Neutrosophic Pre-compactness

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Abstract

The purpose of this article is to study some covering properties in neutrosophic topological spaces via neutrosophic pre-open sets. We define neutrosophic pre-open cover, neutrosophic pre-compactness, neutrosophic countably pre-compactness and neutrosophic pre-Lindelöfness and study various properties connecting them. We study some properties involving neutrosophic continuous and neutrosophic pre-continuous functions. We also define neutrosophic pre-base, neutrosophic pre-subbase, neutrosophic pre*-open function, neutrosophic pre-irresolute function and study some properties. In addition to that, we define and study neutrosophic local pre-compactness.

Keywords: Neutrosophic pre-compact space; Neutrosophic countably pre-compact space; Neutrosophic pre-Lindelöf space; Neutrosophic Np-base; Neutrosophic Np-subbase; Neutrosophic pre-irresolute function; Neutrosophic local pre-compact space.

1 Introduction

The notion of neutrosophic set was coined by Florentin Smarandache.[13–15] Since then, Smarandache and other researchers[6,18,20] have studied and further developed the theory of neutrosophic sets. Neutrosophic set is an extended form of intuitionistic fuzzy set developed by K.Atanassov[1] in 1986. The concept of neutrosophic sets has found various applications in different fields, particularly in situations where uncertainty, vagueness, and indeterminacy are present.

In 2012, Salama & Alblowi[16] developed the concept of neutrosophic topological space, which is a generalization of the intuitionistic fuzzy topological space that was originally proposed by D.Coker[5] in 1997. In 2016, Karatas and Kuru[9] redefined the single-valued neutrosophic set operations and introduced neutrosophic topology. The authors then investigated some important properties of neutrosophic topological spaces. Since then, many researchers[2,4,10,11,17,19] have further developed various aspects of neutrosophic topology. The idea of fuzzy pre-compact space was introduced by Jaber[8] in 2020. Rao & Rao,[12] in 2017, developed the concepts of neutrosophic pre-open and pre-closed sets and thereafter, Arokiarani et.al.[3] developed the idea of neutrosophic pre-open, pre-closed, pre-continuous functions. Recently, Dey & Ray[7] studied separation properties using neutrosophic pre-open sets.

In this write-up, we first define neutrosophic pre-open cover using neutrosophic pre-open sets. After that, we define neutrosophic pre-compact space, neutrosophic countably pre-compact space and neutrosophic pre-Lindelöf space via neutrosophic pre-open covers and study various covering properties involving them. We also define neutrosophic pre-base, neutrosophic pre-subbase, neutrosophic pre*-open function, neutrosophic pre-irresolute function and investigate some covering properties. In the long run, we define and study neutrosophic local pre-compactness.

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2 Preliminaries

2.1 Definition:[13]

Let X be the universe of discourse. A neutrosophic set A over X is defined as $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle : x \in X\}$, where the functions $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A$ are real standard or non-standard subsets of $]^{-}0, 1^{+}[$, i.e., $\mathcal{T}_A : X \to]^{-}0, 1^{+}[$, $\mathcal{I}_A : X \to]^{-}0, 1^{+}[$, $\mathcal{F}_A : X \to]^{-}0, 1^{+}[$ and $^{-}0 \leq \mathcal{T}_A(x) + \mathcal{I}_A(x) + \mathcal{F}_A(x) \leq 3^{+}$.

The neutrosophic set A is characterized by the truth-membership function $\mathcal{T}_A$, indeterminacy-membership function $\mathcal{I}_A$, falsehood-membership function $\mathcal{F}_A$.

2.2 Definition:[20]

Let X be the universe of discourse. A single valued neutrosophic set A over X is defined as $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle : x \in X\}$, where $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A$ are functions from X to $[0, 1]$ and $0 \leq \mathcal{T}_A(x)+\mathcal{I}_A(x)+\mathcal{F}_A(x) \leq 3$.

The set of all single valued neutrosophic sets over X is denoted by $\mathcal{N}(X)$.

Throughout this article, a neutrosophic set (NS, for short) will mean a single-valued neutrosophic set.

2.3 Definition:[9]

Let $A, B \in \mathcal{N}(X)$. Then

(i) (Inclusion): If $\mathcal{T}_A(x) \leq \mathcal{T}_B(x), \mathcal{I}_A(x) \geq \mathcal{I}_B(x), \mathcal{F}_A(x) \geq \mathcal{F}_B(x)$ for all $x \in X$ then A is said to be a neutrosophic subset of B and which is denoted by $A \subseteq B$.

(ii) (Equality): If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(iii) (Intersection): The intersection of A and B, denoted by $A \cap B$, is defined as $A \cap B = \{\langle x, \mathcal{T}_A(x) \wedge \mathcal{T}_B(x), \mathcal{I}_A(x) \vee \mathcal{I}_B(x), \mathcal{F}_A(x) \vee \mathcal{F}_B(x)\rangle : x \in X\}$.

(iv) (Union): The union of A and B, denoted by $A \cup B$, is defined as $A \cup B = \{\langle x, \mathcal{T}_A(x) \vee \mathcal{T}_B(x), \mathcal{I}_A(x) \wedge \mathcal{I}_B(x), \mathcal{F}_A(x) \wedge \mathcal{F}_B(x)\rangle : x \in X\}$.

(v) (Complement): The complement of the NS A, denoted by A^c, is defined as $A^c = \{\langle x, \mathcal{F}_A(x), 1 - \mathcal{I}_A(x), \mathcal{T}_A(x)\rangle : x \in X\}$

(vi) (Universal Set): If $\mathcal{T}_A(x) = 1, \mathcal{I}_A(x) = 0, \mathcal{F}_A(x) = 0$ for all $x \in X$ then A is said to be neutrosophic universal set and which is denoted by $\tilde{X}$.

(vii) (Empty Set): If $\mathcal{T}_A(x) = 0, \mathcal{I}_A(x) = 1, \mathcal{F}_A(x) = 1$ for all $x \in X$ then A is said to be neutrosophic empty set and which is denoted by $\tilde{\emptyset}$.

2.4 Definition:[11]

Let $\mathcal{N}(X)$ be the set of all neutrosophic sets over X. An NS $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ is called a neutrosophic point (NP, for short) iff for any element $y \in X$, $\mathcal{T}_P(y) = \alpha, \mathcal{I}_P(y) = \beta, \mathcal{F}_P(y) = \gamma$ for $y = x$ and $\mathcal{T}_P(y) = 0, \mathcal{I}_P(y) = 1, \mathcal{F}_P(y) = 1$ for $y \neq x$, where $0 < \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1$. A neutrosophic point $P = \{\langle x, \mathcal{T}_P(x), \mathcal{I}_P(x), \mathcal{F}_P(x)\rangle : x \in X\}$ will be denoted by by $x_{\alpha,\beta,\gamma}$. For the NP $x_{\alpha,\beta,\gamma}$, x will be called its support. The complement of the NP $x_{\alpha,\beta,\gamma}$ will be denoted by $x^c_{\alpha,\beta,\gamma}$ or $(x_{\alpha,\beta,\gamma})^c$.

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2.5 Definition:[17]

Let X and Y be two non-empty sets and $f : X \to Y$ be a function. Also let $A \in \mathcal{N}(X)$ and $B \in \mathcal{N}(Y)$. Then

(1) Image of A under f is defined by
$f(A) = \{\langle y, f(\mathcal{T}_A)(y), f(\mathcal{I}_A)(y), (1 - f(1 - \mathcal{F}_A))(y)\rangle : y \in Y\}$, where

$$f(\mathcal{T}_A)(y) = \begin{cases} \sup\{\mathcal{T}_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

$$f(\mathcal{I}_A)(y) = \begin{cases} \inf\{\mathcal{I}_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

$$(1 - f(1 - \mathcal{F}_A))(y) = \begin{cases} \inf\{\mathcal{F}_A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

(2) Pre-image of B under f is defined by
$f^{-1}(B) = \{\langle x, f^{-1}(\mathcal{T}_B)(x), f^{-1}(\mathcal{I}_B)(x), f^{-1}(\mathcal{F}_B)(x)\rangle : x \in X\}$

2.6 Theorem:[17]

Let $f : X \to Y$ be a function. Also let $A, A_i \in \mathcal{N}(X), i \in I$ and $B, B_j \in \mathcal{N}(Y), j \in J$. Then the following hold.

