)$
$= I^{-}_{A^*_N}(\delta, ba, \zeta)$
$F^{-}_{A^*_N}(\delta, ab, \zeta)$
$= F^{-}_{A^*_N}(\delta, acp, \zeta)$
$= \sup_{r\in Q_N}[F^{-}_{A^*_N}(\delta, ac, r) \wedge F^{-}_{A_N}(r, p, \zeta)]$
$= \sup_{r\in Q_N}[F^{-}_{A^*_N}(\delta, ca, r) \wedge F^{-}_{A_N}(r, p, \zeta)]$
$= \sup_{r\in Q_N}[F^{-}_{A^*_N}(r, ca, \delta) \wedge F^{-}_{A_N}(\zeta, p, r)]$
$= \sup_{r\in Q_N}[F^{-}_{A_N}(\zeta, p, r) \wedge F^{-}_{A^*_N}(r, ca, \delta)]$

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$= F^{-}_{A^*_N}(\zeta, pca, \delta)$
$= \sup_{r\in Q_N}[F^{-}_{A^*_N}(\zeta, pc, r) \wedge F^{-}_{A^*_N}(r, a, \delta)]$
$= \sup_{r\in Q_N}[F^{-}_{A^*_N}(\zeta, cp, r) \wedge F^{-}_{A^*_N}(r, a, \delta)]$
$= F^{+}_{A^*_N}(\zeta, cpa, \delta)$
$= F^{-}_{A^*_N}(\zeta, pa, \delta)$
$= F^{-}_{A^*_N}(\delta, ba, \zeta)$.
This shows that the outcome is valid for $|b| = n$. □

Definition 3.8. Let $M_1 = (Q_{N1}, X_{N1}, A_{N1})$ and $M_2 = (Q_{N2}, X_{N2}, A_{N2})$ be two BNFSMs. A pair (ς, ϱ) of mappings $\varsigma : Q_{N1} \to Q_{N2}$ and $\varrho : X_{N1} \to X_{N2}$ is said to be a homomorphism written as $(\varsigma, \varrho) : M_1 \to M_2$ if it satisfies
$T^{+}_{A_{N1}}(\delta, \nu, \zeta) \leq T^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta))$
$I^{+}_{A_{N1}}(\delta, \nu, \zeta) \leq I^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta))$
$F^{+}_{A_{N1}}(\delta, \nu, \zeta) \geq F^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta))$
$T^{-}_{A_{N1}}(\delta, \nu, \zeta) \geq T^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta))$
$I^{-}_{A_{N1}}(\delta, \nu, \zeta) \geq I^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta))$
$F^{-}_{A_{N1}}(\delta, \nu, \zeta) \leq F^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta))$,
$\forall \zeta, \delta \in Q_{N1}$ and $\nu \in X_{N1}$.

Example 3.9. Define $\alpha : Q1 \to Q2$ and $\beta : X1 \to X2$ as follows $\alpha(p) = r, \alpha(q) = s, \beta(a) = a$ and $\beta(b) = b$. Then $(\alpha, \beta) : M1 \to M2$ is a homomorphism.

