Continuity and Compactness via Hypersoft Open Sets

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Abstract

Hypersoft topology (HST) is the study of a structure based on all hypersoft (HS) sets on a given set of alternatives. In continuation of this concern, in this article, we introduce new maps namely HS continuous, HS open, HS closed, and HS homomorphism. We examine the main characteristics of each of these maps. Furthermore, we study HS compact space and discuss some of its properties. We point out that HS compactness preserved under HS continuous map.

Keywords: HS continuous map; HS open (closed) map; HS homeomorphism map; HS compact space

1 Introduction

Data with ambiguity or vagueness is a common component of difficulties in many fields, including engineering, artificial intelligence, economics, environmental science, and social science. Since they were predicated on an exact situation, conventional approaches may not be practical for modeling or solving them.

New ideas that aid in overcoming these kinds of instabilities become necessary as a result. With time, engineers and mathematicians developed other methods to address ambiguous or vague problems, including probability theory, fuzzy sets, intuitionistic fuzzy sets, and rough sets.

However, in order to use any of these tools, you must first pre-specify a few parameters. Soft sets are a novel strategy that Molodtsov[1] presented. Comparing the soft set theory to the probability theory and fuzzy set theory, he showed how effective soft sets are in handling complex issues. Since its establishment in 1999, soft set theory has been a growing subject of research and interactions with other fields. Despite the fact that many studies followed its direction, the expansion of the literature has been inconsistent, and major papers and outcomes are uncommon (,[2],[3],[4],[56]).

In a variety of real-world situations, unique attributes must be categorized into parametric non-overlapping sets; unfortunately, the soft set is inappropriate to manage such sets. Due to HS set theory,[7] it is anticipated that the soft set will be compatible with such settings. The HS set is a soft set expansion that converts an approximate function into a multi-argument function. Researchers have set up the foundations of HS sets (e.g.,,[8],[9],[10],[1112]). Recently, Musa and Asaad[13] introduced the notion of bipolar HS set and study some of its topological concepts,[14],[15],[16],[17],[18].[19]

Musa and Asaad,[20] came up with the concept of hypersoft topological space (HSTS). They defined some topological structures in the frame of HS sets such as connectedness[21] and separation axioms.[22] In keeping with this concept, in this article, we present various maps as well as the concept of compact spaces.

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The rest of the paper is structured as follows. Section 2 briefly reviews some fundamental concepts in HS sets and HSTs. Section 3 presents new maps namely HS continuous map, HS open map, HS closed map, and HS homeomorphism. Some of the properties of proposed maps are discussed. Section 4 gives the concept of HS compact spaces. Lastly, in section 5, we conclude the study and suggest for future works.

2 Preliminary Concepts

Throughout this paper, $\Re$ denotes the set of alternatives, $2^{\Re}$ denotes the power set of $\Re$, and Σ denotes the universal attribute set where $\Sigma = \sigma_1 \times \sigma_2 \times \ldots \times \sigma_n$ with $\sigma_i \cap \sigma_j = \varphi$ and $i \neq j$. Also, Λ and Δ are non-empty subsets of Σ.

Definition 2.1. [7] A pair (g, Λ) is called an HS set over $\Re$, where g is a mapping given by $g : \Lambda \to 2^{\Re}$. We write $(g, \Lambda) = \{(\ell, g(\ell)) : \ell \in \Lambda\}$.

Henceforth, $\mho_{(\Re,\Sigma)}$ represents the set of all HS sets on $\Re$ with the set of attributes Σ.

Definition 2.2. [8,10] Let (g_1, Λ), $(g_2, \Delta) \in \mho_{(\Re,\Sigma)}$. Then

i. (g_1, Λ) is an HS subset of (g_2, Δ), denoted by $(g_1, \Lambda) \,\tilde{\sqsubseteq}\, (g_2, \Delta)$, if $\Lambda \subseteq \Delta$ and $g_1(\ell) \subseteq g_2(\ell)$ for all $\ell \in \Lambda$.

ii. (g_1, Λ) and (g_2, Δ) are HS equal, if $(g_1, \Lambda) \,\tilde{\sqsubseteq}\, (g_2, \Delta)$ and $(g_2, \Delta) \,\tilde{\sqsubseteq}\, (g_1, \Lambda)$.

iii. If $g_1(\ell) = \varphi$ for all $\ell \in \Lambda$, then (g_1, Λ) is called a relative null HS set and denoted by (Φ, Λ).

iv. If $g_1(\ell) = \Re$ for all $\ell \in \Lambda$, then (g_1, Λ) is called a relative whole HS set and denoted by $(\widetilde{\Re}, \Lambda)$.

v. The complement of (g_1, Λ) is an HS set $(g_1, \Lambda)^c = (g_1^c, \Lambda)$ where $g_1^c(\ell) = \Re \setminus g_1(\ell)$ for all $\ell \in \Lambda$.

vi. The intersection of (g_1, Λ) and (g_2, Δ), denoted by $(g_1, \Lambda) \,\tilde{\sqcap}\, (g_2, \Delta)$, is an HS set (g, C), where $C = \Lambda \cap \Delta$ and for all $\ell \in C$, $g(\ell) = g_1(\ell) \cap g_2(\ell)$.

vii. The union of (g_1, Λ) and (g_2, Δ), denoted by $(g_1, \Lambda) \,\tilde{\sqcup}\, (g_2, \Delta)$, is an HS set (g, C), where $C = \Lambda \cap \Delta$ and for all $\ell \in C$, $g(\ell) = g_1(\ell) \cup g_2(\ell)$.

Definition 2.3. [20] Let (g, Σ) be an HS set over $\Re$ and Υ be a non-empty subset of $\Re$. Then the sub HS set of (g, Σ) over Υ denoted by (g_Υ, Σ), is defined as $g_\Upsilon(\ell) = \Upsilon \cap g(\ell)$ for all $\ell \in \Sigma$.

Definition 2.4. [11] Let $\gamma : \Re \to \aleph$ and $\delta : \Sigma \to \acute{\Sigma}$ be two maps. Let $\Psi_{\gamma\delta} : \mho_{(\Re,\Sigma)} \to \mho_{(\aleph,\acute{\Sigma})}$ be an HS map. Then:

1. The HS image of (g, Λ), $\Psi_{\gamma\delta}((g, \Lambda)) = (\Psi_{\gamma\delta}(g), \acute{\Sigma})$ is an HS set in $\mho_{(\aleph,\acute{\Sigma})}$ given as, for all $\acute{\ell} \in \acute{\Sigma}$

$$\Psi_{\gamma\delta}(g)(\acute{\ell}) = \begin{cases} \gamma\left(\bigcup_{\ell\in\delta^{-1}(\acute{\ell})\cap\Lambda} g(\ell)\right), & \text{if } \delta^{-1}(\acute{\ell}) \cap \Lambda \neq \varphi \\ \varphi, & \text{otherwise} \end{cases}$$

2. The HS inverse image of $(f, \acute{\Lambda})$, $\Psi^{-1}_{\gamma\delta\acute{\Lambda}}((f, \acute{\Lambda})) = (\Psi^{-1}_{\gamma\delta}(f), \Sigma)$ is an HS set in $\mho_{(\Re,\Sigma)}$ given as, for all $\ell \in \Sigma$

$$\Psi^{-1}_{\gamma\delta}(f)(\ell) = \begin{cases} \gamma^{-1}\left(f(\delta(\ell))\right), & \text{if } \delta(\ell) \in \acute{\Lambda} \\ \varphi, & \text{if } \delta(\ell) \notin \acute{\Lambda} \end{cases}$$

Definition 2.5. [11] We call an HS map $\Psi_{\gamma\delta}$ HS surjective (resp., HS injective, HS bijective) if the maps γ and δ are surjective (resp., injective, bijective).

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Definition 2.6. [20] Let $\mathcal{T}_{\Re}$ be the collection of HS sets over $\Re$, then $\mathcal{T}_{\Re}$ is said to be an HST on $\Re$ if

1. (Φ, Σ), $(\widetilde{\Re}, \Sigma)$ belong to $\mathcal{T}_{\Re}$;
2. the intersection of any two HS sets in $\mathcal{T}_{\Re}$ belongs to $\mathcal{T}_{\Re}$;
3. the union of any number of HS sets in $\mathcal{T}_{\Re}$ belongs to $\mathcal{T}_{\Re}$.

