A Study on Compact Operators in Locally K -Convex Spaces

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Abstract

In this paper we give an equivalent definition of continuous and compact linear operators by using orthogonal bases in non-archimedean locally K - convex spaces. We also show that if E is a d_1 space and F is a semi-Montel d_2 space, then every continuous linear operator $T{:}E{\to}F$ is compact.

Keywords: Operator; Convex space; Compact set

1. Introduction

In archimedean functional analysis V. P. Zahariuta showed in [7] that if E, F are locally convex space with absolute bases, E is d_1 space and F is a Montel d_2 space, then every continuous linear operator $T{:}E{\to}F$ is compact.

In this paper we prove this result in non-archimedean functional analysis with E, F are locally K - convex spaces with orthogonal basis.
Throughout K is a non-archimedean valued field that is complete under the metric induced by non-trivial valuation $|.|: K \to [0, \infty)$.
Let E be a K - vector space. A non-archimedean seminorm on E is a map $\|.\|: E \to R$ satisfying

(i) $\|x\| \in \overline{|K|}$ Where $\overline{|K|}$ is the closure of $\{|\lambda|: \lambda \in K\}$ in R).
(ii) $\|\lambda x\| = |\lambda|\|x\|$
(iii) $\|x + y\| \leq \max(\|x\|, \|y\|)$
For all $x, y \in E, \lambda \in K$.

In the sequel E, F, ... are locally K - convex Hausdroff spaces over K, the topology of which is determined by a family $(\|.\|)_k$ of non-archimedean seminorms. For the elementary notion and notations concerning locally K - vector spaces we refer to [2], [5] and [6]. By $L(E,F)$ will denote a K - vector spaces consisting of all continuous linear maps.

1.1 Definition Let E be a locally K - convex space. A sequence (x_n) in E is called a (topological) base of E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \lambda_n x_n$ with $\lambda_n \in K$. If the coefficient functionals $f_n: x \in E \to \lambda_n \in N, (n \in N)$ are continuous, then (x_n) is called a Schauder basis (see[4]).

1.2 Definitions (see [1])

A basis (x_i) of a locally K - convex space E is said to be an orthogonal basis if the topology of E can be determined by a sequences $(|.|_p)$ of nonarchimedean seminorms satisfying the conditions:
If $x \in E$, $x = \sum_{i=1}^{\infty} \lambda_i x_i$, then $|x|_p = \max_i |\lambda_i x_i|_p \ \forall p \in N$.

Note that every orthogonal basis is a Schauder basis.

2. Compact operators:

A compactoid in a locally K - convex space E is a subset B of E such that for every zero-neighborhood U in E, there exists a finite set $A \subset E$ such that $B \subset U + co(A)$, where $co(A)$ is the absolutely convex hull of A. E is called semi-Montel if every bounded subset of E is compctoid. An operator $T \in L(E, F)$ is called compact if there exists a zero- neighborhood U in E such that $T(U)$ is compactoid in F (see[3]).

2.1 Theorem: Let E , F be two locally K - convex space and let$(|.|_k)$, $(\|.\|_k)$, be two sequences of non-archimedean seminorms, defining the topologies on E and F respectively. Then
a) $T \in L(E, F)$ if and only if $\forall m \exists k = k(m)$ and $C(m) \in K$ with $|C(m)| > 0$ such that $\forall x \in E$
$\|Tx\|_m \leq |C(m)|\, |x|_k$
b) If (e_n) is any orthogonal basis in E, then $T \in L(E, F)$ if and only if $\forall m \exists k = k(m)$ such that

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$$\sup_n \frac{\|Te_n\|_m}{|e_n|_k} < \infty$$

c) If F is a semi-Montel space, then $T: E \to F$ is a compact operator if and only if $\exists k\ \forall m\ \exists D(m) \in K$ with $|D(m)| > 0$ such that $\forall x \in E$
$\|Tx\|_m \leq |D(m)||x|_k$
d) If e_n is any orthogonal basis in E and F is a semi-Montel space, then $T: E \to F$ is a compact operator if and only if $\exists k\ \forall m$