(i) $A_1 \subseteq A_2 \Leftrightarrow f(A_1) \subseteq f(A_2)$, $B_1 \subseteq B_2 \Leftrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

(ii) $A \subseteq f^{-1}(f(A))$ and if f is injective then $A = f^{-1}(f(A))$.

(iii) $f^{-1}(f(B)) \subseteq B$ and if f is surjective then $f^{-1}(f(B)) = B$.

(iv) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$ and $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$.

(v) $f(\cup A_i) = \cup f(A_i)$, $f(\cap A_i) \subseteq \cap f(A_i)$ and if f is injective then $f(\cap A_i) = \cap f(A_i)$.

(vi) $f^{-1}(\tilde{\emptyset}_Y) = \tilde{\emptyset}_X$, $f^{-1}(\tilde{Y}) = \tilde{X}$.

(vii) $f(\tilde{\emptyset}_X) = \tilde{\emptyset}_Y$, $f(\tilde{X}) = \tilde{Y}$ if f is surjective.

2.7 Definition:[9]

Let $\tau \subseteq \mathcal{N}(X)$. Then τ is called a neutrosophic topology on X if

(i) $\tilde{\emptyset}$ and $\tilde{X}$ belong to τ.

(ii) An arbitrary union of neutrosophic sets in τ is in τ.

(iii) The intersection of any two neutrosophic sets in τ is in τ.

If τ is a neutrosophic topology on X then the pair (X, τ) is called a neutrosophic topological space (NTS, for short) over X. The members of τ are called neutrosophic open sets in X. If for a neutrosophic set A, $A^c \in \tau$ then A is said to be a neutrosophic closed set in X.

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2.8 Definition:[9]

Let (X, τ) be a NTS and $A \in \mathcal{N}(X)$. Then the neutrosophic

(i) interior of A, denoted by $int(A)$, is defined as $int(A) = \cup\{G : G \in \tau$ and $G \subseteq A\}$.

(ii) closure of A, denoted by $cl(A)$, is defined as $cl(A) = \cap\{G : G$ is a neutrosophic closed set and $G \supseteq A\}$.

2.9 Definition:[12]

Let (X, τ) be an NTS and $A \in \mathcal{N}(X)$. Then

(i) A is called a neutrosophic pre-open set (NPO, for short) in X iff $A \subseteq int(cl(A))$.

(ii) A is called a neutrosophic pre-closed (NPC, for short) set in X iff $cl(int(A)) \subseteq A$.

If G is an NPO (resp. NPC) set in X then we may also say that G is a τ-NPO (resp. τ-NPC) set.

2.10 Theorem:[12]

Let (X, τ) be an NTS and $A \in \mathcal{N}(X)$. Then

(i) A is an NPC set in X if and only if A^c is an NPO set in X.

(ii) Every neutrosophic open set in an NTS is an NPO set.

(iii) Every neutrosophic closed set in an NTS is an NPC set.

2.11 Definition:[19]

Let f be a function from an NTS (X, τ) to the NTS (Y, σ). Then

(i) f is called a neutrosophic open function if $f(G) \in \sigma$ for all $G \in \tau$

(ii) f is called a neutrosophic continuous function if $f^{-1}(G) \in \tau$ for all $G \in \sigma$.

2.12 Definition:[3]

Let f be a function from an NTS (X, τ) to the NTS (Y, σ). Then f is called a neutrosophic

(i) pre-open function if $f(G)$ is an NPO set in Y for every neutrosophic open set G in X.

(ii) pre-continuous function if $f^{-1}(G)$ is an NPO set in X for every $G \in \sigma$.

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2.13 Proposition:[7]

Let $(Y, \tau\mid_Y)$ be a neutrosophic subspace of the NTS (X, τ). Then

(i) $G\mid_Y$ is a $\tau\mid_Y$-NPO set in Y for every τ-NPO set G in X.

(ii) $G\mid_Y$ is a $\tau\mid_Y$-NPC set in Y for every τ-NPC set G in X.

3 Neutrosophic pre-compactness

3.1 Definition:

Let (X, τ) be an NTS and $A \in \mathcal{N}(X)$. A collection $C = \{G_\lambda : \lambda \in \triangle\}$ of NPO sets of X is called a neutrosophic pre-open cover (NPOC, in short) of A iff $A \subseteq \cup_{\lambda\in\triangle} G_\lambda$. In particular, C is said to be an NPOC of X iff $\tilde{X} = \cup_{\lambda\in\triangle} G_\lambda$.

Let C be an NPOC of the NS A and $C' \subseteq C$. Then C' is called a neutrosophic pre-open subcover (NPOSC, in short) of C if C' is also an NPOC of A.

An NPOC C of an NS A is said to be countable (resp. finite) if C consists of a countable (resp. finite) number of NPO sets.

3.2 Definition:

An NS A in an NTS (X, τ) is said to be a neutrosophic pre-compact set iff every NPOC of A has a finite NPOSC. In particular, the space X is said to be a neutrosophic pre-compact space iff every NPOC of X has a finite NPOSC.

An NTS (X, τ) is said to be a neutrosophic countably pre-compact space iff every countable NPOC of X has a finite NPOSC.

An NTS (X, τ) is said to be a neutrosophic pre-Lindel$\ddot{o}$f space iff every NPOC of X has a countable NPOSC.

3.3 Proposition:

Every neutrosophic pre-compact space is neutrosophic countably pre-compact.

Proof: Obvious.

3.4 Proposition:

In an NTS, every neutrosophic pre-compact set is neutrosophic compact.

Proof: Let A be a neutrosophic pre-compact set of an NTS (X, τ). Let $\mathcal{C} = \{G_i : i \in\triangle\}$ be an NOC of A. Since every neutrosophic open set is an NPO set[by 2.10], so G_i is an NPO set for each $i \in\triangle$. Therefore $\mathcal{C}$ is an NPOC of A. Since A is pre-compact, so there exists a finite subcollection $\{G_i^1, G_i^2, ..., G_i^m\}$, say, of $\mathcal{C}$ such that $A \subseteq G_i^1 \cup G_i^2 \cup ... \cup G_i^m$. Thus the NOC $\mathcal{C}$ of A has a finite NOSC $\{G_i^1, G_i^2, ..., G_i^m\}$. Hence A is a neutrosophic compact set.

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3.5 Example :

Converse of 3.4 is not true. We establish it by the following example.

Let $X = \{a, b\}, A = \{\langle a, 1, 0, 0\rangle, \langle b, 0, 1, 1\rangle\}, \tau = \{\tilde{X}, \tilde{\emptyset}, A\}$ and for each $n \in \mathsf{N} = \{1, 2, 3, \cdots\}$, we define $H_n = \{\langle a, \frac{n}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}\rangle, \langle b, 0, 1, 1\rangle\}$. Obviously (X, τ) is an NTS and H_n is an NPO set in X. Clearly A is a neutrosophic compact set. Now $\cup_{n\in\mathsf{N}} H_n = A$. So, $\{H_n : n \in \mathsf{N}\}$ is an NPOC of A. It is clear that the NPOC $\{H_n : n \in \mathsf{N}\}$ of A has no finite NPOSC. Therefore A is not a neutrosophic pre-compact set in X.

3.6 Proposition:

Every neutrosophic pre-compact space is a neutrosophic compact space.

Proof: Obvious from 3.4.

3.7 Remark:

Converse of 3.6 is not true. We establish it by the following example.

Let us consider the NTS (N, τ), where $\tau = \{\tilde{\emptyset}, \tilde{\mathsf{N}}\}, \mathsf{N} = \{1, 2, 3, \cdots\}$. Clearly (N, τ) is a neutrosophic compact space. We show that (N, τ) is not a neutrosophic pre-compact space.

For $n \in \mathsf{N}$, we define $G_n = \{\langle x, \mathcal{T}_{G_n}(x), \mathcal{I}_{G_n}(x), \mathcal{F}_{G_n}(x) : x \in X\}$, where $\mathcal{T}_{G_n}(x) = 1, \mathcal{I}_{G_n}(x) = 0, \mathcal{F}_{G_n}(x) = 0$ if $x = n$ and $\mathcal{T}_{G_n}(x) = 0, \mathcal{I}_{G_n}(x) = 1, \mathcal{F}_{G_n}(x) = 1$ if $x \neq n$. Clearly, for each $n \in \mathsf{N}$, G_n is an NPO set in (N, τ). Obviously the collection $\mathcal{C} = \{G_n : n \in \mathsf{N}\}$ is an NPOC of N but it has no finite NPOSC. Therefore (N, τ) is not a neutrosophic pre-compact space.