We call $(\Re, \mathcal{T}_{\Re}, \Sigma)$ an HSTS over $\Re$.

Definition 2.7. [20] Let $(\Re, \mathcal{T}_{\Re}, \Sigma)$ be an HSTS over $\Re$ and Υ be a non-empty subset of $\Re$. Then

$$\mathcal{T}_{\Re_\Upsilon} = \{(g_\Upsilon, \Sigma) \mid (g, \Sigma) \widetilde{\in} \mathcal{T}_{\Re}\}$$

is said to be the relative HST on $\Re$ and $(\Upsilon, \mathcal{T}_{\Re_\Upsilon}, \Sigma)$ is called an HS subspace of $(\Re, \mathcal{T}_{\Re}, \Sigma)$.

Definition 2.8. [20] Let $(\Re, \mathcal{T}_{\Re}, \Sigma)$ be an HSTS and (g, Σ) be an HS set over $\Re$. Then:

1. the intersection of all HS closed supersets of (g, Σ) is called the HS closure of (g, Σ) and is denoted by $\overline{(g, \Sigma)}$.
2. the union of all HS open subsets of (g, Σ) is called the HS interior of (g, Σ) and is denoted by $(g, \Sigma)^o$.

Proposition 2.9. *If $\Psi_{\gamma\delta} : \mho_{(\Re,\Sigma)} \to \mho_{(\aleph,\acute{\Sigma})}$ is an HS bijective map, then $\Psi^{-1}_{\gamma\delta} : \mho_{(\aleph,\acute{\Sigma})} \to \mho_{(\Re,\Sigma)}$ is also an HS bijective map.*

Proof. Let $(f_1, \acute{\Sigma}) \neq (f_2, \acute{\Sigma}) \in \mho_{(\aleph,\acute{\Sigma})}$. Since $\Psi_{\gamma\delta}$ is an HS bijective map, then there exist $(g_1, \Sigma) \neq (g_2, \Sigma) \in \mho_{(\Re,\Sigma)}$ such that $\Psi_{\gamma\delta}(g_1, \Sigma) = (f_1, \acute{\Sigma})$ and $\Psi_{\gamma\delta}(g_2, \Sigma) = (f_2, \acute{\Sigma})$. Also, we have $(g_1, \Sigma) = \Psi^{-1}_{\gamma\delta}(f_1, \acute{\Sigma})$ and $(g_2, \Sigma) = \Psi^{-1}_{\gamma\delta}(f_2, \acute{\Sigma})$. Hence, $\Psi^{-1}_{\gamma\delta}(f_1, \acute{\Sigma}) \neq \Psi^{-1}_{\gamma\delta}(f_2, \acute{\Sigma})$ and $\Psi^{-1}_{\gamma\delta}$ is HS injective map. Now, let $(g, \Sigma) \in \mho_{(\Re,\Sigma)}$. Then, there exists $(f, \acute{\Sigma}) \in \mho_{(\aleph,\acute{\Sigma})}$ such that $\Psi_{\gamma\delta}(g, \Sigma) = (f, \acute{\Sigma})$. Since $\Psi_{\gamma\delta}$ is HS injective map, then $(g, \Sigma) = \Psi^{-1}_{\gamma\delta}(f, \acute{\Sigma})$. Hence, $\Psi^{-1}_{\gamma\delta}$ is HS surjective map. Consequently, $\Psi^{-1}_{\gamma\delta}$ is HS bijective map. □

Definition 2.10. Let $\Psi_{\gamma\delta} : \mho_{(\Re,\Sigma)} \to \mho_{(\aleph,\acute{\Sigma})}$ and $\Theta_{\gamma\delta} : \mho_{(\aleph,\acute{\Sigma})} \to \mho_{(\mathrm{Im},\acute{\acute{\Sigma}})}$ be two HS maps. Then the HS composite map $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta} : \mho_{(\Re,\Sigma)} \to \mho_{(\mathrm{Im},\acute{\acute{\Sigma}})}$ is defined by $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})((g, \Sigma)) = \Theta_{\gamma\delta}(\Psi_{\gamma\delta}((g, \Sigma)))$ for $(g, \Sigma) \in \mho_{(\Re,\Sigma)}$.

Proposition 2.11. *Let $\Psi_{\gamma\delta} : \mho_{(\Re,\Sigma)} \to \mho_{(\aleph,\acute{\Sigma})}$ and $\Theta_{\gamma\delta} : \mho_{(\aleph,\acute{\Sigma})} \to \mho_{(\mathrm{Im},\acute{\acute{\Sigma}})}$ be HS bijective maps. Then $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta} : \mho_{(\Re,\Sigma)} \to \mho_{(\mathrm{Im},\acute{\acute{\Sigma}})}$ is also an HS bijective map and $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})^{-1} = \Psi^{-1}_{\gamma\delta} \circ \Theta^{-1}_{\gamma\delta}$.*

Proof. Let $(g_1, \Sigma) \neq (g_2, \Sigma) \in \mho_{(\Re,\Sigma)}$. Since $\Psi_{\gamma\delta}$ is HS injective map, then $\Psi_{\gamma\delta}((g_1, \Sigma)) \neq \Psi_{\gamma\delta}((g_2, \Sigma))$. Again, Since $\Theta_{\gamma\delta}$ is HS injective map, then $\Theta_{\gamma\delta}(\Psi_{\gamma\delta}((g_1, \Sigma))) \neq \Theta_{\gamma\delta}(\Psi_{\gamma\delta}((g_2, \Sigma)))$. Then, $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})((g_1, \Sigma)) \neq (\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})((g_2, \Sigma))$ and hence $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS injective map. Now, let $(h, \acute{\acute{\Sigma}}) \in \mho_{(\mathrm{Im},\acute{\acute{\Sigma}})}$, then there exists $(f, \acute{\Sigma}) \in \mho_{(\aleph,\acute{\Sigma})}$ such that $\Theta_{\gamma\delta}(f, \acute{\Sigma}) = (h, \acute{\acute{\Sigma}})$ as $\Theta_{\gamma\delta}$ is HS surjective map. Again, since $\Psi_{\gamma\delta}$ is HS surjective map, then there exists $(g, \Sigma) \in \mho_{(\Re,\Sigma)}$ such that $\Psi_{\gamma\delta}(g, \Sigma) = (f, \acute{\Sigma})$. Then, $\Theta_{\gamma\delta}(\Psi_{\gamma\delta}(g, \Sigma) = (h, \acute{\acute{\Sigma}})$ and hence $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS surjective map. Therefore, $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is HS bijective map. Next, let $(g, \Sigma) \in \mho_{(\Re,\Sigma)}$, $(f, \acute{\Sigma}) \in \mho_{(\aleph,\acute{\Sigma})}$, and $(h, \acute{\acute{\Sigma}}) \in \mho_{(\mathrm{Im},\acute{\acute{\Sigma}})}$ such that $\Psi_{\gamma\delta}(g, \Sigma) = (f, \acute{\Sigma})$ and $\Theta_{\gamma\delta}(f, \acute{\Sigma}) = (h, \acute{\acute{\Sigma}})$. Since $\Psi_{\gamma\delta}$ and $\Theta_{\gamma\delta}$ are HS injective maps, then $(g, \Sigma) = \Psi^{-1}_{\gamma\delta}(f, \acute{\Sigma})$ and $(f, \acute{\Sigma}) = \Theta^{-1}_{\gamma\delta}(h, \acute{\acute{\Sigma}})$. Now, $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})((g, \Sigma)) = \Theta_{\gamma\delta}(\Psi_{\gamma\delta}((g, \Sigma))) = \Theta_{\gamma\delta}(f, \acute{\Sigma}) = (h, \acute{\acute{\Sigma}})$. Since, $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is HS injective map, then $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})^{-1}((h, \acute{\acute{\Sigma}})) = (g, \Sigma)$. Also, $(\Psi^{-1}_{\gamma\delta} \circ \Theta^{-1}_{\gamma\delta})((h, \acute{\acute{\Sigma}})) = \Psi^{-1}_{\gamma\delta}(\Theta^{-1}_{\gamma\delta}((h, \acute{\acute{\Sigma}}))) = \Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})) = (g, \Sigma)$. Hence, $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})^{-1} = \Psi^{-1}_{\gamma\delta} \circ \Theta^{-1}_{\gamma\delta}$. □

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3 Hypersoft Homeomorphism Maps

This section focuses on HS continuous, HS open, HS closed, and HS homeomorphism maps, which are new forms of HS maps. We investigate their characterizations and identify key characteristics.

