A Contribution to the Group of Units' Problem in Some 2-Cyclic Refined Neutrosophic Rings

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Abstract:

This paper is dedicated to study the group of units problem in some 2-cyclic refined neutrosophic rings, where it presents a full classification of the group of units in 2-cyclic refined neutrosophic ring of integers $Z_2(I)$, 2-cyclic refined neutrosophic ring of integers modulo 2 and 3 $(Z_2)_2(I), (Z_3)_2(I)$ and 2-cyclic refined neutrosophic ring of real numbers $R_2(I)$, as direct products of the cyclic groups. On the other hand, this work provides a necessary and sufficient condition for the invertibility of any square 2-cyclic refined neutrosophic real matrix.Also, we will illustrate some examples to clarify the validity of our work.

Keywords: 2-cyclic refined rings; group of units; invertible; neutrosophic ring

1.Introduction

In the theory of rings, the problem of determining units (invertible elements) in a ring with unity is still open. The classification of the group of units is still unknown in general. After the arrival of neutrosophy in 1995, it was used to define some new kinds of algebraic structures, where we find concepts such as neutrosophic rings, refined neutrosophic modules and spaces, refined neutrosophic ring of matrices, n-refined neutrosophic rings [3-6,7-12], and n-cyclic refined neutrosophic rings [2]. The condition of invertibility in n-cyclic refined neutrosophic rings is still an open question in general for any positive integer n.

This open question has motivated us to study the case of 2-cyclic refined neutrosophic rings. We determine the units in 2-cyclic refined neutrosophic ring of integers $Z_2(I)$, 2-cyclic refined neutrosophic ring of integers modulo 2 and 3 $(Z_2)_2(I), (Z_3)_2(I)$ and 2-cyclic refined neutrosophic ring of real numbers $R_2(I)$, with the corresponding structure of the group of units by using a computational method depends only on the definition of multiplication operation. Also, we have determined a necessary and sufficient condition for the invertibility of any square 2-cyclic refined neutrosophic real matrix. All rings through this paper are considered with unity 1.

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2. Preliminaries

Definition 1: [2]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n sub-indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-cyclic refined neutrosophic ring.

Operations on R_n(I) are defined as:

$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n}(x_i + y_i) I_i$, $\sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i,j=0}^{n}(x_i \times y_j) I_i I_j = \sum_{i,j=0}^{n}(x_i \times y_j) I_{(i+j\, mod n)}$.

× is the multiplication on the ring R.

Example 2:

(a) The 2-cyclic refined neutrosophic ring of integers is defined as follows:

$Z_2(I) = \{t_0 + t_1 I_1 + t_2 I_2;\ t_i \in Z\}$.

(b) Addition on $Z_2(I)$ can be clarified as follows:

$(a + bI_1 + cI_2) + \ (m + nI_1 + tI_2) = (a + m) + I_1(b + n) + I_2(c + t)$.

(c) Multiplication on $Z_2(I)$ can be clarified as follows:

$(a + bI_1 + cI_2)(\ m + nI_1 + tI_2) = \quad am + anI_1 + atI_2 + bmI_1 + bnI_2 + btI_1 + cmI_2 + cnI_1 + ctI_2$

$= am + I_1(an + bm + bt + cn) + I_2(at + bn + cm + ct)$.

Where $I_1 I_1 = I_{(1+1\, mod 2)} = I_2, I_2 I_2 = I_{(2+2\, mod\, 2)} = I_2, I_1 I_2 = I_{(1+2\, mod\, 2)} = I_1$.

The elements of Z_2 are taken by the form {1,2} instead of {0,1} in the definition of indices I in sub-indeterminacies I_i. See [2].

Definition 3:

Let R be any ring with unity. An arbitrary element $x \in R$ is called a unit if and only if there exists $y \in R$ such that $xy = yx = 1$. The element y is called the inverse of x.

Example 4:

Let $Z_{4_2}(I) = \{a + bI_1 + cI_2; a, b, c \in Z_4\}$ be the 2-cyclic refined neutrosophic ring of integers modulo 4. The element $x = 1 + 2I_2$ is a unit, that is because there exists $y = x = 1 + 2I_2$, such that $xy = yx = 1 + 4I_2 + 4I_2 = 1$. It is clear that the inverse of x is x itself.