Thus (N, τ) is a neutrosophic compact space but not a neutrosophic pre-compact space.

3.8 Proposition:

In an NTS, union of two neutrosophic pre-compact sets is neutrosophic pre-compact.

Proof: Let A and B be two neutrosophic pre-compact sets of an NTS (X, τ). Let $\mathcal{C} = \{G_i : i \in \triangle\}$ be an NPOC of $A \cup B$. Then $A \cup B \subseteq \cup_{i\in\triangle} G_i$. Since $A \subseteq A \cup B$, so $\mathcal{C}$ is an NPOC of A. Since A is neutrosophic pre-compact, so there exists a finite subcollection $\{G_i^1, G_i^2, ..., G_i^m\}$ of $\mathcal{C}$ such that $A \subseteq G_i^1 \cup G_i^2 \cup ... \cup G_i^m$. Similarly, since B is neutrosophic pre-compact, so there exists a finite subcollection $\{H_i^1, H_i^2, ..., H_i^n\}$ of $\mathcal{C}$ such that $B \subseteq H_i^1 \cup H_i^2 \cup ... \cup H_i^n$. Therefore $A \cup B \subseteq G_i^1 \cup G_i^2 \cup ... \cup G_i^m \cup H_i^1 \cup H_i^2 \cup ... \cup H_i^n$. Thus there exists a finite subcollection $\{G_i^1, G_i^2, ..., G_i^m, H_i^1, H_i^2, ..., H_i^n\}$ of $\mathcal{C}$ such that $A \cup B \subseteq G_i^1 \cup G_i^2 \cup ... \cup G_i^m \cup H_i^1 \cup H_i^2 \cup ... \cup H_i^n$. Therefore $A \cup B$ is neutrosophic pre-compact. Hence proved.

3.9 Proposition:

In an NTS, finite union of neutrosophic pre-compact sets is neutrosophic pre-compact.

Proof: Immediate from 3.8.

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3.10 Proposition:

In an NTS, union of a neutrosophic pre-compact set and a neutrosophic compact set is a neutrosophic compact set.

Proof: Obvious.

3.11 Definition:

Let $(Y, \tau|_Y)$ be a neutrosophic subspace of the NTS (X, τ). Then the set of all NPO sets $G|_Y$ in Y for which G is an NPO set in X will be denoted by $NPO(Y)$, i.e., $NPO(Y) = \{G|_Y \subseteq Y : G|_Y$ is an NPO set in Y and $G \subseteq X$ is an NPO set in $X\}$.

3.12 Proposition:

Let $(Y, \tau|_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. Then A is neutrosophic pre-compact in X iff every cover of A by the NPO sets in $NPO(Y)$ has a finite subcover.

Proof: Necessary part: Let $\mathcal{C} = \{G_i|_Y : i \in \triangle\}$ be a cover of A, where $G_i|_Y \in NPO(Y)$ for each $i \in \triangle$. Then $A \subseteq \cup_{i\in\triangle}(G_i|_Y) \Rightarrow A \subseteq (\cup_{i\in\triangle}G_i)|_Y \Rightarrow A \subseteq \cup_{i\in\triangle}G_i$. Clearly G_i is an NPO set in X [by 3.11] for each $i \in \triangle$ and so, $\mathcal{C}^* = \{G_i : i \in \triangle\}$ is an NPOC of A in X. Since A is pre-compact in X, so there exists a finite subcollection $\{G_{i_k} : k = 1, 2, 3, ..., n\}$ of $\mathcal{C}^*$ such that $A \subseteq \cup_{k=1}^n G_{i_k} \Rightarrow A \subseteq (\cup_{k=1}^n G_{i_k})|_Y \Rightarrow A \subseteq \cup_{k=1}^n (G_{i_k}|_Y)$. Thus the cover $\mathcal{C}$ of A has a finite subcover $\{G_{i_k} : k = 1, 2, 3, ..., n\}$.

Sufficient part: Let $\mathcal{B} = \{G_i : i \in \triangle\}$ be an NPOC of A in X, where G_i is an NPO set in X for each $i \in \triangle$. Then $A \subseteq \cup_{i\in\triangle}G_i \Rightarrow A \subseteq (\cup_{i\in\triangle}G_i)|_Y \Rightarrow A \subseteq \cup_{i\in\triangle}(G_i|_Y)$. Since $G_i|_Y \in NPO(Y)$ for each $i \in \triangle$[by 2.13], so $\mathcal{B}^* = \{G_i|_Y : i \in \triangle\}$ is a cover of A by the NPO sets in $NPO(Y)$. Therefore, by hypothesis, there exists a finite subcollection $\{G_{i_k}|_Y : k = 1, 2, 3, ..., n\}$ of $\mathcal{B}^*$ such that $A \subseteq \cup_{k=1}^n (G_{i_k}|_Y) \Rightarrow A \subseteq (\cup_{k=1}^n G_{i_k})|_Y \Rightarrow A \subseteq \cup_{k=1}^n G_{i_k}$. Thus the NPOC $\mathcal{B}$ of A has a finite NPOSC $\{G_{i_k} : k = 1, 2, 3, ..., n\}$. Therefore, A is neutrosophic pre-compact in X.

3.13 Proposition:

Let $(Y, \tau|_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. Then A is neutrosophic pre-Lindelöf (resp. neutrosophic countably pre-compact) in X iff every cover (resp. countable cover) of A by the NPO sets in $NPO(Y)$ has a countable (resp. finite) subcover.

Proof: Obvious from 3.12.

3.14 Proposition:

Let $(Y, \tau|_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. If A is neutrosophic pre-compact in X then A is neutrosophic compact in Y.

Proof: Let $\mathcal{C} = \{G_i|_Y : i \in \triangle\}$ be an NOC of A in Y, where $G_i|_Y \in \tau|_Y$ for each $i \in \triangle$. Then $A \subseteq \cup_{i\in\triangle}(G_i|_Y) \Rightarrow A \subseteq \cup_{i\in\triangle}G_i$. Obviously $G_i \in \tau$ and so, G_i is an NPO set in X for each $i \in \triangle$. Therefore, $\mathcal{C}^* = \{G_i : i \in \triangle\}$ is an NPOC of A in X. Since A is pre-compact in X, so there exists a finite subcollection $\{G_{i_k} : k = 1, 2, 3, ..., n\}$ of $\mathcal{C}^*$ such that $A \subseteq \cup_{k=1}^n G_{i_k} \Rightarrow A \subseteq (\cup_{k=1}^n G_{i_k})|_Y \Rightarrow A \subseteq \cup_{k=1}^n (G_{i_k}|_Y)$. Thus the NOC $\mathcal{C}$ of A has a finite NOSC $\{G_{i_k} : k = 1, 2, 3, ..., n\}$. Therefore A is neutrosophic compact in Y.

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3.15 Proposition:

Let $(Y, \tau\mid_Y)$ be a neutrosophic subspace of the NTS (X, τ) and $A \subseteq Y$. If A is pre-compact in Y then A is pre-compact in X.

Proof: Obvious.

3.16 Proposition:

If G is an NPC subset of a neutrosophic pre-compact space (X, τ) such that $G \cap G^c = \tilde{\emptyset}$ then G is neutrosophic pre-compact.

Proof: Let $\mathcal{C} = \{H_i : i \in \triangle\}$ be an NPOC of G. Then $G \subseteq \cup_{i\in\triangle} H_i$. Since G^c is an NPO set and since $G \cap G^c = \tilde{\emptyset}$, i.e., $G \cup G^c = \tilde{X}$, so $\mathcal{D} = \{H_i : i \in\triangle\} \cup \{G^c\}$ is an NPOC of X. As X is neutrosophic pre-compact, so there exists a finite subcollection $\mathcal{D}' = \{H_{i_1}, H_{i_2}, ..., H_{i_n}\} \cup \{G^c\}$ of $\mathcal{D}$ such that $X \subseteq H_{i_1} \cup H_{i_2} \cup ... \cup H_{i_n} \cup G^c$. Therefore $G \subseteq H_{i_1} \cup H_{i_2} \cup ... \cup H_{i_n} \cup G^c$. But $G \cap G^c = \tilde{\emptyset}$, so $G \subseteq H_{i_1} \cup H_{i_2} \cup ... \cup H_{i_n}$. Thus the NPOC $\mathcal{C}$ of G has a finite NPOSC $\{H_{i_1}, H_{i_2}, ..., H_{i_n}\}$. Hence G is neutrosophic pre-compact set.