Definition 3.1. An HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is said to be HS continuous if $\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})) \in \mathcal{T}_{\Re}$ for every $(f, \acute{\Sigma}) \in \mathcal{T}_{\aleph}$.

Example 3.2. Let $\Re = \{r_1, r_2, r_3\}$ and $\aleph = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ be two sets, $\sigma_1 = \{\ell_1, \ell_2, \ell_3, \ell_4\}, \sigma_2 = \{\ell_5\}, \sigma_3 = \{\ell_6\}$, and $\acute{\sigma_1} = \{\acute{\ell_1}, \acute{\ell_2}, \acute{\ell_3}, \acute{\ell_4}\}, \acute{\sigma_2} = \{\acute{\ell_5}\}, \acute{\sigma_3} = \{\acute{\ell_6}\}$ be sets of parameters, $\gamma : \Re \to \aleph$ be a mapping defined as $\gamma(r_i) = \eta_i$ for $i = 1, 2, 3$, the mapping $\delta : \Sigma \to \acute{\Sigma}$ be defined as $\delta((\ell_1, \ell_5, \ell_6)) = \delta((\ell_2, \ell_5, \ell_6)) = (\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6})$, $\delta((\ell_3, \ell_5, \ell_6)) = (\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6})$, $\delta((\ell_4, \ell_5, \ell_6)) = (\acute{\ell_4}, \acute{\ell_5}, \acute{\ell_6})$. Let $\mathcal{T}_{\Re} = \{(\Phi, \Sigma), (\widetilde{\Re}, \Sigma), (g, \Sigma)\}$ and $\mathcal{T}_{\aleph} = \{(\Phi, \acute{\Sigma}), (\widetilde{\aleph}, \acute{\Sigma}), (f, \acute{\Sigma})\}$ be two HSTs defined respectively on $\Re$ and $\aleph$ where (g, Σ) and $(f, \acute{\Sigma})$ are HS sets defined as follows

$(g, \Sigma) = \{((\ell_1, \ell_5, \ell_6), \varphi), ((\ell_2, \ell_5, \ell_6), \varphi), ((\ell_3, \ell_5, \ell_6), \{r_1\}), ((\ell_4, \ell_5, \ell_6), \varphi)\}$, and

$(f, \acute{\Sigma}) = \{((\acute{\ell_1}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_4\}), ((\acute{\ell_2}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_1\}), ((\acute{\ell_3}, \acute{\ell_5}, \acute{\ell_6}), \{\eta_1\}), ((\acute{\ell_4}, \acute{\ell_5}, \acute{\ell_6}), \varphi)\}$.

Then, it is easy to see that $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is HS continuous map.

Proposition 3.3. *Let $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ be an HS map, then the following statements are equivalent:*

i. $\Psi_{\gamma\delta}$ *is an HS continuous map.*

ii. The HS inverse image of each HS closed set is HS closed set.

iii. $\overline{\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma}))} \tilde{\sqsubseteq} \Psi_{\gamma\delta}^{-1}(\overline{(f, \acute{\Sigma})})$ *for all* $(f, \acute{\Sigma}) \tilde{\sqsubseteq} (\widetilde{\aleph}, \acute{\Sigma})$.

iv. $\Psi_{\gamma\delta}(\overline{(g, \Sigma)}) \tilde{\sqsubseteq} \overline{\Psi_{\gamma\delta}((g, \Sigma))}$ *for all* $(g, \Sigma) \tilde{\sqsubseteq} (\widetilde{\Re}, \Sigma)$.

v. $\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})^o) \tilde{\sqsubseteq} (\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})))^o$ *for all* $(f, \acute{\Sigma}) \tilde{\sqsubseteq} (\widetilde{\aleph}, \acute{\Sigma})$.

Proof. (i.) $\Rightarrow$ (ii.): Let $(f, \acute{\Sigma})$ be an HS closed set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$, then $\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma}))^c$ is an HS open subset of $(\widetilde{\Re}, \Sigma)$. But, $\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma}))^c = (\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})))^c$, then $\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma}))$ is an HS closed subset of $(\widetilde{\Re}, \Sigma)$.
(ii.) $\Rightarrow$ (iii.): Obviously, $\overline{(f, \acute{\Sigma})}$ is an HS closed subset of $(\widetilde{\aleph}, \Phi, \acute{\Sigma})$. From (ii.), $\Psi_{\gamma\delta}^{-1}(\overline{(f, \acute{\Sigma})})$ is an HS closed subset of $(\widetilde{\Re}, \Sigma)$ and hence $\overline{\Psi_{\gamma\delta}^{-1}(\overline{(f, \acute{\Sigma})})} = \Psi_{\gamma\delta}^{-1}(\overline{(f, \acute{\Sigma})})$. Obviously, $(f, \acute{\Sigma}) \tilde{\sqsubseteq} \overline{(f, \acute{\Sigma})}$ and $\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})) \tilde{\sqsubseteq} \Psi_{\gamma\delta}^{-1}(\overline{(f, \acute{\Sigma})})$. Then, $\overline{\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma}))} \tilde{\sqsubseteq} \overline{\Psi_{\gamma\delta}^{-1}(\overline{(f, \acute{\Sigma})})} = \Psi_{\gamma\delta}^{-1}(\overline{(f, \acute{\Sigma})})$.
(iii.) $\Rightarrow$ (iv.): Let $(g, \Sigma) \tilde{\sqsubseteq} (\widetilde{\Re}, \Sigma)$, then $\Psi_{\gamma\delta}((g, \Sigma)) \tilde{\sqsubseteq} (\widetilde{\aleph}, \acute{\Sigma})$. From (iii.), we have $\overline{\Psi_{\gamma\delta}^{-1}(\Psi_{\gamma\delta}((g, \Sigma)))} \tilde{\sqsubseteq} \Psi_{\gamma\delta}^{-1}(\overline{\Psi_{\gamma\delta}((g, \Sigma))})$. Then, $\overline{(g, \Sigma)} \tilde{\sqsubseteq} \overline{\Psi_{\gamma\delta}^{-1}(\Psi_{\gamma\delta}((g, \Sigma)))} \tilde{\sqsubseteq} \Psi_{\gamma\delta}^{-1}(\overline{\Psi_{\gamma\delta}((g, \Sigma))})$. It follows that, $\Psi_{\gamma\delta}(\overline{(g, \Sigma)}) \tilde{\sqsubseteq} \Psi_{\gamma\delta}(\Psi_{\gamma\delta}^{-1}(\overline{\Psi_{\gamma\delta}((g, \Sigma))})) \tilde{\sqsubseteq} \overline{\Psi_{\gamma\delta}((g, \Sigma))}$.
(iv.) $\Rightarrow$ (v.): Let $(f, \acute{\Sigma}) \tilde{\sqsubseteq} (\widetilde{\aleph}, \Phi, \acute{\Sigma})$, then apply (iv.) to $(f, \acute{\Sigma})^c$ yields $\Psi_{\gamma\delta}(\overline{\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})^c)}) \tilde{\sqsubseteq} \overline{\Psi_{\gamma\delta}(\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})^c))}$. Hence, $\Psi_{\gamma\delta}(((\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})))^o)^c) \tilde{\sqsubseteq} \overline{(f, \acute{\Sigma})^c} = ((f, \acute{\Sigma})^o)^c$. Thus, $((\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})))^o)^c \tilde{\sqsubseteq} \Psi_{\gamma\delta}^{-1}(((f, \acute{\Sigma})^o)^c)$. Therefore, $\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})^o) \tilde{\sqsubseteq} (\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})))^o$.
(v.) $\Rightarrow$ (i.): Let $(f, \acute{\Sigma})$ be an HS open subset of $(\widetilde{\aleph}, \Phi, \acute{\Sigma})$. From (v.), $\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})^o) = \Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})) \tilde{\sqsubseteq} (\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})))^o$. But, $(\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})))^o \tilde{\sqsubseteq} \Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma}))$. Hence, $(\Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma})))^o = \Psi_{\gamma\delta}^{-1}((f, \acute{\Sigma}))$ and $\Psi_{\gamma\delta}^{-1}(f, \acute{\Sigma})$ is an HS open subset of $(\widetilde{\Re}, \Sigma)$. Therefore, $\Psi_{\gamma\delta}$ is an HS continuous map. □