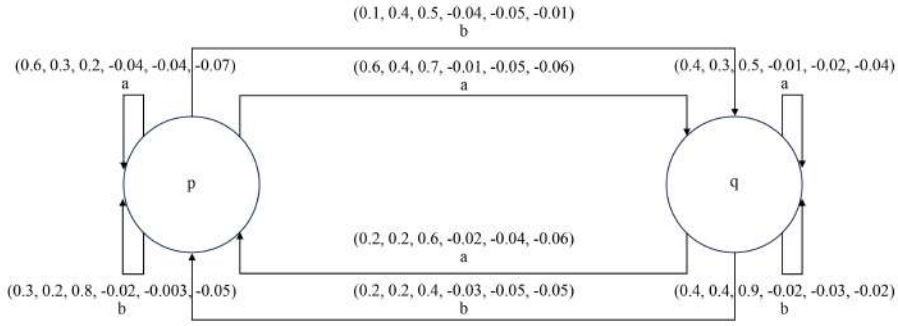

Figure 3: M1

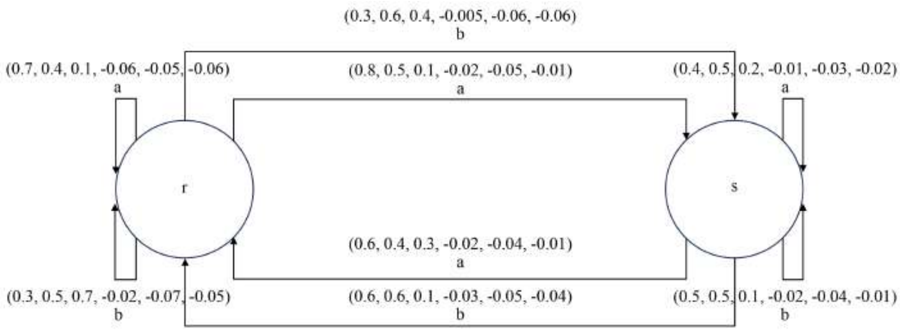

Figure 4: M2 (Homomorphic image of M1)

Definition 3.10. Let $M_1 = (\delta_1, X_{N1}, A_{N1})$ and $M_2 = (\delta_2, X_{N2}, A_{N2})$ be two BNFSMs. A pair (ς, ϱ) of mappings $\varsigma : Q_{N1} \to Q_{N2}$ and $\varrho : X_{N1} \to X_{N2}$ is said to be a Strong homomorphism written as $(\varsigma, \varrho) : M_1 \to M_2$ if it satisfies
$T^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = \vee\{T^{+}_{A_{N1}}(\delta, \nu, t) \mid t \in Q_{N1}, \varsigma(t) = \varsigma(\zeta)\}$
$I^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = \vee\{I^{+}_{A_{N1}}(\delta, \nu, t) \mid t \in Q_{N1}, \varsigma(t) = \varsigma(\zeta)\}$
$F^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = \wedge\{F^{+}_{A_{N1}}(\delta, \nu, t) \mid t \in Q_{N1}, \varsigma(t) = \varsigma(\zeta)\}$
$T^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = \wedge\{T^{-}_{A_{N1}}(\delta, \nu, t) \mid t \in Q_{N1}, \varsigma(t) = \varsigma(\zeta)\}$
$I^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = \wedge\{I^{-}_{A_{N1}}(\delta, \nu, t) \mid t \in Q_{N1}, \varsigma(t) = \varsigma(\zeta)\}$

$F^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = \vee\{F^{-}_{A_{N1}}(\delta, \nu, t) \mid t \in Q_{N1}, \varsigma(t) = \varsigma(\zeta)\}$,
$\forall \zeta, \delta \in Q_{N1}$ and $\nu \in X_{N1}$.
If $X_{N1} = X_{N2}$ and ϱ is the identity map, then a homomorphism or strong homomorphism $\varsigma : M_1 \to M_2$ is one-one and
$T^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = T^{+}_{A_{N1}}(\delta, \nu, \zeta)$
$I^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = I^{+}_{A_{N1}}(\delta, \nu, \zeta)$
$F^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = F^{+}_{A_{N1}}(\delta, \nu, \zeta)$
$T^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = T^{-}_{A_{N1}}(\delta, \nu, \zeta)$
$I^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = I^{-}_{A_{N1}}(\delta, \nu, \zeta)$
$F^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(\zeta)) = F^{-}_{A_{N1}}(\delta, \nu, \zeta)$,
$\forall \zeta, \delta \in Q_{N1}$ and $\nu \in X_{N1}$.