3.17 Proposition:

If G is an NPC subset of a neutrosophic pre-compact space (X, τ) such that $G \cap G^c = \tilde{\emptyset}$ then G is neutrosophic compact.

Proof: Immediate from 3.16.

3.18 Proposition:

If G is a neutrosophic closed subset of a neutrosophic pre-compact space (X, τ) such that $G \cap G^c = \tilde{\emptyset}$ then G is neutrosophic pre-compact.

Proof: Immediate from 3.16.

3.19 Proposition:

If G is a neutrosophic closed subset of a neutrosophic pre-compact space (X, τ) such that $G \cap G^c = \tilde{\emptyset}$ then G is neutrosophic compact.

Proof: Immediate from 3.18.

3.20 Proposition:

Let (X, τ) be an NTS. An NS $A = \{\langle x, \mathcal{T}_A(x), \mathcal{I}_A(x), \mathcal{F}_A(x)\rangle : x \in X\}$ in X is neutrosophic pre-compact iff for every collection $C = \{G_\lambda : \lambda \in\triangle\}$ of NPO sets of X satisfying $\mathcal{T}_A(x) \leq \bigvee_{\lambda\in\triangle} \mathcal{T}_{G_\lambda}(x)$, $1 - \mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1 - \mathcal{I}_{G_\lambda}(x))$ and $1 - \mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1 - \mathcal{F}_{G_\lambda}(x))$, there exists a finite subcollection $\{G_{\lambda_k} : k = 1, 2, 3, ..., n\}$ such that $\mathcal{T}_A(x) \leq \bigvee_{k=1}^{n} \mathcal{T}_{G_{\lambda_k}}(x)$, $1 - \mathcal{I}_A(x) \leq \bigvee_{k=1}^{n}(1 - \mathcal{I}_{G_{\lambda_k}}(x))$ and $1 - \mathcal{F}_A(x) \leq \bigvee_{k=1}^{n}(1 - \mathcal{F}_{G_{\lambda_k}}(x))$.

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Proof: Necessary Part : Let $C = \{G_\lambda : \lambda \in\triangle\}$ be any collection of NPO sets of X satisfying $\mathcal{T}_A(x) \leq \bigvee_{\lambda\in\triangle} \mathcal{T}_{G_\lambda}(x), 1-\mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x))$ and $1-\mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x))$. Now $1-\mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x)) \Rightarrow 1-\mathcal{I}_A(x) \leq 1-\mathcal{I}_{G_\beta}(x)$ for some $\beta \in\triangle \Rightarrow \mathcal{I}_A(x) \geq \mathcal{I}_{G_\beta}(x) \Rightarrow \mathcal{I}_A(x) \geq \bigwedge_{\lambda\in\triangle} \mathcal{I}_{G_\lambda}(x)$. Similarly $1-\mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x)) \Rightarrow \mathcal{F}_A(x) \geq \bigwedge_{\lambda\in\triangle} \mathcal{F}_{G_\lambda}(x)$. Therefore $A \subseteq \cup_{\lambda\in\triangle}G_\lambda$, i.e., C is an NPOC of A. Since A is neutrosophic pre-compact, so C has a finite NPOSC $\{G_{\lambda_k} : k = 1, 2, 3, \cdots, n\}$, say. Therefore $A \subseteq \cup_{k=1}^n G_{\lambda_k}$. Then $\mathcal{T}_A(x) \leq \bigvee_{k=1}^n \mathcal{T}_{G_{\lambda_k}}(x), \mathcal{I}_A(x) \geq \bigwedge_{k=1}^n \mathcal{I}_{G_{\lambda_k}}(x)$ and $\mathcal{F}_A(x) \geq \bigwedge_{k=1}^n \mathcal{F}_{G_{\lambda_k}}(x)$. Now $\mathcal{I}_A(x) \geq \bigwedge_{k=1}^n \mathcal{I}_{G_{\lambda_k}}(x) \Rightarrow \mathcal{I}_A(x) \geq \mathcal{I}_{G_{\lambda_m}}(x)$ for some $m, 1 \leq m \leq n$ $\Rightarrow 1-\mathcal{I}_A(x) \leq 1-\mathcal{I}_{G_{\lambda_m}}(x)$ for some $m, 1 \leq m \leq n \Rightarrow 1-\mathcal{I}_A(x) \leq \bigvee_{k=1}^n(1-\mathcal{I}_{G_{\lambda_k}}(x))$. Similarly $\mathcal{F}_A(x) \geq \bigwedge_{k=1}^n \mathcal{F}_{G_{\lambda_k}}(x) \Rightarrow 1-\mathcal{F}_A(x) \leq \bigvee_{k=1}^n(1-\mathcal{F}_{G_{\lambda_k}}(x))$. Thus $\mathcal{T}_A(x) \leq \bigvee_{k=1}^n \mathcal{T}_{G_{\lambda_k}}(x), 1-\mathcal{I}_A(x) \leq \bigvee_{k=1}^n(1-\mathcal{I}_{G_{\lambda_k}}(x))$ and $1-\mathcal{F}_A(x) \leq \bigvee_{k=1}^n(1-\mathcal{F}_{G_{\lambda_k}}(x))$.

Sufficient Part : Let $C = \{G_\lambda : \lambda \in\triangle\}$ be an NPOC of A. Then $A \subseteq \cup_{\lambda\in\triangle}G_\lambda$, i.e., $\mathcal{T}_A(x) \leq \bigvee_{\lambda\in\triangle} \mathcal{T}_{G_\lambda}(x)$, $\mathcal{I}_A(x) \geq \bigwedge_{\lambda\in\triangle} \mathcal{I}_{G_\lambda}(x)$ and $\mathcal{F}_A(x) \geq \bigwedge_{\lambda\in\triangle} \mathcal{F}_{G_\lambda}(x)$. Now $\mathcal{I}_A(x) \geq \bigwedge_{\lambda\in\triangle} \mathcal{I}_{G_\lambda}(x) \Rightarrow \mathcal{I}_A(x) \geq \mathcal{I}_{G_\alpha}(x)$ for some $\alpha \in\triangle \Rightarrow 1-\mathcal{I}_A(x) \leq 1-\mathcal{I}_{G_\alpha}(x) \Rightarrow 1-\mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x))$. Similarly $\mathcal{F}_A(x) \geq \bigwedge_{\lambda\in\triangle} \mathcal{F}_{G_\lambda}(x) \Rightarrow 1-\mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x))$. Thus the collection C satisfies the condition $\mathcal{T}_A(x) \leq \bigvee_{\lambda\in\triangle} \mathcal{T}_{G_\lambda}(x), 1-\mathcal{I}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x))$ and $1-\mathcal{F}_A(x) \leq \bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x))$. By the hypothesis, there exists a finite subcollection $\{G_{\lambda_k} : k = 1, 2, 3, ..., n\}$ such that $\mathcal{T}_A(x) \leq \bigvee_{k=1}^n \mathcal{T}_{G_{\lambda_k}}(x), 1-\mathcal{I}_A(x) \leq \bigvee_{k=1}^n(1-\mathcal{I}_{G_{\lambda_k}}(x))$ and $1-\mathcal{F}_A(x) \leq \bigvee_{k=1}^n(1-\mathcal{F}_{G_{\lambda_k}}(x))$. Now $1-\mathcal{I}_A(x) \leq \bigvee_{k=1}^n(1-\mathcal{I}_{G_{\lambda_k}}(x)) \Rightarrow 1-\mathcal{I}_A(x) \leq 1-\mathcal{I}_{G_{\lambda_m}}(x)$ for some $m, 1 \leq m \leq n \Rightarrow \mathcal{I}_A(x) \geq \mathcal{I}_{G_{\lambda_m}}(x) \Rightarrow \mathcal{I}_A(x) \geq \bigwedge_{k=1}^n \mathcal{I}_{G_{\lambda_k}}(x)$. Similarly we shall have $\mathcal{F}_A(x) \geq \bigwedge_{k=1}^n \mathcal{F}_{G_{\lambda_k}}(x)$. Therefore $A \subseteq \cup_{k=1}^n G_{\lambda_k}$, i.e., the NPOC C of A has a finite NPOSC $\{G_{\lambda_k} : k = 1, 2, 3, \cdots, n\}$. Therefore, A is neutrosophic pre-compact set.

Hence proved.