Proposition 3.4. *An HS bijective map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS continuous if and only if $(\Psi_{\gamma\delta}((g, \Sigma)))^o \tilde{\sqsubseteq} \Psi_{\gamma\delta}((g, \Sigma)^o)$ for all $(g, \Sigma) \tilde{\sqsubseteq} (\widetilde{\Re}, \Sigma)$.*

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Proof. Let $\Psi_{\gamma\delta}$ be an HS continuous map. Let $(g, \Sigma) \tilde{\sqsubseteq} (\widetilde{\Re}, \Sigma)$, then $\Psi_{\gamma\delta}((g, \Sigma)) \tilde{\sqsubseteq} (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. Obviously, $(\Psi_{\gamma\delta}((g, \Sigma)))^o$ is an HS open in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. As, $\Psi_{\gamma\delta}$ is an HS continuous map, then $\Psi^{-1}_{\gamma\delta}((\Psi_{\gamma\delta}((g, \Sigma)))^o)$ is an HS open in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ and $(\Psi^{-1}_{\gamma\delta}((\Psi_{\gamma\delta}((g, \Sigma)))^o))^o = \Psi^{-1}_{\gamma\delta}((\Psi_{\gamma\delta}((g, \Sigma)))^o)$. Obviously, $(\Psi_{\gamma\delta}((g, \Sigma)))^o \tilde{\sqsubseteq} \Psi_{\gamma\delta}((g, \Sigma))$. As, $\Psi_{\gamma\delta}$ is an HS injective map, then $\Psi^{-1}_{\gamma\delta}((\Psi_{\gamma\delta}((g, \Sigma)))^o) \tilde{\sqsubseteq} (g, \Sigma)$ and $(\Psi^{-1}_{\gamma\delta}((\Psi_{\gamma\delta}((g, \Sigma)))^o))^o = \Psi^{-1}_{\gamma\delta}((\Psi_{\gamma\delta}((g, \Sigma)))^o) \tilde{\sqsubseteq} (g, \Sigma)^o$. Again, as $\Psi_{\gamma\delta}$ is an HS surjective map, then $(\Psi_{\gamma\delta}(g, \Sigma))^o \tilde{\sqsubseteq} \Psi_{\gamma\delta}((g, \Sigma)^o)$. Conversely, let $(f, \acute{\Sigma})$ be any HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$, then $\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})) \tilde{\sqsubseteq} (\widetilde{\Re}, \Sigma)$. By hypothesis, $(\Psi_{\gamma\delta}(\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma}))))^o \tilde{\sqsubseteq} \Psi_{\gamma\delta}((\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})))^o)$. Since, $\Psi_{\gamma\delta}$ is an HS surjective map, then $(f, \acute{\Sigma})^o = (f, \acute{\Sigma}) \tilde{\sqsubseteq} \Psi_{\gamma\delta}((\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})))^o)$. Again, since $\Psi_{\gamma\delta}$ is an HS injective map, then $\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})) \tilde{\sqsubseteq} (\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})))^o$. But, $(\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})))^o \tilde{\sqsubseteq} \Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma}))$ and hence $(\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})))^o = \Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma}))$. Therefore, $\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma}))$ is an HS open in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ and consequently $\Psi_{\gamma\delta}$ is an HS continuous map. □

The next two results are straightforward, so we cancel their proofs.

Proposition 3.5. *An HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS continuous map if:*

i. $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ *is an HS indiscerte space.*

ii. $(\Re, \mathcal{T}_{\Re}, \Sigma)$ *is an HS discerte space.*

Proposition 3.6. *Let $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ be an HS continuous map:*

i. If $\mathcal{T}^*_{\Re}$ is a finer than $\mathcal{T}_{\Re}$, then $\Psi_{\gamma\delta} : (\Re, \mathcal{T}^*_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS continuous map.

ii. If $\mathcal{T}^*_{\aleph}$ is a coarser than $\mathcal{T}_{\aleph}$, then $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}^*_{\aleph}, \acute{\Sigma})$ is an HS continuous map.

Definition 3.7. Let $(\Re, \mathcal{T}_{\Re}, \Sigma)$ be an HSTS. A subcollection β of $\mathcal{T}_{\Re}$ is called an HS base for $\mathcal{T}_{\Re}$ if every element of $\mathcal{T}_{\Re}$ can be expressed as the union of members of β. Each element of β is called HS basis element.

Proposition 3.8. *An HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS continuous if and only if the HS inverse image of every member of an HS base β for $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS open in $(\Re, \mathcal{T}_{\Re}, \Sigma)$.*

Proof. Let $\Psi_{\gamma\delta}$ be an HS continuous and $(b, \acute{\Sigma})$ be any HS basis element for $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. Since $(b, \acute{\Sigma})$ is an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ and $\Psi_{\gamma\delta}$ is an HS continuous map, then $\Psi^{-1}_{\gamma\delta}((b, \acute{\Sigma}))$ is an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Conversely, let $\Psi^{-1}_{\gamma\delta}((b, \acute{\Sigma}))$ be an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ for every $(b, \acute{\Sigma}) \in \beta$, and let $(f, \acute{\Sigma})$ be any HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. Then

$$(f, \acute{\Sigma}) = \tilde{\sqcup}\{(b, \acute{\Sigma}) : (b, \acute{\Sigma}) \in \beta\}$$
$$\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma})) = \Psi^{-1}_{\gamma\delta}(\tilde{\sqcup}\{(b, \acute{\Sigma}) : (b, \acute{\Sigma}) \in \beta\}) = \tilde{\sqcup}\{\Psi^{-1}_{\gamma\delta}(\{(b, \acute{\Sigma}) : (b, \acute{\Sigma}) \in \beta\})\}$$

Hence $\Psi^{-1}_{\gamma\delta}((f, \acute{\Sigma}))$ is an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ since each $\Psi^{-1}_{\gamma\delta}((b, \acute{\Sigma}))$ is an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ by hypothesis. Therefore, $\Psi_{\gamma\delta}$ is an HS continuous map. □

Proposition 3.9. *Let $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ and $\Theta_{\gamma\delta} : (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}) \to (\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$ be HS continuous maps, then $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$ is also an HS continuous map.*

Proof. Let $(h, \acute{\acute{\Sigma}})$ be any HS open set in $(\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$. Since $\Theta_{\gamma\delta}$ is HS continuous map, then $\Theta^{-1}_{\gamma\delta}((h, \acute{\acute{\Sigma}}))$ is HS open in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. Again, since $\Psi_{\gamma\delta}$ is HS continuous, then $\Psi^{-1}_{\gamma\delta}(\Theta^{-1}_{\gamma\delta}(h, \acute{\acute{\Sigma}}))$ is HS open in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. But

$$\Psi^{-1}_{\gamma\delta}(\Theta^{-1}_{\gamma\delta}(h, \acute{\acute{\Sigma}})) = (\Psi^{-1}_{\gamma\delta} \circ \Theta^{-1}_{\gamma\delta})(h, \acute{\acute{\Sigma}}) = (\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})^{-1}(h, \acute{\acute{\Sigma}})$$

Thus, the HS inverse image under $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ of every HS open set in $(\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$ is HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ and therefore $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS continuous map. □

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Definition 3.10. An HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is said to be:

1. HS open if the HS image of every HS open set is HS open.
2. HS closed if the HS image of every HS closed set is HS closed.