$$\sup_n \frac{\|Te_n\|_m}{|e_n|_k} < \infty$$

Proof : a)
$T \in L(E, F)$ if and only if for the a zero-neighborhood $B_{\|.\|_m}(0,1)$ in F, there exist zero-neighborhood $B_{|.|_k}(0,1)$ in E and $C'(m) \in K$ with such $|C'(m)| > 0$ that $T(C'(m).B_{|.|_k}(0,1)) \subseteq B_{\|.\|_m}(0,1)$. Now if $x \in E$, then $\left|\frac{C'(m)}{2}\frac{x}{|x|_k}\right|_k = \left|\frac{C'(m)}{2}\right|\frac{|x|_k}{|x|_k} = \left|\frac{C'(m)}{2}\right| < |C'(m)|$, and so $\frac{C'(m)}{2}\frac{x}{|x|_k} \in C'(m).B_{|.|_k}(0,1)$ and this implies that
$T(\frac{C'(m)}{2}\frac{x}{|x|_k}) \in T(C'(m).B_{|.|_k}(0,1)) \subseteq B_{\|.\|_m}(0,1)$.
Thus, $\left\|\frac{C'(m)}{2}\frac{x}{|x|_k}\right\|_m \leq 1$, and so $\|Tx\|_m \leq |C(m)||x|_k$ where $|C(m)| = \left|\frac{2}{C'(m)}\right|$.
b) ($\Rightarrow$) Let $T \in L(E, F)$. Then by part (a) $\forall m\ \exists k = k(m)$ and $C(m) \in K$ with $|C(m)| > 0$ such that
$\|Tx\|_m \leq |C(m)||x|_k \quad \forall x \in E$
Since $e_n \in E$, then $\forall m\ \exists k = k(m)$ and $C(m) \in K$ with $|C(m)| > 0$ such that
$\|Te_n\|_m \leq |C(m)||e_n|_k \quad \forall n \in N$
Hence $\forall m\ \exists k = k(m)$ and $C(m) \in K$ with $|C(m)| > 0$ such that
$\frac{\|Te_n\|_m}{|e_n|_k} \leq |C(m)| \quad \forall n \in N$
It follows that, $\forall m\ \exists k = k(m)$ such that
$\sup_n \frac{\|Te_n\|_m}{|e_n|_k} < \infty$
($\Leftarrow$) Let $T: E \to F$ be any linear operator and (e_n) be any orthogonal basis in E. Suppose that $\forall m\ \exists k = k(m)$ such that
$\sup_n \frac{\|Te_n\|_m}{|e_n|_k} < \infty$
Then $\forall m\ \exists k = k(m)$ and $C(m) \in K$ with $|C(m)| > 0$ such that
$\|Te_n\|_m \leq |C(m)||e_n|_k \quad \forall n \in N$
Now if $x \in E$. Then x has a unique representation $x = \sum_{n=1}^{\infty} \lambda_n e_n$ .Since
$Tx = \sum_{n=1}^{\infty} \lambda_n Te_n$, then
$\|Tx\|_m = \|\sum_{n=1}^{\infty} \lambda_n Te_n\|_m \leq \max_n|\lambda_n|\ \|Te_n\|_m \leq \max_n|C(m)|\ |\lambda_n||e_n|_k = |C(m)||x|_k$.
and hence $T \in L(E, F)$
c) $T: E \to F$ is compact if and only if there exists a zero-neighborhood $B_{|.|_k}(0,1)$ in E, such that $T(B_{|.|_k}(0,1))$ is compactiod in F. Since F is semi-Montel space, then $T(B_{|.|_k}(0,1))$ is bounded in F. So there exist a zero-neighborhood $B_{\|.\|_m}(0,1)$ in F and $D'(m) \in K$ with $|D'(m)| > 0$ such that $T\left(B_{|.|_k}(0,1)\right) \subseteq D'(m).B_{\|.\|_m}(0,1)$. Now let $x \in E$, then $\left|\frac{1}{2}\frac{x}{|x|_k}\right|_k = \frac{1}{2} < 1$, and so $\frac{1}{2}\frac{x}{|x|_k} \in B_{|.|_k}(0,1)$ and this implies that $T(\frac{1}{2}\frac{x}{|x|_k}) \in T(B_{|.|_k}(0,1)) \subseteq D'(m).B_{\|.\|_m}(0,1)$. Thus $\left\|\frac{1}{2}\frac{Tx}{|x|_k}\right\|_m \leq D'(m)$, and so $\|Tx\|_m \leq |D(m)||x|_k$ where $|D(m)| = |2D'(m)|$.
d) ($\Rightarrow$)let $T: E \to F$ be any compact operator. Then by part (c) $\exists k\ \forall m\ \exists D(m) \in K$ with $|D(m)| > 0$ such that $\forall x \in E$
$\|Tx\|_m \leq |D(m)||x|_k$
Since $e_n \in E$, then $\exists k\ \forall m\ \exists D(m) \in K$ with $|D(m)| > 0$ such that
$\|Te_n\|_m \leq |D(m)||e_n|_k \quad \forall n \in N$.
Hence $\exists k\ \forall m\ \exists D(m) \in K$ with $|D(m)| > 0$ such that
$\frac{\|Te_n\|_m}{|e_n|_k} \leq |D(m)| \quad \forall n \in N$
It follows that, $\exists k\ \forall m$ such that
$\sup_n \frac{\|Te_n\|_m}{|e_n|_k} < \infty$
($\Leftarrow$) Let $T: E \to F$ be any linear operator and (e_n) be any orthogonal basis in E. Suppose that $\exists k\ \forall m$ such that
$\sup_n \frac{\|Te_n\|_m}{|e_n|_k} < \infty$