Theorem 3.11. *Suppose $M_1 = (\delta_1, X_{N1}, A_{N1})$ and $M_2 = (\delta_2, X_{N2}, A_{N2})$ be two BNFSMs. Let $(\varsigma, \varrho) : M_1 \to M_2$ be an onto strong homomorphism (SH). If M_1 is commutative, then so is M_2.*

Proof. Suppose $r_2, s_2 \in Q_{N2}$. Thus, $r_1, s_1 \in Q_{N1} \ni \varsigma(r_1) = r_2$ and $\varsigma(s_1) = s_2$. Suppose $x_2, \varepsilon_2 \in X_{N2}$. Then there exist $\nu_1, \varepsilon_1 \in X_{N1} \ni \varrho(\nu_1) = x_2, \varrho(\varepsilon_1) = \varepsilon_2$. Since M_1 is commutative , we have
$T^{+}_{A^*_{N2}}(r_2, \nu_2\varepsilon_2, s_2) = T^{+}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1)\varrho(\varepsilon_1), \varsigma(s_1))$
$= T^{+}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1\varepsilon_1), \varsigma(s_1))$
$=\vee\{T^{+}_{A^*_{N1}}(r_1, \nu_1\varepsilon_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=\vee\{T^{+}_{A^*_{N1}}(r_1, \varepsilon_1\nu_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=T^{+}_{A_{N2}}(\varsigma(r_1), \varrho(\varepsilon_1\nu_1), \varsigma(s_1))$
$=T^{+}_{A^*_{N2}}(r_2, \varepsilon_2\nu_2, s_2)$
$I^{+}_{A^*_{N2}}(r_2, \nu_2\varepsilon_2, s_2) = I^{+}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1)\varrho(\varepsilon_1), \varsigma(s_1))$
$= I^{+}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1\varepsilon_1), \varsigma(s_1))$
$=\vee\{I^{+}_{A^*_{N1}}(r_1, \nu_1\varepsilon_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=\vee\{I^{+}_{A^*_{N1}}(r_1, \varepsilon_1\nu_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=I^{+}_{A_{N2}}(\varsigma(r_1), \varrho(\varepsilon_1\nu_1), \varsigma(s_1))$
$=I^{+}_{A^*_{N2}}(r_2, \varepsilon_2\nu_2, s_2)$
$F^{+}_{A^*_{N2}}(r_2, \nu_2\varepsilon_2, s_2) = F^{+}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1)\varrho(\varepsilon_1), \varsigma(s_1))$
$= F^{+}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1\varepsilon_1), \varsigma(s_1))$
$=\wedge\{F^{+}_{A^*_{N1}}(r_1, \nu_1\varepsilon_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=\wedge\{F^{+}_{A^*_{N1}}(r_1, \varepsilon_1\nu_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=F^{+}_{A_{N2}}(\varsigma(r_1), \varrho(\varepsilon_1\nu_1), \varsigma(s_1))$
$=F^{+}_{A^*_{N2}}(r_2, \varepsilon_2\nu_2, s_2)$
$T^{-}_{A^*_{N2}}(r_2, \nu_2\varepsilon_2, s_2) = T^{-}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1)\varrho(\varepsilon_1), \varsigma(s_1))$
$= T^{-}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1\varepsilon_1), \varsigma(s_1))$
$=\wedge\{T^{-}_{A^*_{N1}}(r_1, \nu_1\varepsilon_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=\wedge\{T^{-}_{A^*_{N1}}(r_1, \varepsilon_1\nu_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=T^{-}_{A_{N2}}(\varsigma(r_1), \varrho(\varepsilon_1\nu_1), \varsigma(s_1))$
$=T^{-}_{A^*_{N2}}(r_2, \varepsilon_2\nu_2, s_2)$
$I^{-}_{A^*_{N2}}(r_2, \nu_2\varepsilon_2, s_2) = I^{-}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1)\varrho(\varepsilon_1), \varsigma(s_1))$
$= I^{-}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1\varepsilon_1), \varsigma(s_1))$
$=\wedge\{I^{-}_{A^*_{N1}}(r_1, \nu_1\varepsilon_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=\wedge\{I^{-}_{A^*_{N1}}(r_1, \varepsilon_1\nu_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=I^{-}_{A_{N2}}(\varsigma(r_1), \varrho(\varepsilon_1\nu_1), \varsigma(s_1))$
$=I^{-}_{A^*_{N2}}(r_2, \varepsilon_2\nu_2, s_2)$
$F^{-}_{A^*_{N2}}(r_2, \nu_2\varepsilon_2, s_2) = F^{-}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1)\varrho(\varepsilon_1), \varsigma(s_1))$
$= F^{-}_{A^*_{N2}}(\varsigma(r_1), \varrho(\nu_1\varepsilon_1), \varsigma(s_1))$
$=\vee\{F^{-}_{A^*_{N1}}(r_1, \nu_1\varepsilon_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$