Proposition 3.11. *An HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS open if and only if $\Psi_{\gamma\delta}((g,\Sigma)^o) \,\widetilde{\sqsubseteq}\, (\Psi_{\gamma\delta}(g,\Sigma))^o$ for every $(g,\Sigma) \,\widetilde{\sqsubseteq}\, (\widetilde{\Re}, \Sigma)$.*

Proof. Let $\Psi_{\gamma\delta}$ be an HS open map and let $(g,\Sigma) \,\widetilde{\sqsubseteq}\, (\widetilde{\Re}, \Sigma)$. We know that, $(g,\Sigma)^o$ is an HS open set. Since $\Psi_{\gamma\delta}$ is an HS open map, then $\Psi_{\gamma\delta}((g,\Sigma)^o)$ is an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ and $(\Psi_{\gamma\delta}((g,\Sigma)^o))^o = \Psi_{\gamma\delta}((g,\Sigma)^o)$. Obviously, $(g,\Sigma)^o \,\widetilde{\sqsubseteq}\, (g,\Sigma)$ and $\Psi_{\gamma\delta}((g,\Sigma)^o) \,\widetilde{\sqsubseteq}\, \Psi_{\gamma\delta}(g,\Sigma)$. Therefore, $(\Psi_{\gamma\delta}((g,\Sigma)^o))^o = \Psi_{\gamma\delta}((g,\Sigma)^o) \,\widetilde{\sqsubseteq}\, (\Psi_{\gamma\delta}(g,\Sigma))^o$. Conversely, let $\Psi_{\gamma\delta}((g,\Sigma)^o) \,\widetilde{\sqsubseteq}\, (\Psi_{\gamma\delta}((g,\Sigma)))^o$ for every $(g,\Sigma) \,\widetilde{\sqsubseteq}\, (\widetilde{\Re}, \Sigma)$ and let (g,Σ) be any HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ so that $(g,\Sigma)^o = (g,\Sigma)$. This implies $\Psi_{\gamma\delta}((g,\Sigma)^o) = \Psi_{\gamma\delta}((g,\Sigma))$ $\widetilde{\sqsubseteq}\, (\Psi_{\gamma\delta}((g,\Sigma)))^o$. But $(\Psi_{\gamma\delta}((g,\Sigma)))^o \,\widetilde{\sqsubseteq}\, \Psi_{\gamma\delta}((g,\Sigma))$ and hence $\Psi_{\gamma\delta}((g,\Sigma)) = (\Psi_{\gamma\delta}((g,\Sigma)))^o$. Therefore, $\Psi_{\gamma\delta}((g,\Sigma))$ is HS open set and hence $\Psi_{\gamma\delta}$ is HS open map. □

Proposition 3.12. *An HS map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS closed if and only if $\overline{\Psi_{\gamma\delta}((g,\Sigma))} \,\widetilde{\sqsubseteq}\, \Psi_{\gamma\delta}(\overline{(g,\Sigma)})$ for every $(g,\Sigma) \,\widetilde{\sqsubseteq}\, (\widetilde{\Re}, \Sigma)$.*

Proof. Let $\Psi_{\gamma\delta}$ be an HS closed map and let $(g,\Sigma) \,\widetilde{\sqsubseteq}\, (\widetilde{\Re}, \Sigma)$. We know that, $\overline{(g,\Sigma)}$ is an HS closed set. Since $\Psi_{\gamma\delta}$ is an HS closed map, then $\Psi_{\gamma\delta}(\overline{(g,\Sigma)})$ is an HS closed set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ and $\overline{\Psi_{\gamma\delta}(\overline{(g,\Sigma)})} = \Psi_{\gamma\delta}(\overline{(g,\Sigma)})$. Obviously, $(g,\Sigma) \,\widetilde{\sqsubseteq}\, \overline{(g,\Sigma)}$ and $\Psi_{\gamma\delta}((g,\Sigma)) \,\widetilde{\sqsubseteq}\, \Psi_{\gamma\delta}(\overline{(g,\Sigma)})$. Therefore, $\overline{\Psi_{\gamma\delta}((g,\Sigma))} \,\widetilde{\sqsubseteq}\, \overline{\Psi_{\gamma\delta}(\overline{(g,\Sigma)})} = \Psi_{\gamma\delta}(\overline{(g,\Sigma)})$. Conversely, let $\overline{\Psi_{\gamma\delta}((g,\Sigma))} \,\widetilde{\sqsubseteq}\, \Psi_{\gamma\delta}(\overline{(g,\Sigma)})$ for every $(g,\Sigma) \,\widetilde{\sqsubseteq}\, (\widetilde{\Re}, \Sigma)$ and let (g,Σ) be any HS closed set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ so that $\overline{(g,\Sigma)} = (g,\Sigma)$. This implies $\overline{\Psi_{\gamma\delta}((g,\Sigma))} \,\widetilde{\sqsubseteq}\, \Psi_{\gamma\delta}(\overline{(g,\Sigma)}) = \Psi_{\gamma\delta}((g,\Sigma))$. But $\Psi_{\gamma\delta}((g,\Sigma)) \,\widetilde{\sqsubseteq}\, \overline{\Psi_{\gamma\delta}((g,\Sigma))}$ and hence $\Psi_{\gamma\delta}((g,\Sigma)) = \overline{\Psi_{\gamma\delta}((g,\Sigma))}$. Therefore, $\Psi_{\gamma\delta}((g,\Sigma))$ is HS closed set and hence $\Psi_{\gamma\delta}$ is HS closed map. □

Proposition 3.13. *Let $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ and $\Theta_{\gamma\delta} : (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}) \to (\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$ be two HS maps, then:*

i. If $\Psi_{\gamma\delta}$ and $\Theta_{\gamma\delta}$ are HS open maps, then $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is also an HS open map.

ii. If $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS open map and $\Psi_{\gamma\delta}$ is an HS surjective HS continuous map, then $\Theta_{\gamma\delta}$ is an HS open map.

iii. If $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS open map and $\Theta_{\gamma\delta}$ is an HS injective HS continuous map, then $\Psi_{\gamma\delta}$ is an HS open map.

Proof.

i. Let (g,Σ) be an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Since $\Psi_{\gamma\delta}$ is an HS open map, then $\Psi_{\gamma\delta}((g,\Sigma))$ is an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. Again, since $\Theta_{\gamma\delta}$ is an HS open map, then $\Theta_{\gamma\delta}(\Psi_{\gamma\delta}(g,\Sigma)) = (\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})((g,\Sigma))$ is an HS open set in $(\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$. Hence, $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS open map.

ii. Let $(f, \acute{\Sigma})$ be an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. Since $\Psi_{\gamma\delta}$ is an HS continuous map, then $\Psi^{-1}_{\gamma\delta}((f,\acute{\Sigma}))$ is an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Again, since $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS open map, then $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})(\Psi^{-1}_{\gamma\delta}((f,\acute{\Sigma})))$ is an HS open set in $(\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$. As, $\Psi_{\gamma\delta}$ is an HS surjective, then $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})(\Psi^{-1}_{\gamma\delta}((f,\acute{\Sigma}))) = \Theta_{\gamma\delta}(\Psi_{\gamma\delta}(\Psi^{-1}_{\gamma\delta}(f,\acute{\Sigma}))) = \Theta_{\gamma\delta}((f,\acute{\Sigma}))$ is an HS open set in $(\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$. Hence, $\Theta_{\gamma\delta}$ is an HS open map.

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iii. Let (g, Σ) be an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Since $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS open map, then $(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})((g, \Sigma))$ is an HS open set in $(\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$. Again, since $\Theta_{\gamma\delta}$ is an HS continuous map, then $\Theta_{\gamma\delta}^{-1}((\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta})((g, \Sigma)))$ is an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. As, $\Theta_{\gamma\delta}$ is an HS injective, then $\Theta_{\gamma\delta}^{-1}(\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}((g, \Sigma)))$ $= \Theta_{\gamma\delta}^{-1}(\Theta_{\gamma\delta}(\Psi_{\gamma\delta}(g, \Sigma))) = \Psi_{\gamma\delta}((g, \Sigma))$ is an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. Hence, $\Psi_{\gamma\delta}$ is an HS open map.