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then $\exists k\ \forall m\ \exists D(m) \in K$ with $|D(m)| > 0$ such that
$\|Te_n\|_m \leq |D(m)||e_n|_k \qquad \forall n \in N.$
Now if $x \in E$, then x has a unique representation $x = \sum_{n=1}^{\infty} \lambda_n e_n$. Since
$Tx = \sum_{n=1}^{\infty} \lambda_n e_n$,

$$\|Tx\|_m = \left\|\sum_{n=1}^{\infty} \lambda_n Te_n\right\|_m \leq \max_n |\lambda_n|\,\|Te_n\|_m \leq \max_n |D(m)|\,|\lambda_n||e_n|_k = |D(m)||x|_k$$

And so $T: E \to F$ is a compact operator.

2.2 Definition

We shall say that a locally K - convex space $E \in d_i, i = 1,2$ if there exists in E an orthogonal basisx_n such that
$\exists p\ \forall q\ \exists r$ such that $|x_k|_q^2 \leq |x_k|_p |x_k|_r\ \forall k \in N$ for i=1, and
$\forall q\ \exists p\ \forall r$ such that $|x_k|_q^2 \geq |x_k|_p |x_k|_r\ \forall k \in N$ for i=2.

2.3 Theorem

Let $E \in d_2$, $F \in d_1$ and F be a semi-Montel space. Then every continuous linear operator $T: E \to F$ is compact.

Proof: Let $T: E \to F$ be an arbitrary continuous linear operator. Since $E \in d_2, F \in d_1$, then there exist two orthogonal bases $(x_n), (y_n)$ in E, F respectively and sequences of non-archimedean seminorms $(|.|_p), (\|.\|_q)$ defining the topology on E , F respectively, satisfying the conditions:
if $x \in E$, $x = \sum_{i=1}^{\infty} \lambda_i x_i$, then $|x|_p = \max_i |\lambda_i x_i|_p \quad \forall p \in N$, and
if $x \in F$, $x = \sum_{i=1}^{\infty} \lambda_i x_i$, then $\|x\|_p = \max_i \|\lambda_i x_i\|_p \quad \forall p \in N$.
Now let $Tx_k = \sum_{i=1}^{\infty} t_{ik} y_i$. Since $x = \sum_{k=1}^{\infty} \lambda_k x_k$ and T is continuous, then
$Tx = \sum_{i=1}^{\infty} \eta_i y_i$, where $\eta_i = \sum_{k=1}^{\infty} t_{ik} \lambda_k$, that is T can be represented by the
Matrix (t_{ik}) Now Since T is continuous, by proposition (2.1(a)) $\forall p\ \exists q = q(p)$ and $C(p) \in K$ with $|C(p)| > 0$ such that $\forall x \in E$
$\|Tx\|_p \leq |C(p)||x|_q$.
It follows that

$$\left\|\sum_{i=1}^{\infty} \eta_i y_i\right\|_p = \left\|\sum_{i=1}^{\infty}\left(\sum_{k=1}^{\infty} t_{ik}\lambda_k\right) y_i\right\|_p \leq |C(p)| \left|\sum_{i=1}^{\infty} \lambda_i x_i\right|_q$$