3.21 Proposition:

Let (X, τ) be an NTS. Then X is neutrosophic compact iff for every collection $C = \{G_\lambda : \lambda \in\triangle\}$ of NPO sets of X satisfying $\bigvee_{\lambda\in\triangle} \mathcal{T}_{G_\lambda}(x) = 1, \bigvee_{\lambda\in\triangle}(1-\mathcal{I}_{G_\lambda}(x)) = 1$ and $\bigvee_{\lambda\in\triangle}(1-\mathcal{F}_{G_\lambda}(x)) = 1$, there exists a finite subcollection $\{G_{\lambda_k} : k = 1, 2, 3, ..., n\}$ such that $\bigvee_{k=1}^n \mathcal{T}_{G_{\lambda_k}}(x) = 1, \bigvee_{k=1}^n(1-\mathcal{I}_{G_{\lambda_k}}(x)) = 1$ and $\bigvee_{k=1}^n(1-\mathcal{F}_{G_{\lambda_k}}(x)) = 1$.

Proof: Immediate from 3.20.

3.22 Definition:

Let (X, τ) be an NTS. A collection $\{G_\lambda : \lambda \in \Delta\}$ of neutrosophic sets of X is said to have the finite intersection property (FIP, in short) iff every finite subcollection $\{G_{\lambda_k} : k = 1, 2, \cdots, n\}$ of $\{G_\lambda : \lambda \in \Delta\}$ satisfies the condition $\bigcap_{k=1}^n G_{\lambda_k} \neq \tilde{\emptyset}$, where $\triangle$ is an index set.

3.23 Proposition:

An NTS (X, τ) is neutrosophic pre-compact iff every collection of NPC sets with the FIP has a non-empty intersection.

Proof: Necessary part: Let $\mathcal{A} = \{N_i : i \in\triangle\}$ be an arbitrary collection of NPC sets with the FIP. We show that $\cap_{i\in\triangle}N_i \neq \tilde{\emptyset}$. On the contrary, suppose that $\cap_{i\in\triangle}N_i = \tilde{\emptyset}$. Then $(\cap_{i\in\triangle}N_i)^c = (\tilde{\emptyset})^c \Rightarrow \cup_{i\in\triangle}N_i^c = \tilde{X}$. Therefore $\mathcal{B} = \{N_i^c : N_i \in \mathcal{A}\}$ is an NPOC of X and so, $\mathcal{B}$ has a finite NPOSC $\{N_{i_1}^c, N_{i_2}^c, ..., N_{i_k}^c\}$, say. Then $\cup_{j=1}^k N_{i_j}^c = \tilde{X} \Rightarrow \cap_{j=1}^k N_{i_j} = \tilde{\emptyset}$, which is a contradiction as $\mathcal{A}$ has FIP. Therefore $\cap_{i\in\triangle}N_i \neq \tilde{\emptyset}$.

Sufficient part: On the contrary, suppose that X is not neutrosophic pre-compact. Then there must exist an NPOC of X which will have no finite NPOSC. Let $\mathcal{C} = \{G_i : i \in\triangle\}$ be an NPOC of X which has no

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finite subcover. Then for every finite subcollection $\{G_{i_1}, G_{i_2}, ..., G_{i_k}\}$ of $\mathcal{C}$, we have $\cup_{j=1}^{k} G_{i_j} \neq \tilde{X} \Rightarrow \cap_{j=1}^{k} G_{i_j}^c \neq \tilde{\emptyset}$. Therefore $\{G_i^c : G_i \in \mathcal{C}\}$ is a collection of NPC sets having the FIP. By the assumption, $\cap_{i\in\triangle} G_i^c \neq \tilde{\emptyset} \Rightarrow \cup_{i\in\triangle} G_i \neq \tilde{X}$. This implies that $\mathcal{C}$ is not an NPOC of X, which is a contradiction. Therefore, NPOC $\mathcal{C}$ of X must have a finite NPOSC. Therefore X is neutrosophic pre-compact.

Hence proved.

3.24 Proposition:

Let f be a neutrosophic pre-continuous function from an NTS (X, τ) to the NTS (Y, σ). If A is neutrosophic pre-compact set in X then $f(A)$ is neutrosophic compact set in Y.

Proof: Let $\mathcal{B} = \{G_\lambda : \lambda \in \triangle\}$ be an NOC of $f(A)$. Then $f(A) \subseteq \cup_{\lambda\in\triangle} G_\lambda \Rightarrow f^{-1}(f(A)) \subseteq f^{-1}(\cup_{\lambda\in\triangle} G_\lambda) \Rightarrow f^{-1}(f(A)) \subseteq \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) \Rightarrow A \subseteq \cup_{\lambda\in\Delta} f^{-1}(G_\lambda)$. Since G_λ is σ-open NS in Y, so $f^{-1}(G_\lambda)$ is τ-NPO set in X as f is pre-continuous. Therefore $C = \{f^{-1}(G_\lambda) : \lambda \in \Delta\}$ is an NPOC of A. Since A is neutrosophic pre-compact, so C has a finite NPOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \cdots, f^{-1}(G_{\lambda_n})\}$, say. Therefore $A \subseteq \cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) \Rightarrow f(A) \subseteq f(\cup_{i=1}^{n} f^{-1}(G_{\lambda_i})) \Rightarrow f(A) \subseteq \cup_{i=1}^{n} f(f^{-1}(G_{\lambda_i})) \Rightarrow f(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NOC $\mathcal{B}$ of $f(A)$ has a finite NOSC. Therefore $f(A)$ is neutrosophic compact. Hence proved.

3.25 Proposition:

Let f be a neutrosophic continuous function from an NTS (X, τ) to the NTS (Y, σ). If A is neutrosophic pre-compact in X then $f(A)$ is neutrosophic compact in Y.

Proof: Obvious from 3.24.

3.26 Proposition:

Let f be a neutrosophic pre-continuous function from an NTS (X, τ) onto the NTS (Y, σ). If (X, τ) is neutrosophic pre-compact then (Y, σ) is neutrosophic compact.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $\mathcal{B} = \{G_\lambda : \lambda \in \Delta\}$ be an NOC of Y. Then $\cup_{\lambda\in\triangle} G_\lambda = \tilde{Y} \Rightarrow f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) = f^{-1}(\tilde{Y}) \Rightarrow \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) = \tilde{X}$. Since G_λ is σ-open NS in Y, so $f^{-1}(G_\lambda)$ is τ-NPO set in X as f is pre-continuous. Therefore $C = \{f^{-1}(G_\lambda) : \lambda \in \Delta\}$ is an NPOC of X. Since X is pre-compact, so C has a finite NPOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \cdots, f^{-1}(G_{\lambda_n}\})$, say. Therefore $\cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) = \tilde{X} \Rightarrow f(\cup_{i=1}^{n} f^{-1}(G_{\lambda_i})) = f(\tilde{X}) \Rightarrow \cup_{i=1}^{n} f(f^{-1}(G_{\lambda_i})) = \tilde{Y} \Rightarrow \cup_{i=1}^{n} G_{\lambda_i} = \tilde{Y}$. Thus the NOC $\mathcal{B}$ of Y has a finite NOSC. Therefore Y is neutrosophic compact. Hence proved.

3.27 Proposition:

Let f be a neutrosophic continuous function from an NTS (X, τ) onto the NTS (Y, σ). If (X, τ) is neutrosophic pre-compact then (Y, σ) is neutrosophic compact.

Proof: Obvious from 3.26.

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3.28 Proposition:

Let f be a neutrosophic pre-continuous function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic countably pre-compact then Y is neutrosophic countably compact.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $\mathcal{A} = \{G_\lambda : \lambda \in \Delta\}$ be a countable NOC of Y. Then $\cup_{\lambda\in\Delta} G_\lambda = \tilde{Y} \Rightarrow f^{-1}(\cup_{\lambda\in\Delta} G_\lambda) = f^{-1}(\tilde{Y}) \Rightarrow \cup_{\lambda\in\Delta} f^{-1}(G_\lambda) = \tilde{X}$. Since G_λ is σ-open NS in Y, so $f^{-1}(G_\lambda)$ is τ-NPO set in X as f is pre-continuous. Therefore $C = \{f^{-1}(G_\lambda) : \lambda \in \Delta\}$ is an NPOC of X. Obviously C is countable as A is countable. Again since X is neutrosophic countably pre-compact, so C has a finite NPOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \cdots, f^{-1}(G_{\lambda_n})\}$, say. Therefore $\cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) = \tilde{X} \Rightarrow f(\cup_{i=1}^{n} f^{-1}(G_{\lambda_i})) = f(\tilde{X}) \Rightarrow \cup_{i=1}^{n} f(f^{-1}(G_{\lambda_i})) = \tilde{Y} \Rightarrow \cup_{i=1}^{n} G_{\lambda_i} = \tilde{Y}$. Thus the countable NOC $\mathcal{A}$ of Y has a finite NPOSC. Hence Y is neutrosophic countably compact.