3. Main concepts and results

Theorem

Let R be any ring with unity, let $R_2(I)$ be its corresponding 2-cyclic refined neutrosophic ring.

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Let $A = a_0 + a_1I_1 + a_2I_2$ be a unit in $R_2(I)$, then there exists $B = b_0 + b_1I_1 + b_2I_2 \in R_2(I)$ such that

1). $a_0b_0 = 1$.

2). $(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 1$.

Proof.

Suppose that A is a unit, there exists $B = b_0 + b_1I_1 + b_2I_2 \in R_2(I)$, such that $A.B = 1$.

Thus, $a_0b_0 + I_1(a_1b_2 + a_2b_1 + a_0b_1 + a_1b_0) + I_2(a_2b_0 + a_0b_2 + a_2b_2 + a_1b_1) = 1$, then

$$\begin{cases} a_0b_0 = 1 \ldots (1) \\ a_1b_2 + a_2b_1 + a_0b_1 + a_1b_0 = 0 \ldots (2) \\ a_2b_0 + a_0b_2 + a_2b_2 + a_1b_1 = 0 \ldots (3) \end{cases}$$

By Adding (1) and (2) and (3),we get.

$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 1$, hence the proof is complete.

Theorem:

Let $R_2(I) = \{t_0 + t_1I_1 + t_2I_2;\ t_i \in Z\}$ be the 2-cyclic refined neutrosophic ring of integers, then it has exactly 8 units.

Proof.

Suppose that $A = a_0 + a_1I_1 + a_2I_2 \in Z_2(I)$is a unit, then $B = A^{-1} = b_0 + b_1I_1 + b_2I_2$, where $a_0b_0 = 1$ (I) and $(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 1(II)$.

From equation (II), we get:

$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 1$ or

$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = -1$.

We discuss the possible cases.

Case 1: if $a_0 = b_0 = 1$ and $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 1$, hence $a_1 + a_2 = b_1 + b_2 = 0$, thus $a_2 = -a_1$,$b_2 = -b_1$

The possible unit according to this case has the form $A = 1 + a_1I_1 - a_1I_2$, $A^{-1} = 1 + b_1I_1 - b_1I_2$.

$A.A^{-1} = 1 \Longrightarrow 1 + b_1I_1 - b_1I_2 + a_1I_1 + a_1b_1I_2 - a_1b_1I_1 - a_1I_2 - a_1b_1I_1 + a_1b_1I_2 = 1$

$\Longrightarrow I_1(b_1 + a_1 - 2a_1b_1) + I_2(-a_1 - b_1 + 2a_1b_1) = 0$, so that $b_1 + a_1 - 2a_1b_1 = 0 \Longrightarrow b_1 = a_1(2b_1 - 1)$.

This means that $2b_1 - 1 / b_1$ which is possible if and only if $b_1 = 0$ or $b_1 = 1$.

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If $b_1 = 0$ then $0 = a_1(-1) \Longrightarrow a_1 = 0$.

If $b_1 = 1$ then $1 = a_1(1) \Longrightarrow a_1 = 1$.

Thus, we get the following two units $A_1 = 1, A_2 = 1 + I_1 - I_2$, where ${A_2}^{-1} = A_2$

Case 2: if $a_0 = b_0 = 1$ and $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = -1$, hence $a_2 = -2 - a_1$ and $b_2 = -2 - b_1$.

The possible units are $A = 1 + a_1I_1 + (-2 - a_1)I_2, A^{-1} = B = 1 + b_1I_1 + (-2 - b_1)I_2$.

$A.A^{-1} = A.B = 1 \Longrightarrow 1 + b_1I_1 - 2I_2 - b_1I_2 + a_1I_1 + a_1b_1I_2 - 2a_1I_1 - a_1I_2 - a_1b_1I_1 - 2I_2 - a_1I_2 - 2b_1I_1 - a_1b_1I_1 + 4I_2 + a_1b_1I_2 + 2a_1I_2 + 2b_1I_2 = 1$,

$\Longrightarrow I_1(-b_1 - a_1 - 2a_1b_1) + I_2(a_1 + b_1 + 2a_1b_1) = 0$, so that $a_1 + b_1 + 2a_1b_1 = 0 \Longrightarrow b_1 = a_1(-2b_1 - 1)$.