And so,

$$\max_i \left\|\left(\sum_{k=1}^{\infty} t_{ik}\lambda_k\right) y_i\right\|_p \leq |C(p)| \max_i |\lambda_i x_i|_q$$

Hence

$$\frac{\max_i \|(\sum_{k=1}^{\infty} t_{ik}\lambda_k) y_i\|_p}{\max_i |\lambda_i x_i|_q} \leq |C(p)|$$

Now if $x = x_k$, then

$$\frac{\max_i |t_{ik}|\|y_i\|_p}{|x_k|_q} \leq |C(p)|$$

Taking the supremum over k, we have

$$\sup_k \frac{\max_i |t_{ik}|\|y_i\|_p}{|x_k|_q} \leq |C(p)| \qquad (1)$$

Now since F is a semi-Montel space, to prove that T is compact, it sufficient to show that for some neighborhood $U_{q_0} = \{x \in X: |x|_{q_0} \leq 1\}$ in E, $T(U_{q_0})$ is bounded in F .i.e. for all $V_p = \{y \in F: |y|_p \leq 1\}, p \in N$
There exists $M(p) \in K$ with $|M(p)| > 0$ such that $T(U_{q_0}) = \{Tx \in F: |x|_{q_0} \leq 1\} \subseteq M(p)V_p$, that is if $x \in U_{q_0}$then for all $p \in N$,

$$\|Tx\|_p = \max_i \left\|\left(\sum_{k=1}^{\infty} t_{ik}\lambda_k\right) y_i\right\|_p \leq |M(p)|$$

Now let $x \in E$, then $\left|\frac{x}{2|x|_{q_0}}\right|_{q_0} = \frac{1}{2} < 1$. Hence,
$T(\frac{x}{2|x|_{q_0}}) \in T(U_{q_0}) \subseteq M(p)V_p$. So $\frac{\|Tx\|_p}{|x|_{q_0}} \leq 2M(p)$, and hence
$\frac{\max_i \|(\sum_{k=1}^{\infty} t_{ik}\lambda_k) y_i\|_p}{\max_i |\lambda_i x_i|_{q_0}} \leq 2|M(p)|$. Now if we take $x = x_k$, then

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$$\frac{\max_i |t_{ik}| \|y_i\|_p}{|x_k|_{q_0}} \leq 2|M(p)|$$

Taking the supremum over k we have

$$\sup_k \frac{\max_i |t_{ik}| \|y_i\|_p}{|x_k|_q} \leq 2|M(p)| \qquad (2)$$

To prove $T(U_{q_0})$ is bounded in F we want to prove that (2) holds.

Now, since $F \in d_1$, then

$\exists p_1 \,\forall p \,\exists p_2 = p_2(p)$ such that $\|y_i\|_p^2 \leq \|y_i\|_{p_1} \|y_i\|_{p_2} \quad \forall i \in N, \quad (3)$

And since $E \in d_2$, then

$\forall q_1 = q(p_1) \,\exists q = q_0 \,\forall q_2 = q(p_2)$ such that $|x_k|_{q_0}^2 \geq |x_k|_{q_1} |x_k|_{q_2} \quad \forall k \in N, \quad (4)$

Now by (1), (3), (4), we have

$$\max_i \frac{|t_{ik}| \|y_i\|_p}{|x_k|_{q_0}} \leq (\max_i \frac{|t_{ik}|^2 (\|y_i\|_p)^2}{(|x_k|_{q_0})^2})^{\frac{1}{2}} \leq (\max_i \frac{|t_{ik}|^2 \|y_i\|_{p_1} \|y_i\|_{p_2}}{|x_k|_{q_1} |x_k|_{q_2}})^{\frac{1}{2}}$$

$$\leq (\max_i \frac{|t_{ik}| \|y_i\|_{p_1}}{|x_k|_{q_1}})^{\frac{1}{2}} (\max_i \frac{|t_{ik}| \|y_i\|_{p_2}}{|x_k|_{q_2}})^{\frac{1}{2}} \leq \left(|C(p_1)|^{\frac{1}{2}}\right)\left(|C(p_2)|^{\frac{1}{2}}\right) < \infty \;\forall p \in N$$

If we take the supremum over k , then the inequality (2) holds, and hence T is a compact operator.

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