□

Proposition 3.14. *Let $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ and $\Theta_{\gamma\delta} : (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}) \to (\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$ be two HS maps, then:*

i. If $\Psi_{\gamma\delta}$ and $\Theta_{\gamma\delta}$ are HS closed maps, then $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is also an HS closed map.

ii. If $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS closed map and $\Psi_{\gamma\delta}$ is an HS surjective HS continuous map, then $\Theta_{\gamma\delta}$ is an HS closed map.

iii. If $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta}$ is an HS closed map and $\Theta_{\gamma\delta}$ is an HS injective HS continuous map, then $\Psi_{\gamma\delta}$ is an HS closed map.

Proof. Similar to the proof of Proposition 3.13. □

Definition 3.15. An HS bijective map $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is said to be HS homeomorphism if $\Psi_{\gamma\delta}$ and $\Psi_{\gamma\delta}^{-1}$ are HS continuous maps.

Proposition 3.16. *If $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS bijective map, then the following statements are equivalent:*

i. $\Psi_{\gamma\delta}$ is HS homeomorphism.

ii. $\Psi_{\gamma\delta}$ is HS continuous and HS open.

iii. $\Psi_{\gamma\delta}$ is HS continuous and HS closed.

Proof. (i.) $\Rightarrow$ (ii.): Let $\Theta_{\gamma\delta}$ be the HS inverse map of $\Psi_{\gamma\delta}$ so that $\Psi_{\gamma\delta}^{-1} = \Theta_{\gamma\delta}$ and $\Theta_{\gamma\delta}^{-1} = \Psi_{\gamma\delta}$. Since $\Psi_{\gamma\delta}$ is HS bijective map, then $\Theta_{\gamma\delta}$ is also HS bijective. Let (g, Σ) be an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Since $\Theta_{\gamma\delta}$ is HS continuous, then $\Theta_{\gamma\delta}^{-1}((g, \Sigma))$ is an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. But $\Theta_{\gamma\delta}^{-1} = \Psi_{\gamma\delta}$ so that $\Theta_{\gamma\delta}^{-1}((g, \Sigma)) = \Psi_{\gamma\delta}((g, \Sigma))$ is an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. It follows that $\Psi_{\gamma\delta}$ is HS open map. Also, $\Psi_{\gamma\delta}$ is HS continuous by hypothesis.
(ii.) $\Rightarrow$ (iii.): Let (g, Σ) be an HS closed set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$, then $(g, \Sigma)^c$ is an HS open set. Since $\Psi_{\gamma\delta}$ is HS open map, then $\Psi_{\gamma\delta}((g, \Sigma)^c)$ is an HS open set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$. As $\Psi_{\gamma\delta}$ is an HS bijective map, then $\Psi_{\gamma\delta}((g, \Sigma)^c)$ $= (\Psi_{\gamma\delta}((g, \Sigma)))^c$. Hence, $\Psi_{\gamma\delta}((g, \Sigma))$ is an HS closed set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ and consequently $\Psi_{\gamma\delta}$ is HS closed map.
(iii.) $\Rightarrow$ (i.): Let (g, Σ) be an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$, then $(g, \Sigma)^c$ is an HS closed set. Since $\Psi_{\gamma\delta}$ is HS closed map, then $\Psi_{\gamma\delta}((g, \Sigma)^c) = \Theta_{\gamma\delta}^{-1}((g, \Sigma)^c) = (\Theta_{\gamma\delta}^{-1}((g, \Sigma)))^c$ is an HS closed set in $(\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$, that is, $\Theta_{\gamma\delta}^{-1}((g, \Sigma))$ is an HS open set. Hence, $\Theta_{\gamma\delta} = \Psi_{\gamma\delta}^{-1}$ is HS continuous map. □

Proposition 3.17. *Let $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ and $\Theta_{\gamma\delta} : (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma}) \to (\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$ be HS homeomorphism maps, then $\Theta_{\gamma\delta} \circ \Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\mathrm{Im}, \mathcal{T}_{\mathrm{Im}}, \acute{\acute{\Sigma}})$ is also an HS homeomorphism map.*

Proof. Follows from Proposition 3.9, Proposition 3.13 (i.), and Proposition 3.16. □

Proposition 3.18. *If $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS homeomorphism map, then the following statements hold for all $(g, \Sigma) \mathrel{\widetilde{\sqsubseteq}} (\widetilde{\Re}, \Sigma)$.*

i. $\Psi_{\gamma\delta}(\overline{(g, \Sigma)}) = \overline{\Psi_{\gamma\delta}((g, \Sigma))}$.

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ii. $\Psi_{\gamma\delta}((g,\Sigma)^o) = (\Psi_{\gamma\delta}(g,\Sigma))^o$.

Proof. i. Follows from Proposition 3.3, Proposition 3.12, and Proposition 3.16.

ii. Follows from Proposition 3.4, Proposition 3.11, and Proposition 3.16. □

4 Hypersoft Compact Spaces

We'll look at HS compactness in this section, which is another important property of HSTSs. The subject of HS compact spaces is investigated, and conclusions are derived.

Definition 4.1. A collection $\{(g_i,\Sigma) : i \in I\}$ of HS sets is called the HS cover of an HS set (g,Σ) if $(g,\Sigma) \tilde{\sqsubseteq} \tilde{\sqcup} \{(g_i,\Sigma) : i \in I\}$. If each member of $\{(g_i,\Sigma) : i \in I\}$ is HS open set, then it is called the HS open cover of (g,Σ). An HS subcover is a subcollection of $\{(g_i,\Sigma) : i \in I\}$ which is also an HS cover.

Definition 4.2. An HSTS $(\Re, \mathcal{T}_\Re, \Sigma)$ is said to be HS compact space if each HS open cover of $(\tilde{\Re}, \Sigma)$ has finite HS subcover.

Example 4.3. Let $\Re = \{r_1, r_2, r_3\}$, $\sigma_1 = \{\ell_1, \ell_2\}$, $\sigma_2 = \{\ell_3\}$, and $\sigma_3 = \{\ell_4\}$. Let $\mathcal{T}_\Re = \{(\Phi,\Sigma), (\tilde{\Re},\Sigma), (g_1,\Sigma), (g_2,\Sigma), (g_3,\Sigma), (g_4,\Sigma), (g_5,\Sigma), (g_6,\Sigma), (g_7,\Sigma)\}$ be an HST defined on $\Re$, where

$(g_1,\Sigma) = \{((\ell_1,\ell_3,\ell_4),\{r_2\},),((\ell_2,\ell_3,\ell_4),\{r_2\})\}$.

$(g_2,\Sigma) = \{((\ell_1,\ell_3,\ell_4),\{r_1\},),((\ell_2,\ell_3,\ell_4),\{r_1\})\}$.

$(g_3,\Sigma) = \{((\ell_1,\ell_3,\ell_4),\{r_1,r_2\}),((\ell_2,\ell_3,\ell_4),\{r_1,r_2\})\}$.

$(g_4,\Sigma) = \{((\ell_1,\ell_3,\ell_4),\{r_2,r_3\},\varphi),((\ell_2,\ell_3,\ell_4),\{r_2,r_3\},\varphi)\}$.

$(g_5,\Sigma) = \{((\ell_1,\ell_3,\ell_4),\{r_1,r_3\}),((\ell_2,\ell_3,\ell_4),\{r_1,r_3\})\}$.

$(g_6,\Sigma) = \{((\ell_1,\ell_3,\ell_4),\varphi),((\ell_2,\ell_3,\ell_4),\varphi)\}$.

$(g_7,\Sigma) = \{((\ell_1,\ell_3,\ell_4),\{r_3\}),((\ell_2,\ell_3,\ell_4),\{r_3\})\}$.

It is easy to see that $(\Re, \mathcal{T}_\Re, \Sigma)$ is an HS compact space.