This is possible if and only if $a_1 = b_1 = 0$ or $b_1 = -1$, which implies that $-1 = a_1(1) = a_1$.

The possible unit is $A_3 = 1 - I_1 + I_2$ with ${A_3}^{-1} = A_3$, and $A_4 = 1 - 2I_2$ with ${A_4}^{-1} = A_4$.

Case 3: if $a_0 = b_0 = -1$ and $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = -1$, then $a_2 = -a_1$ and $b_2 = -b_1$.

By a similar argument to the case 1, we get:

$A_5 = -1$ or $A_5 = -1 + I_1 - I_2$, where ${A_6}^{-1} = A_6$.

Case 4: if $a_0 = b_0 = -1$ and $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 1$, then $a_2 = 2 - a_1$ and $b_2 = 2 - b_1$.

By a similar argument to the case 1, we get:

$A_5 = -1$ or $A_5 = -1 + I_1 - I_2$, where ${A_6}^{-1} = A_6$

The possible units have the form:

$A = -1 + a_1I_1 + (2 - a_1)I_2, B = A^{-1} = -1 + b_1I_1 + (2 - b_1)I_2$.

$A.B = 1 \Longrightarrow 1 - b_1I_1 + (2 - a_1)I_2 - a_1I_1 + a_1b_1I_2 + I_1(2a_1 - a_1b_1) + (-2 + a_1)I_2 - 2b_1I_1 - a_1b_1I_1 + (4 - 2b_1 + a_1b_1)I_2 = 1$,

$\Longrightarrow I_1(b_1 + a_1 - 2a_1b_1) + I_2(-b_1 - a_1 + 2a_1b_1) = 0 \Longrightarrow b_1 + a_1 - 2a_1b_1 = 0 \Longrightarrow b_1 = a_1(2b_1 - 1)$,

$\Longrightarrow 2b_1 - 1 \,/\, b_1$ so that $a_1 + b_1 + 2a_1b_1 = 0 \Longrightarrow b_1 = a_1(-2b_1 - 1)$.

Which is possible if and only if $a_1 = b_1 = 0$ or $b_1 = 1 = a_1$

Hence $A_7 = -1 + I_1 + I_2, {A_7}^{-1} = A_7$ and $A_8 = 1 + 2I_2$and ${A_8}^{-1} = A_8$.

Theorem .

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The group of units of $Z_2(I)$ is isomorphic to $Z_2 \times Z_2 \times Z_2$.

Proof. According to the previous theorem we get 8 units. Each one has order 2. This implies that the group of units is isomorphic to $Z_2 \times Z_2 \times Z_2$.

Theorem.

The group of units of the finite 2-cyclic refined neutrosophic ring $(Z_2)_2(I)$is isomorphic to Z_2.

Proof.

Let $A = a_0 + a_1 I_1 + a_2 I_2$ be a unit in $(Z_2)_2(I)$, then $A^{-1} = b_0 + b_1 I_1 + b_2 I_2$such that $a_0 b_0 = 1$ and

$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 1$.

This means that $a_0 = b_0 = 1$ and $a_0 + a_1 I_1 + a_2 I_2 = b_0 + b_1 I_1 + b_2 I_2 = 1$, thus $a_2 = -a_1$, $b_2 = -b_1$.

$A = 1 + a_1 I_1 - a_2 I_2, A^{-1} = 1 + b_1 I_1 - b_2 I_2$.

$A.A^{-1} = 1 \Longrightarrow 1 + b_1 I_1 + a_1 I_1 + a_1 b_1 I_2 - a_1 b_1 I_1 - a_1 I_2 - a_1 b_1 I_1 + a_1 b_1 I_2 = 1$.

Hence.
$\Longrightarrow I_1(a_1 + b_1 - 2a_1 b_1) + I_2(-a_1 - b_1 + 2a_1 b_1) = 0$, so that $a_1 + b_1 - 2a_1 b_1 = 0 \Longrightarrow a_1 + b_1 = 0 \Longrightarrow$

$a_1 = -b_1$.