3.29 Proposition:

Let f be a neutrosophic continuous function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic countably pre-compact then Y is neutrosophic countably compact.

Proof: Immediate from 3.28.

3.30 Proposition:

Let f be a neutrosophic pre-continuous function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic pre-Lindelöf then Y is neutrosophic Lindelöf.

Proof: Since f is onto, so $f(\tilde{X}) = \tilde{Y}$. Let $\mathcal{A} = \{A_i : i \in\Delta\}$ be an NOC of Y. Then $\tilde{Y} = \cup_{i\in\Delta} A_i \Rightarrow f^{-1}(\tilde{Y}) = f^{-1}(\cup_{i\in\Delta} A_i) \Rightarrow \tilde{X} = \cup_{i\in\Delta} f^{-1}(A_i) \Rightarrow \{f^{-1}(A_i) : i \in\Delta\}$ is an NPOC of X. Since X is neutrosophic pre-Lindelöf, so $\{f^{-1}(A_i) : i \in\Delta\}$ has a countable NPOSC $\mathcal{B} = \{f^{-1}(A_{i_k}) : k = 1, 2, 3, \ldots\}$, say. Therefore $\tilde{X} = f^{-1}(A_{i_1}) \cup f^{-1}(A_{i_2}) \cup f^{-1}(A_{i_3}) \cup \ldots$. This gives $f(\tilde{X}) = f[f^{-1}(A_{i_1}) \cup f^{-1}(A_{i_2}) \cup f^{-1}(A_{i_3}) \cup \ldots] \Rightarrow \tilde{Y} = f(f^{-1}(A_{i_1})) \cup f(f^{-1}(A_{i_2})) \cup f(f^{-1}(A_{i_3})) \cup \ldots \Rightarrow \tilde{Y} = A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup \ldots \Rightarrow \{A_{i_k} : k = 1, 2, 3, \cdots\}$ an NOC of Y. Since $\mathcal{B}$ is countable, so $\{A_{i_k} : k = 1, 2, 3, \cdots\}$ is also countable. Therefore the NOC $\mathcal{A}$ of Y has a countable NOSC $\{A_{i_k} : k = 1, 2, 3, \cdots\}$ and so, Y is neutrosophic Lindelöf. Hence proved.

3.31 Proposition:

Let f be a neutrosophic continuous function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic pre-Lindelöf then Y is neutrosophic Lindelöf.

Proof: Immediate from 3.30.

3.32 Proposition:

Let f be a neutrosophic pre-open function from an NTS (X, τ) to the NTS (Y, σ). If $A \subseteq Y$ is neutrosophic pre-compact in Y then $f^{-1}(A)$ is neutrosophic compact in X.

Proof: Let $\mathcal{B} = \{G_\lambda : \lambda \in\Delta\}$ be an NOC of $f^{-1}(A)$. Then $f^{-1}(A) \subseteq \cup_{\lambda\in\Delta} G_\lambda \Rightarrow A \subseteq f(\cup_{\lambda\in\Delta} G_\lambda) \Rightarrow A \subseteq \cup_{\lambda\in\Delta} f(G_\lambda)$. Since G_λ is τ-open set, so $f(G_\lambda)$ is σ-NPO set for each $\lambda \in\Delta$ as f is a pre-open function. Therefore, $\mathcal{C} = \{f(G_\lambda) : \lambda \in\Delta\}$ is an NPOC of A. Since A is neutrosophic pre-compact, so $\mathcal{C}$ has a finite NPOSC $\{f(G_{\lambda_1}), f(G_{\lambda_2}), f(G_{\lambda_3}), ..., f(G_{\lambda_n})\}$, say. Therefore $A \subseteq \cup_{i=1}^{n} f(G_{\lambda_i}) \Rightarrow A \subseteq f(\cup_{i=1}^{n} G_{\lambda_i}) \Rightarrow f^{-1}(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NOC $\mathcal{B}$ of $f^{-1}(A)$ has a finite NOSC $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, ..., G_{\lambda_n}\}$. Therefore $f^{-1}(A)$ is neutrosophic compact in X. Hence proved.

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3.33 Proposition:

Let f be a neutrosophic pre-open function from an NTS (X, τ) onto the NTS (Y, σ). If (Y, σ) is neutrosophic pre-compact (resp. neutrosophic countably pre-compact, neutrosophic pre-Lindelöf) then (X, τ) is neutrosophic compact (resp. neutrosophic countably compact, neutrosophic Lindelöf).

Proof: Immediate from 3.32 as f is onto.

3.34 Proposition:

Let f be a neutrosophic open function from an NTS (X, τ) onto the NTS (Y, σ). If (Y, σ) is neutrosophic pre-compact (resp. neutrosophic countably pre-compact, neutrosophic pre-Lindelöf) then (X, τ) is neutrosophic compact (resp. neutrosophic countably compact, neutrosophic Lindelöf).

Proof: Obvious from 3.33.

3.35 Definition:

Let f be a function from an NTS (X, τ) to the NTS (Y, σ). Then f is called a neutrosophic pre*-open function if $f(G)$ is an NPO set in Y for every NPO set G in X.

3.36 Proposition:

Let f be a neutrosophic pre-*-open function from an NTS (X, τ) to the NTS (Y, σ) and $A \in \mathcal{N}(Y)$. If A is neutrosophic pre-compact in Y then $f^{-1}(A)$ is neutrosophic pre-compact in X.

Proof: Let $\mathcal{B}$ = $\{G_\lambda : \lambda \in \triangle\}$ be an NPOC of $f^{-1}(A)$. Then $f^{-1}(A) \subseteq \cup_{\lambda \in \triangle} G_\lambda \Rightarrow A \subseteq f(\cup_{\lambda \in \triangle} G_\lambda) \Rightarrow A \subseteq \cup_{\lambda \in \triangle} f(G_\lambda)$. Since G_λ is τ-NPO set, so $f(G_\lambda)$ is σ-NPO set for each $\lambda \in \triangle$ as f is a pre*-open function. Therefore, $\mathcal{C} = \{f(G_\lambda) : G_\lambda \in \mathcal{B}\}$ is an NPOC of A. Since A is neutrosophic pre-compact, so $\mathcal{C}$ has a finite NPOSC $\{f(G_{\lambda_1}), f(G_{\lambda_2}), f(G_{\lambda_3}), ..., f(G_{\lambda_n})\}$, say. Therefore $A \subseteq \cup_{i=1}^{n} f(G_{\lambda_i}) \Rightarrow A \subseteq f(\cup_{i=1}^{n} G_{\lambda_i}) \Rightarrow f^{-1}(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NPOC $\mathcal{B}$ of $f^{-1}(A)$ has a finite NPOSC $\{G_{\lambda_1}, G_{\lambda_2}, G_{\lambda_3}, ..., G_{\lambda_n}\}$. Therefore $f^{-1}(A)$ is neutrosophic pre-compact in X. Hence proved.

3.37 Proposition:

Let f be a neutrosophic pre-*-open function from an NTS (X, τ) onto the NTS (Y, σ). If (Y, σ) is neutrosophic pre-compact (resp. neutrosophic countably pre-compact, neutrosophic pre-Lindelöf) then (X, τ) is also neutrosophic pre-compact (resp. neutrosophic countably pre-compact, neutrosophic pre-Lindelöf).

Proof: Immediate from 3.36 as f is onto.

3.38 Definition:

Let f be a neutrosophic function from an NTS (X, τ) to the NTS (Y, σ). Then f is called a neutrosophic pre-irresolute function if $f^{-1}(G)$ is an NPO set in X for every NPO set G in Y.

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3.39 Proposition:

Let f be a neutrosophic pre-irresolute function from an NTS (X, τ) to the NTS (Y, σ). If A is neutrosophic pre-compact in X then $f(A)$ is also neutrosophic pre-compact in Y.