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$=\vee\{F^{-}_{A^{*}_{N1}}(r_1, \varepsilon_1\nu_1, \mu_1) \mid \mu_1 \in Q_{N1}, \varsigma(\mu_1) = \varsigma(s_1)\}$
$=F^{-}_{A_{N2}}(\varsigma(r_1), \varrho(\varepsilon_1\nu_1), \varsigma(s_1))$
$=F^{-}_{A^{*}_{N2}}(r_2, \varepsilon_2\nu_2, s_2)$.
Hence M_2 is a commutative BNFSM. □

Theorem 3.12. *Suppose $M_1 = (Q_{N1}, X_{N1}, A_{N1})$ and $M_2 = (Q_{N2}, X_{N2}, A_{N2})$ are two BNFSMs and $(\varsigma, \varrho) : M_1 \to M_2$ a strong homomorphism. Then for all $\delta, r \in Q_{N1}$ and for all $\nu \in X_{N1}$ provided that*
$T^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) > 0$ and $T^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) < 0 \,\exists\, t \in Q_{N1} \ni T^{+}_{A_{N1}}(\delta, \nu, t) > 0, T^{-}_{A_{N1}}(\delta, \nu, t) < 0$ and $\varsigma(t) = \varsigma(r)$,
$I^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) > 0$ and $I^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) < 0 \,\exists\, t \in Q_{N1}$ such that $I^{+}_{A_{N1}}(\delta, \nu, t) > 0, I^{-}_{A_{N1}}(\delta, \nu, t) < 0$ and $\varsigma(t) = \varsigma(r)$,
$F^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) < 0$ and $I^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) > 0 \,\exists\, t \in Q_{N1} \ni F^{+}_{A_{N1}}(\delta, \nu, t) < 0, F^{-}_{A_{N1}}(\delta, \nu, t) > 0$ and $\varsigma(t) = \varsigma(r)$.
Moreover, $\forall p \in Q_{N1}$ if $\varsigma(\zeta) = \varsigma(\delta)$,
$T^{+}_{A_{N1}}(\delta, \nu, t) \geq T^{+}_{A_{N1}}(\delta, \nu, r)$
$I^{+}_{A_{N1}}(\delta, \nu, t) \geq I^{+}_{A_{N1}}(\delta, \nu, r)$
$F^{+}_{A_{N1}}(\delta, \nu, t) \leq F^{+}_{A_{N1}}(\delta, \nu, r)$
$T^{-}_{A_{N1}}(\delta, \nu, t) \leq T^{-}_{A_{N1}}(\delta, \nu, r)$
$I^{-}_{A_{N1}}(\delta, \nu, t) \leq I^{-}_{A_{N1}}(\delta, \nu, r)$
$F^{-}_{A_{N1}}(\delta, \nu, t) \geq F^{-}_{A_{N1}}(\delta, \nu, r)$.