This means that. $A = 1 + a_1 I_1 - a_1 I_2, A^{-1} = 1 - a_1 I_1 + a_1 I_2$.

If $a_1 = 0$, then $A_1 = {A_1}^{-1} = 1$.

If $a_1 = 1$, then $A_2 = 1 + I_1 - I_2, {A_2}^{-1} = 1 - I_1 + I_2 = 1 + I_1 - I_2 = A_2$.

$A_2.{A_2}^{-1} = 1 \Longrightarrow 1 - I_1 + I_2 + I_1 - I_2 + I_1 - I_2 + I_1 - I_2 = 1 \Longrightarrow 2I_1 - 2I_2 = 0$ it is true.

Remark 2.1. $R_n(I) \ncong \underbrace{R \times R \times \dots R}_{n-times}$ in general.

Theorem

The group of units of $(Z_3)(I)$ is isomorphic to $Z_2 \times Z_2 \times Z_2$.

Proof.

If $A = a_0 + a_1 I_1 + a_2 I_2 \in Z_2(I)$ is a unit, then $B = A^{-1} = b_0 + b_1 I_1 + b_2 I_2$, such that $a_0 b_0 = 1\ (I)$ and

$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 1(II)$.

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From equation (II), we get:

$a_0 = b_0 = 1$ or $a_0 = b_0 = 2$

From equation (II), we get:

$a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 1$ or $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 2$.

We discuss the possible cases.

Case 1: if $a_0 = b_0 = 1$ and $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 1$, we get $a_1 + a_2 = b_1 + b_2 = 0$, hence

$a_2 = -a_1, b_2 = -b_1$

The possible units have the form $A = 1 + a_1I_1 - a_1I_2$, $A^{-1} = B = 1 + b_1I_1 - b_1I_2$.

$\Rightarrow AA^{-1} = 1 \Rightarrow I_1(b_1 + a_1 - 2a_1b_1) + I_2(-a_1 - b_1 + 2a_1b_1) = 0$, so that $a_1 + b_1 - 2a_1b_1 = 0$

$\Rightarrow a_1 + b_1 = 2a_1b_1\ (*)$.

Equation $(*)$ has the following solutions:

$a_1 = b_1 = 0$, $a_1 = b_1 = 1$, hence

$A_1 = 1$, $A_2 = 1 + I_1 - I_2$, where ${A_2}^{-1} = A_2$.

Case 2: if $a_0 = b_0 = 1$ and $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 2 \Rightarrow a_1 + a_2 = b_1 + b_2 = 1$,so that

$A = 1 + a_1I_1 + (1 - a_1)I_2$, $A^{-1} = B = 1 + b_1I_1 + (1 - b_1)I_2$.

$A.A^{-1} = 1 \Rightarrow 1 + a_1I_1 + (1 - a_1)I_2 + a_1b_1I_2 + I_2(b_1 - b_1a_1) + (1 - b_1)I_2 + (a_1 - a_1b_1)I_1 + I_2(1 + a_1b_1 - a_1 - b_1) = 1$,

$$\Rightarrow I_1(a_1 + b_1 + b_1 - b_1a_1 + a_1 - a_1b_1) + I_2(1 - a_1 + a_1b_1 + 1 - b_1 + 1 + a_1b_1 - a_1 - b_1) = 0$$

$$\Rightarrow \begin{cases} 2a_1 + 2b_1 - 2a_1b_1 = 0 \\ -2a_1 - 2b_1 + 2a_1b_1 = 0 \end{cases}$$

$\Rightarrow 2(a_1 + b_1 - a_1b_1) = 0$, so $a_1 + b_1 - a_1b_1$ is a zero divisor, which is possible if and only if $a_1 + b_1 - a_1b_1 = 0 \Rightarrow a_1 + b_1 = a_1b_1$.

The possible solutions are($a_1 = b_1 = 0$), ($a_1 = b_1 = 2$)

$A_3 = 1 + I_2, {A_3}^{-1} = A_3, A_4 = 1 + 2I_1 - I_2$ and ${A_4}^{-1} = A_4$.