Proof: Let $\mathcal{B} = \{G_\lambda : \lambda \in \Delta\}$ be an NPOC of $f(A)$ in Y. Then $f(A) \subseteq \cup_{\lambda \in \Delta} G_\lambda \Rightarrow A \subseteq f^{-1}(\cup_{\lambda \in \Delta} G_\lambda) \Rightarrow A \subseteq \cup_{\lambda \in \Delta} f^{-1}(G_\lambda)$. Since G_λ is σ-NPO set in Y, so $f^{-1}(G_\lambda)$ is a τ-NPO set in X as f is a neutrosophic pre-irresolute function. Therefore $C = \{f^{-1}(G_\lambda) : \lambda \in \Delta\}$ is an NPOC of A in X. Since A is neutrosophic pre-compact in X, so C has a finite NPOSC $\{f^{-1}(G_{\lambda_1}), f^{-1}(G_{\lambda_2}), \cdots, f^{-1}(G_{\lambda_n})\}$, say. Therefore $A \subseteq \cup_{i=1}^{n} f^{-1}(G_{\lambda_i}) \Rightarrow f(A) \subseteq f(\cup_{i=1}^{n} f^{-1}(G_{\lambda_i})) \Rightarrow f(A) \subseteq \cup_{i=1}^{n} f(f^{-1}(G_{\lambda_i})) \Rightarrow f(A) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$. Thus the NPOC $\mathcal{B}$ of $f(A)$ has a finite NPOSC $\{G_{\lambda_1}, G_{\lambda_2}, \cdots, G_{\lambda_n}\}$. Therefore $f(A)$ is neutrosophic pre-compact. Hence proved.

3.40 Proposition:

Let f be a neutrosophic pre-irresolute function from an NTS (X, τ) onto the NTS (Y, σ). If (X, τ) is neutrosophic pre-compact (resp. neutrosophic countably pre-compact, neutrosophic pre-Lindelöf) then (Y, σ) is also neutrosophic pre-compact (resp. neutrosophic countably pre-compact, neutrosophic pre-Lindelöf).

Proof: Obvious from 3.39.

3.41 Definition :

Let(X, τ) be an NTS and $NPO(X)$ be the collection of all NPO sets in X. A subcollection $\mathcal{B}$ of $NPO(X)$ is called a neutrosophic pre-base (Np-base, for short) for X iff for each $A \in NPO(X)$, there exists a subcollection $\{A_i : i \in \Delta\}$ of $\mathcal{B}$ such that $A = \cup\{A_i : i \in \triangle\}$, where $\triangle$ is an index set.

A subcollection $\mathcal{B}_*$ of $NPO(X)$ is called a neutrosophic pre-subbase (Np-subbase, for short) for X iff the finite intersection of members of $\mathcal{B}_*$ forms a neutrosophic pre-base for X.

3.42 Definition:

An NTS (X, τ) is said to be neutrosophic pre-C_{II} space iff X has a countable neutrosophic pre-base, i.e., an NTS (X, τ) is said to be pre-C_{II} space iff there exists a countable subcollection $\mathcal{B}$ of $NPO(X)$ such that every member of $NPO(X)$ can be expressed as the union of some members of $\mathcal{B}$.

3.43 Proposition:

Let $\mathcal{B}$ be an Np-base for an NTS (X, τ). Then X is neutrosophic pre-compact iff every NPOC of X by the members of $\mathcal{B}$ has a finite NPOSC.

Proof: Necessary Part : Obvious.

Sufficient Part : Let $\mathcal{B} = \{B_\alpha : \alpha \in \triangle\}$ be the Np-base. Also let $\mathcal{C} = \{G_\lambda : \lambda \in \triangle\}$ be an NPOC of X. Then each member G_λ of $\mathcal{C}$ is the union of some members of $\mathcal{B}$ and the totality of such members of $\mathcal{B}$ is evidently an NPOC of X. By the hypothesis, this collection of members of $\mathcal{B}$ has a finite NPOSC $\mathcal{D} = \{B_{\alpha_j} : j = 1, 2, 3, \cdots, n\}$, say. Clearly for each B_{α_j} in $\mathcal{D}$, there is a G_{λ_j} in $\mathcal{C}$ such that $B_{\alpha_j} \subseteq G_{\lambda_j}$. Therefore the finite subcollection $\{G_{\lambda_j} : j = 1, 2, 3, \cdots, n\}$ of $\mathcal{C}$ is an NPOC of X, i.e., the NPOC $\mathcal{C}$ of X has a finite NPOSC $\{G_{\lambda_j} : j = 1, 2, 3, \cdots, n\}$. Therefore X is neutrosophic pre-compact.

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3.44 Proposition:

Let (X, τ) be a neutrosophic countably pre-compact space. If X is pre-C_{II} then X neutrosophic pre-compact.

Proof: Let $\mathcal{D} = \{A_i : i \in \triangle\}$ be any NPOC of X. Since X is pre-C_{II}, so there exists a countable Np-base $\mathcal{B} = \{B_n : n = 1, 2, 3, \cdots\}$ for X. Then each $A_i \in \mathcal{D}$ can be expressed as the union of some members of $\mathcal{B}$. Let $A_i = \bigcup_{k=1}^{i_0} B_{n_k}$, where $B_{n_k} \in \mathcal{B}$ and i_0 may be infinity. Clearly $\mathcal{B}_0 = \{B_{n_k}\}$ is an NPOC of X. Also $\mathcal{B}_0$ is countable as $\mathcal{B}_0 \subseteq \mathcal{B}$. Therefore, $\mathcal{B}_0$ is a countable NPOC of X. Since X is countably pre-compact, so $\mathcal{B}_0$ has a finite NPOSC $\mathcal{B}'$, say. Since by construction, each member of $\mathcal{B}'$ is contained in one member A_i of $\mathcal{D}$, so these A_i's form a finite NPOC of X. Thus the NPOC $\mathcal{D}$ of X has a finite NPOSC. Therefore X is neutrosophic pre-compact. Hence Proved.

3.45 Remark:

In view of 3.3 and 3.44, it is evident that if an NTS (X, τ) is pre-C_{II} then neutrosophic pre-compactness and neutrosophic countably pre-compactness are equivalent.

3.46 Proposition:

If an NTS (X, τ) is pre-C_{II} then it is neutrosophic pre-Lindel$\ddot{o}$f.

Proof: Let $\mathcal{A} = \{A_i : i \in \triangle\}$ be an NPOC of X. Since X is pre-C_{II}, so there exists a countable Np-base $\mathcal{B} = \{B_n : n = 1, 2, 3, \cdots\}$ for X. Then each $A_i \in \mathcal{A}$ can be expressed as the union of some members of $\mathcal{B}$. Let $A_i = \bigcup_{k=1}^{i_0} B_{n_k}$, where $B_{n_k} \in \mathcal{B}$ and i_0 may be infinity. Let $\mathcal{B}_0 = \{B_{n_k}\}$. Then $\mathcal{B}_0$ is an NPOC of X. Also $\mathcal{B}_0$ is countable as $\mathcal{B}_0 \subseteq \mathcal{B}$. Therefore, $\mathcal{B}_0$ is a countable NPOC of X. By construction, each member of $\mathcal{B}_0$ is contained in one A_i of $\mathcal{A}$. So, these A_i's form a countable NPOC of X. Thus the NPOC $\mathcal{A}$ of X has a countable NPOSC. Therefore X is neutrosophic pre-Lindel$\ddot{o}$f.

3.47 Proposition:

Let β be an Np-subbase of an NTS (X, τ). Then X is neutrosophic pre-compact iff for every collection of NPC sets chosen from β^c having the FIP, there is a non-empty intersection.

Proof: Necessary part : Immediate from 3.23.

Sufficient Part : On the contrary, let us suppose that X is not pre-compact. Then by 3.23, there exists a collection $\mathcal{C} = \{G_i : i \in I\}$ of NPC of X having the FIP such that $\cap_{i\in\triangle} G_i = \tilde{\emptyset}$. The collection of all such collections $\mathcal{C}$ can be arranged in an order by using the classical inclusion($\subseteq$) and the collection will certainly have an upper bound. Therefore by Zorn's lemma, there will be a maximal collection of all the collections $\mathcal{C}$. Let $\mathcal{P} = \{P_j : j \in J\}$ be the maximal collection. This collection $\mathcal{P}$ has the following properties :
(i) $\tilde{\emptyset} \notin \mathcal{P}$ (ii) $P \in \mathcal{P}, P \subseteq Q \Rightarrow Q \in \mathcal{P}$ (iii) $P, Q \in \mathcal{P} \Rightarrow P \cap Q \in \mathcal{P}$ (iv) $\cap(\mathcal{P} \cap \beta^c) = \tilde{\emptyset}$. Clearly the property (iv) delivers a contradiction to the hypothesis. Therefore X is pre-compact.