Proof. Let $\zeta, \delta, r \in Q_{N1}$ and $\nu \in X_{N1}$, we have
$T^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = \vee\{T^{+}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} > 0$ and
$T^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = \wedge\{T^{-}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} < 0$ (by strong homomorphism).
Since Q_{N1} is finite, $\exists t \in Q_{N1} \ni \varsigma(t) = \varsigma(r)$ and
$T^{+}_{A_{N1}}(\delta, \nu, t) = \vee\{T^{+}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} > 0$ and
$T^{-}_{A_{N1}}(\delta, \nu, t) = \wedge\{T^{-}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} < 0$.
Suppose $\varsigma(\zeta) = \varsigma(\delta)$. Then
$T^{-}_{A_{N1}}(\delta, \nu, t) = T^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = T^{-}_{A_{N2}}(\varsigma(\zeta), \varrho(\nu), \varsigma(r))$
$\leq T^{-}_{A_{N1}}(\zeta, \nu, r)$ and
$T^{+}_{A_{N1}}(\delta, \nu, t) = T^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = T^{+}_{A_{N2}}(\varsigma(\zeta), \varrho(\nu), \varsigma(r))$
$\geq T^{+}_{A_{N1}}(\zeta, \nu, r)$.
Now, $I^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = \vee\{I^{+}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} > 0$ and
$I^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = \wedge\{I^{-}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} < 0$ (by strong homomorphism).
Since Q_{N1} is finite, $\exists t \in Q_{N1} \ni \varsigma(t) = \varsigma(r)$ and
$I^{+}_{A_{N1}}(\delta, \nu, t) = \vee\{I^{+}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} > 0$ and
$I^{-}_{A_{N1}}(\delta, \nu, t) = \wedge\{I^{-}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} < 0$.
Suppose $\varsigma(\zeta) = \varsigma(\delta)$. Then
$I^{-}_{A_{N1}}(\delta, \nu, t) = I^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = I^{-}_{A_{N2}}(\varsigma(\zeta), \varrho(\nu), \varsigma(r))$
$\leq I^{-}_{A_{N1}}(\zeta, \nu, r)$ and
$I^{+}_{A_{N1}}(\delta, \nu, t) = I^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = I^{+}_{A_{N2}}(\varsigma(\zeta), \varrho(\nu), \varsigma(r))$
$\geq I^{+}_{A_{N1}}(\zeta, \nu, r)$.
Now, $F^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = \wedge\{F^{+}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} < 0$ and
$F^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = \vee\{F^{-}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} > 0$ (by strong homomorphism).
Since Q_{N1} is finite, $\exists t \in Q_{N1} \ni \varsigma(t) = \varsigma(r)$ and
$F^{+}_{A_{N1}}(\delta, \nu, t) = \wedge\{F^{+}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} < 0$ and
$F^{-}_{A_{N1}}(\delta, \nu, t) = \vee\{F^{-}_{A_{N1}}(\delta, \nu, s) \mid s \in Q_{N1}, \varsigma(s) = \varsigma(r)\} > 0$.
Suppose $\varsigma(\zeta) = \varsigma(\delta)$. Then
$F^{-}_{A_{N1}}(\delta, \nu, t) = F^{-}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = F^{-}_{A_{N2}}(\varsigma(\zeta), \varrho(\nu), \varsigma(r))$
$\geq F^{-}_{A_{N1}}(\zeta, \nu, r)$ and
$F^{+}_{A_{N1}}(\delta, \nu, t) = F^{+}_{A_{N2}}(\varsigma(\delta), \varrho(\nu), \varsigma(r)) = F^{+}_{A_{N2}}(\varsigma(\zeta), \varrho(\nu), \varsigma(r))$
$\leq F^{+}_{A_{N1}}(\zeta, \nu, r)$. □

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4 Conclusion

The notion of BNFSSMs, homomorphisms and strong homomorphisms of BNFSMs are introduced and some related results are studied. We will continue to work on BNFSM decomposition, as well as many other concepts, in the future.

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