Example 4.4. Let $\Re = \{r_1, r_2, r_3, ...\}$, $\sigma_1 = \{\ell_1, \ell_2\}$, $\sigma_2 = \{\ell_3\}$, $\sigma_3 = \{\ell_4\}$, and consider the family of HS sets $\{(g_n,\Sigma) : n = 1, 2, 3, ...\}$, where

$(g_n,\Sigma) = \{((\ell_1,\ell_3,\ell_4),\{r_1,r_2,...,r_n\}),((\ell_2,\ell_3,\ell_4),\{r_1,r_2,...,r_n\})\}$.

The family $\mathcal{T}_\Re = \{(\Phi,\Sigma), (\tilde{\Re},\Sigma), (g_n,\Sigma)\}$ is an HST on $\Re$. However, $(\Re, \mathcal{T}_\Re, \Sigma)$ is not an HS compact space.

Proposition 4.5. *Let $(\Upsilon, \mathcal{T}_{\Re_\Upsilon}, \Sigma)$ be an HS subspace of $(\Re, \mathcal{T}_\Re, \Sigma)$ and let $(g,\Sigma) \tilde{\sqsubseteq} (\tilde{\Upsilon},\Sigma)$. Then (g,Σ) is HS compact relative to $(\Re, \mathcal{T}_\Re, \Sigma)$ if and only if (g,Σ) is HS compact relative to $(\Upsilon, \mathcal{T}_\Upsilon, \Sigma)$.*

Proof. Let (g,Σ) is HS compact relative to $(\Re, \mathcal{T}_\Re, \Sigma)$ and let $\{(g_i,\Sigma) : i \in I\}$ be an HS open cover of (g,Σ) so that $(g,\Sigma) \tilde{\sqsubseteq} \tilde{\sqcup} \{(g_i,\Sigma) : i \in I\}$. Then there exists (f_i,Σ) HS open relative to $(\Re, \mathcal{T}_\Re, \Sigma)$ such that $(f_i,\Sigma) = (\tilde{\Upsilon},\Sigma) \tilde{\sqcap} (g_i,\Sigma)$ for all $i \in I$. It then follows that $(g,\Sigma) \tilde{\sqsubseteq} \tilde{\sqcup} \{(f_i,\Sigma) : i \in I\}$. Since (g,Σ) is HS compact relative to $(\Re, \mathcal{T}_\Re, \Sigma)$, there exists finitely many indices $i_1, i_2, ..., i_n$ such that $(g,\Sigma) \tilde{\sqsubseteq} \tilde{\sqcup} \{(f_i,\Sigma) : i = 1, 2, ..., n\}$. Since $(g,\Sigma) \tilde{\sqsubseteq} (\tilde{\Upsilon},\Sigma)$, we have $(g,\Sigma) \tilde{\sqsubseteq} (\tilde{\Upsilon},\Sigma) \tilde{\sqcap} [\tilde{\sqcup} \{(f_i,\Sigma) : i = 1, 2, ..., n\}] = \tilde{\sqcup} [(\tilde{\Upsilon},\Sigma) \tilde{\sqcap} \{(f_i,\Sigma) : i = 1, 2, ..., n\}]$. It follows that $(g,\Sigma) \tilde{\sqsubseteq} \tilde{\sqcup} \{(f_i,\Sigma) : i = 1, 2, ..., n\})$. This shows that (g,Σ) is HS compact relative to $(\Upsilon, \mathcal{T}_\Upsilon, \Sigma)$. Conversely, let (g,Σ) be an HS compact relative to $(\Upsilon, \mathcal{T}_{\Re_\Upsilon}, \Sigma)$ and let $\{(f_i,\Sigma) : i \in I\}$ be an HS open cover of (g,Σ) so that $(g,\Sigma) \tilde{\sqsubseteq} \tilde{\sqcup} \{(f_i,\Sigma) : i \in I\}$. Since $(g,\Sigma) \tilde{\sqsubseteq} (\tilde{\Upsilon},\Sigma)$, we have $(g,\Sigma) \tilde{\sqsubseteq} (\tilde{\Upsilon},\Sigma) \tilde{\sqcap} [\tilde{\sqcup} \{(f_i,\Sigma) : i \in I\}] = \tilde{\sqcup} [(\tilde{\Upsilon},\Sigma) \tilde{\sqcap} \{(f_i,\Sigma) : i \in I\}]$. Since $(\tilde{\Upsilon},\Sigma) \tilde{\sqcap} (f_i,\Sigma)$ is HS open relative to $(\Upsilon, \mathcal{T}_{\Re_\Upsilon}, \Sigma)$, the collection $\{(\tilde{\Upsilon},\Sigma) \tilde{\sqcap} (f_i,\Sigma)\}$ is an HS open cover of (g,Σ) relative to $(\Upsilon, \mathcal{T}_{\Re_\Upsilon}, \Sigma)$. Since (g,Σ) be an HS compact relative to $(\Upsilon, \mathcal{T}_{\Re_\Upsilon}, \Sigma)$, we must have $(g,\Sigma) \tilde{\sqsubseteq} \tilde{\sqcup} [(\tilde{\Upsilon},\Sigma) \tilde{\sqcap} \{(f_i,\Sigma) : i = 1, 2, ..., n\}]$ for some choice of finitely many indices $i_1, i_2, ..., i_n$. This implies $(g,\Sigma) \tilde{\sqsubseteq} \tilde{\sqcup} \{(f_i,\Sigma) : i = 1, 2, ..., n\}]$. It follows that (g,Σ) is HS compact relative to $(\Re, \mathcal{T}_\Re, \Sigma)$. □

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Proposition 4.6. *Every HS closed subset of an HS compact space is HS compact.*

Proof. Let (g, Σ) be an HS closed subset of an HS compact space $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Let $\{(g_i, \Sigma) : i \in I\}$ be an HS open cover of (g, Σ) so that $(g, \Sigma) \widetilde{\sqsubseteq} \widetilde{\sqcup} \{(g_i, \Sigma) : i \in I\}$. Then $(\widetilde{\Re}, \Sigma) = \widetilde{\sqcup} [\{(g_i, \Sigma) : i \in I\} \widetilde{\sqcup} (g, \Sigma)^c]$. It follows that $[\{(g_i, \Sigma) : i \in I\} \widetilde{\sqcup} (g, \Sigma)^c]$ is an HS open cover of $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Since $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is HS compact space, then $(\widetilde{\Re}, \Sigma) = [\{(g_i, \Sigma) : i = 1, 2, ..., n\} \widetilde{\sqcup} (g, \Sigma)^c]$. This implies $(g, \Sigma) \widetilde{\sqsubseteq} \{(g_i, \Sigma) : i = 1, 2, ..., n\}$. Hence (g, Σ) is HS compact. □

Proposition 4.7. *If $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is a compact space and $\mathcal{T}^*_{\Re}$ is coarser than $\mathcal{T}_{\Re}$, then $(\Re, \mathcal{T}^*_{\Re}, \Sigma)$ is HS compact.*

Proof. Let the collection $\{(g_i, \Sigma) : i \in I\}$ be an HS open cover of $(\widetilde{\Re}, \Sigma)$ in $(\Re, \mathcal{T}^*_{\Re}, \Sigma)$. Since $\mathcal{T}_{\Re}$ is finer than $\mathcal{T}^*_{\Re}$, then each member of $\{(g_i, \Sigma) : i \in I\}$ is also an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Hence, $\{(g_i, \Sigma) : i \in I\}$ is also an HS open cover of $(\widetilde{\Re}, \Sigma)$ in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Since $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is an HS compact space, then $\{(g_i, \Sigma) : i \in I\}$ has a finite HS subcover. It follows that $(\Re, \mathcal{T}^*_{\Re}, \Sigma)$ is HS compact space. □

Proposition 4.8. *Let $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma)$ be an HS subspace of $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Then $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma)$ is HS compact space if and only if every HS cover of $(\widetilde{\Upsilon}, \Sigma)$ by HS open sets in $(\Re, \mathcal{T}_{\Re}, \Sigma)$ contains a finite HS subcover.*