Hence proved.

4 Neutrosophic local pre-compactness

4.1 Definition:

An NTS (X, τ) is said to be a neutrosophic locally pre-compact space iff for every NP $x_{\alpha,\beta,\gamma}$ in X, there exists a τ-NPO set G such that $x_{\alpha,\beta,\gamma} \in G$ and G is neutrosophic pre-compact in X.

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4.2 Proposition:

Every neutrosophic pre-compact space is a neutrosophic locally pre-compact space.

Proof: Let (X, τ) be a neutrosophic pre-compact space and let $x_{\alpha,\beta,\gamma}$ be an NP in X. Since X is neutrosophic pre-compact and since $\tilde{X}$ is an NPO set such that $x_{\alpha,\beta,\gamma} \in \tilde{X}$, so, (X, τ) is a neutrosophic locally pre-compact space.

4.3 Proposition:

Let f be a neutrosophic pre-*-open and pre-irresolute function from an NTS space (X, τ) to the NTS (Y, σ). If (Y, σ) neutrosophic locally pre-compact then (X, τ) is also a neutrosophic locally pre-compact space.

Proof: Let $x_{\alpha,\beta,\gamma}$ be any NP in X. Then there exists an NP $y_{p,q,r}$ in Y such that $f(x_{\alpha,\beta,\gamma}) = y_{p,q,r}$. Since $y_{p,q,r} \in Y$ and Y neutrosophic locally pre-compact, so there exists a σ-NPO set G such that $y_{p,q,r} \in G$ and G is neutrosophic pre-compact in Y. Now $y_{p,q,r} \in G \Rightarrow f(x_{\alpha,\beta,\gamma}) \in G \Rightarrow x_{\alpha,\beta,\gamma} \in f^{-1}(G)$. Since f is neutrosophic pre-*-open and G is neutrosophic pre-compact in Y, so by 3.36, $f^{-1}(G)$ is neutrosophic pre-compact in X. Again since f is a neutrosophic pre-irresolute function, so $f^{-1}(G)$ is a τ-NPO set. Thus for any any NP $x_{\alpha,\beta,\gamma}$ in X, there exists a τ-NPO set $f^{-1}(G)$ such that $x_{\alpha,\beta,\gamma} \in f^{-1}(G)$ and $f^{-1}(G)$ is neutrosophic pre-compact in X. Therefore (X, τ) is neutrosophic locally pre-compact space.

4.4 Proposition:

Let f be a neutrosophic pre*-open and pre-irresolute function from an NTS (X, τ) onto the NTS (Y, σ). If X is neutrosophic locally pre-compact then Y is neutrosophic locally pre-compact.

Proof: Let $y_{p,q,r}$ be any NP in Y. Since f is onto, so there is an NP $x_{\alpha,\beta,\gamma}$ in X such that $f(x_{\alpha,\beta,\gamma}) = y_{p,q,r}$. Since $x_{\alpha,\beta,\gamma} \in X$ and X neutrosophic locally pre-compact, so there exists a τ-NPO set G such that $x_{\alpha,\beta,\gamma} \in G$ and G is neutrosophic pre-compact in X. Now $x_{\alpha,\beta,\gamma} \in G \Rightarrow f(x_{\alpha,\beta,\gamma}) \in f(G) \Rightarrow y_{p,q,r} \in f(G)$. Since f is neutrosophic pre-irresolute and G is neutrosophic pre-compact in X, so by 3.39, $f(G)$ is neutrosophic pre-compact in Y. Again since f is a neutrosophic pre*-open function, so $f(G)$ is a σ-NPO set. Thus for any any NP $y_{p,q,r}$ in Y, there exists a σ-NPO set $f(G)$ such that $y_{p,q,r} \in f(G)$ and $f(G)$ is neutrosophic pre-compact in Y. Therefore (Y, σ) is neutrosophic locally pre-compact space.

5 Conclusions

In this article, we have defined neutrosophic pre-open cover with the help of neutrosophic pre-open sets and then we have defined neutrosophic pre-compact space, neutrosophic countably pre-compact space, neutrosophic pre-Lindelöf space and investigated various covering properties. We have proved that every neutrosophic pre-compact space is a neutrosophic compact space but the converse is not true. We have shown that if a neutrosophic topological space is neutrosophic pre-C_{II} then neutrosophic pre-compactness and neutrosophic countably pre-compactness are equivalent. In 3.40, we have established that neutrosophic pre-compactness (resp. neutrosophic countably pre-compactness, neutrosophic pre-Lindelöfness) is preserved under a neutrosophic pre-irresolute function. In 3.47, we have also stated and proved "Alexander subbase lemma" in case of a neutrosophic pre-compact space with the help of neutrosophic pre-subbase. In the long run, we have defined neutrosophic locally pre-compact space and put forward a few propositions with proofs. Hope that the findings in this article will assist the research fraternity to move forward for the development of different aspects of neutrosophic topology.

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References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, vol. 20, pp. 87–96, 1986.

[2] M. Arar, About Neutrosophic Countably Compactness, Neutrosophic Sets and Systems, vol. 36(1), pp. 246–255, 2020.

[3] I. Arokiarani, R. Dhavaseelan, S. Jafari, M. Parimala, On Some New Notions and Functions in Neutrosophic Topological Space, Neutrosophic Sets and Systems, vol. 16, pp. 16–19, 2017.

[4] K. Bageerathi, P. Puvaneswary, Neutrosophic Feebly Connectedness and Compactness, IOSR Journal of Polymer and Textile Engineering, vol. 6(3), pp. 7–13, 2019.

[5] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems, vol. 88, pp. 81–89, 1997.

[6] I. Deli, S. Broumi, Neutrosophic soft relations and some properties, Ann. Fuzzy Math. Inform., vol. 9, pp. 169–182, 2015.

[7] S. Dey, G. C. Ray, Pre-separation axioms in Neutrosophic Topological Spaces, Neutrosophic Sets and Systems (Accepted).

[8] S. M. Jaber, Fuzzy Precompact Space, Journal of Physics: Conference Series 1591 012073, FISCAS 2020, Iraq, 26–27 June 2020, IOP Publishing. doi:10.1088/1742-6596/1591/1/012073.

[9] S. Karatas, C. Kuru, Neutrosophic Topology, Neutrosophic Sets and Systems, vol. 13(1), pp. 90–95, 2016.

[10] T. Y. Ozturk, A. Benek, A. Ozkan, Neutrosophic soft compact spaces, Afrika Matematika, vol. 32, pp. 301–316, 2021.

[11] G. C. Ray, S. Dey, Neutrosophic point and its neighbourhood structure, Neutrosophic Sets and Systems, vol. 43, pp. 156–168, 2021.

[12] V. V. Rao, Y. S. Rao, Neutrosophic Pre-open Sets and Pre-closed Sets in Neutrosophic Topology, International Journal of ChemTech Research, vol. 10(10), pp. 449–458, 2017.

[13] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM, 1999.

[14] F. Smarandache, Neutrosophy and neutrosophic logic, First international conference on neutrosophy, neutrosophic logic, set, probability, and statistics, University of New Mexico, Gallup, NM 87301, USA, 2002.

[15] F. Smarandache, Neutrosophic set - a generalization of the intuitionistic fuzzy set, International Journal of Pure and Applied Mathematics, vol. 24(3), pp. 287–297, 2005.

[16] A. A. Salama, S. Alblowi, Neutrosophic set and Neutrosophic Topological Spaces, IOSR Journal of Mathematics, vol. 3(4), pp. 31–35, 2012.

[17] A. A. Salama, F. Smarandache, V. Kroumov, Closed sets and Neutrosophic Continuous Functions, Neutrosophic Sets and Systems, vol. 4, pp. 4–8, 2014.

[18] A. A. Salama, F. Smarandache, Neutrosophic Set Theory, The Educational Publisher 415 Columbus, Ohio, 2015.

[19] S. Şenyurt, G. Kaya, On Neutrosophic Continuity, Ordu University Journal of Science and Technology, vol. 7(2), pp. 330–339, 2017.

[20] H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct, vol. 4, pp. 410–413, 2010.

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