Proof. Let $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma)$ be an HS compact space and $\{(g_i, \Sigma) : i \in I\}$ be an HS cover of $(\widetilde{\Upsilon}, \Sigma)$ by HS open sets in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Then the collection $\{(g_{i\Upsilon}, \Sigma) : i \in I\}$ be an HS cover of $(\widetilde{\Upsilon}, \Sigma)$ by HS open sets in $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma)$. Since $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma)$ is an HS compact, then the finite subcollection of $\{(g_{i\Upsilon}, \Sigma) : i \in I\}$ is also an HS cover of $(\widetilde{\Upsilon}, \Sigma)$. Thus, the finite subcollection of $\{(g_i, \Sigma) : i \in I\}$ is also an HS cover of $(\widetilde{\Upsilon}, \Sigma)$. Conversely, let $\{(g_{i\Upsilon}, \Sigma) : i \in I\}$ be an HS cover of $(\widetilde{\Upsilon}, \Sigma)$. Obviously, $\{(g_i, \Sigma) : i \in I\}$ be an HS cover of $(\widetilde{\Upsilon}, \Sigma)$ by HS open sets in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. By hypothesis, the finite subcollection of $\{(g_i, \Sigma) : i \in I\}$ is also an HS cover of $(\widetilde{\Upsilon}, \Sigma)$ and hence $\{(g_{i\Upsilon}, \Sigma) : i = 1, 2, ..., n\}$ is an HS subcover of $(\widetilde{\Upsilon}, \Sigma)$. Therefore, $(\Upsilon, \mathcal{T}_{\Upsilon}, \Sigma)$ is HS compact space. □

Proposition 4.9. *An HSTS $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is an HS compact if there exists an HS basis β for $\mathcal{T}_{\Re}$ such that every HS cover of $(\widetilde{\Re}, \Sigma)$ by the elements of β has a finite HS subcover.*

Proof. Let $(\Re, \mathcal{T}_{\Re}, \Sigma)$ be an HS compact space. Obviously, $\mathcal{T}_{\Re}$ is an HS basis for $\mathcal{T}_{\Re}$. Hence, every HS cover of $(\widetilde{\Re}, \Sigma)$ by the elements of $\mathcal{T}_{\Re}$ has a finite HS subcover. Conversely, let $\{(g_i, \Sigma) : i \in I\}$ be an HS cover of $(\widetilde{\Re}, \Sigma)$. Now, $\{(g_i, \Sigma) : i \in I\}$ can be written as the union of some HS basis elements $\{(b_i, \Sigma) : i \in I\}$ of β. These elements form an HS cover of $(\widetilde{\Re}, \Sigma)$. By hypothesis, every HS cover of $(\widetilde{\Re}, \Sigma)$ by the elements of β has a finite HS subcover. Now, we have $\{(b_i, \Sigma) : i = 1, 2, ..., n\} \widetilde{\sqsubseteq} \{(g_i, \Sigma) : i = 1, 2, ..., n\}$. This implies that $\{(g_i, \Sigma) : i = 1, 2, ..., n\}$ is a finite HS subcover of $(\widetilde{\Re}, \Sigma)$. Hence, $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is an HS compact space. □

Proposition 4.10. *If $\Psi_{\gamma\delta} : (\Re, \mathcal{T}_{\Re}, \Sigma) \to (\aleph, \mathcal{T}_{\aleph}, \acute{\Sigma})$ is an HS continuous map and $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is an HS compact space, then $\Psi_{\gamma\delta}((\Re, \mathcal{T}_{\Re}, \Sigma))$ is an HS compact.*

Proof. Let $\{(f_i, \acute{\Sigma}) : i \in I\}$ be an HS cover of $\Psi_{\gamma\delta}((\widetilde{\Re}, \Sigma))$. Since $\Psi_{\gamma\delta}$ is an HS continuous map, then, for each $i \in I$, $\Psi^{-1}_{\gamma\delta}((f_i, \acute{\Sigma}))$ is an HS open set in $(\Re, \mathcal{T}_{\Re}, \Sigma)$. Then, the collection $\{\Psi^{-1}_{\gamma\delta}((f_i, \acute{\Sigma})) : i \in I\}$ forms an HS cover of $(\widetilde{\Re}, \Sigma)$. Since $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is an HS compact, then we have $(\widetilde{\Re}, \Sigma) = \widetilde{\sqcup} \{\Psi^{-1}_{\gamma\delta}((f_i, \acute{\Sigma})) : i = 1, 2, ..., n\} = \Psi^{-1}_{\gamma\delta}(\widetilde{\sqcup}\{(f_i, \acute{\Sigma}) : i = 1, 2, ..., n\})$ so that $\Psi_{\gamma\delta}((\widetilde{\Re}, \Sigma)) \widetilde{\sqsubseteq} \widetilde{\sqcup}\{(f_i, \acute{\Sigma}) : i = 1, 2, ..., n\}$. Hence, $\Psi_{\gamma\delta}((\widetilde{\Re}, \Sigma))$ is an HS compact space. □

Definition 4.11. A collection ξ of HS sets is said to have the finite intersection property if the HS intersection of members of each finite subcollection of ξ is non-null HS set.

Proposition 4.12. *An HSTS $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is an HS compact if and only if every collection of HS closed subsets of $(\widetilde{\Re}, \Sigma)$ with the finite intersection property has a non-null HS intersection.*

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Proof. Let $(\Re, \mathcal{T}_{\Re}, \Sigma)$ be an HS compact space and let $\mu = \{(g_i, \Sigma) : i \in I\}$ be a collection of HS closed subsets of $(\widetilde{\Re}, \Sigma)$ with the finite intersection property and suppose, if possible, $\widetilde{\sqcap}\{(g_i, \Sigma) : i \in I\} = (\Phi, \widetilde{\Re}, \Sigma)$. Then, $\widetilde{\sqcup}\{(g_i, \Sigma)^c : i \in I\} = (\widetilde{\Re}, \Sigma)$. This means that $\{(g_i, \Sigma)^c : i \in I\}$ is HS open cover of $(\widetilde{\Re}, \Sigma)$. Since $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is an HS compact space, we have that $\widetilde{\sqcup}\{(g_i, \Sigma)^c : i = 1, 2, ..., n\} = (\widetilde{\Re}, \Sigma)$ which implies that $\widetilde{\sqcap}\{(g_i, \Sigma)^c : i = 1, 2, ..., n\} = (\Phi, \widetilde{\Re}, \Sigma)$. But this contradicts the finite intersection property of μ. Hence, we must have $\{(g_i, \Sigma) : i \in I\} \neq (\Phi, \widetilde{\Re}, \Sigma)$. Conversely, let every collection of HS closed subsets of $(\widetilde{\Re}, \Sigma)$ with the finite intersection property has a non-null HS intersection and let $\{(g_i, \Sigma) : i \in I\}$ be an HS open cover of $(\widetilde{\Re}, \Sigma)$ so that $\widetilde{\sqcup}\{(g_i, \Sigma) : i \in I\} = (\widetilde{\Re}, \Sigma)$. Then, $\widetilde{\sqcap}\{(g_i, \Sigma)^c : i \in I\} = (\Phi, \widetilde{\Re}, \Sigma)$. Hence, by hypothesis $\widetilde{\sqcap}\{(g_i, \Sigma)^c : i = 1, 2, ..., n\} = (\Phi, \widetilde{\Re}, \Sigma)$. Then, $\widetilde{\sqcup}\{(g_i, \Sigma) : i = 1, 2, ..., n\} = (\widetilde{\Re}, \Sigma)$. Thus, $(\Re, \mathcal{T}_{\Re}, \Sigma)$ is an HS compact space. □

5 Conclusions

With this study, we have made a contribution to the field of HSTs. We have developed new types of maps in the context of HS maps: continuous map, open map, closed map, and homeomorphism map. The concept of HS compact space has also been investigated. We have produced several examples to validate and illustrate the conclusions and relationships established. The work that is being presented can also be applied to a variety of current HS set hybrids, including fuzzy HS sets, intuitionistic HS sets, Pythagorean HS sets, neutrosophic HS sets and so on.

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