Case 3: if $a_0 = b_0 = 2$ and $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 1$, then $a_2 = 2 - a_1$ and $b_2 = 2 - b_1$.

The possible units have the form:

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$A = 2a_1I_1 + (2 - a_1)I_2$, $A^{-1} = 2 + b_1I_1 + (2 - b_1)I_2$.

$A.A^{-1} = 1 + 2b_1I_1 + (4 - 2b_1)I_2 + 2a_1I_1 + a_1b_1I_2 + (2a_1 - a_1b_1)I_1 + (4 - 2a_1)I_2 + (2b_1 - a_1b_1)I_2 + (4 - 2b_1 - 2a_1 + a_1b_1)I_2 = 1,$

$\Longrightarrow I_1(2b_1 + 2a_1 + 2a_1 - a_1b_1 + 2b_1 - a_1b_1) + I_2(4 - 2b_1 + a_1b_1 + 4 - 2a_1 + 4 - 2b_1 - 2a_1 + a_1b_1) = 0,$

$\Longrightarrow \begin{cases} a_1 + b_1 - 2a_1b_1 = 0 \\ -a_1 - b_1 + 2a_1b_1 = 0 \end{cases} \Longrightarrow a_1 + b_1 = 2a_1b_1,$

The possible solutions are $(a_1 = b_1 = 0 \; or \; a_1 = b_1 = 1)$.

Thus the corresponding units are:

$A_5 = 2 + 2I_2$, $A_6 = 2 + I_1 + I_2$where ${A_5}^{-1} = A_5$,${A_6}^{-1} = A_6$.

Case 4: if $a_0 = b_0 = 2$ and $a_0 + a_1 + a_2 = b_0 + b_1 + b_2 = 2$, we get$a_1 + a_2 = b_1 + b_2 = 0$,hence $a_2 = -a_1 b_2 = -b_1$.

The possible units are:

$A = 2 + a_1I_1 - a_1I_2$, $A^{-1} = 2 + b_1I_1 - b_1I_2$.

$A.A^{-1} = 1 \Longrightarrow 1 + 2b_1I_1 - 2b_1I_2 + 2a_1I_1 + a_1b_1I_2 - a_1b_1I_1 - 2a_1I_2 - a_1b_1I_1 + a_1b_1I_2 = 1$

$\Longrightarrow I_1(2b_1 + 2a_1 - 2a_1b_1) + I_2(-2b_1 - 2a_1 + 2a_1b_1) = 0$

$\Longrightarrow \{2b_1 + 2a_1 - 2a_1b_1 = 0$ so that $2(b_1 + a_1 - a_1b_1) = 0$, hence $b_1 + a_1 = a_1b_1$ is a zero divisor, thus

$b_1 + a_1 - a_1b_1 = 0$, this means that $b_1 + a_1 = a_1b_1$.

The possible solutions are $(a_1 = b_1 = 0 \; or \; a_1 = b_1 = 2)$.

Thus the corresponding units are:

$A_7 = 2$, $A_8 = 2 + 2I_1$ where ${A_8}^{-1} = A_8$.

Since the order of any unit is 2, we get the proof.

Theorem

The 2-cyclic refined neutrosophic ring of real numbers $R_2(I) = \{t_0 + t_1I_1 + t_2I_2;\; t_i \in R\}$ has infinite group of units.

Proof.

Let$A = a_0 + a_1I_1 + a_2I_2 \in R_2(I)$ is a unit, then $B = A^{-1} = b_0 + b_1I_1 + b_2I_2;\; a_0b_0 = 1(I)$, and

$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) = 1(II)$.

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From equation (I), we get:

$b_0 = \frac{1}{a_0}; a_0 \neq 0.$

From equation (II), we get:

$b_0 + b_1 + b_2 = \frac{1}{a_0+a_1+a_2}$, thus implies that $b_0 + b_1 + b_2 \neq 0, a_0 + a_1 + a_2 \neq 0$

and $\frac{1}{a_0} + b_1 + b_2 = \frac{1}{a_0+a_1+a_2}$, thus $b_2 = \frac{1}{a_0+a_1+a_2} - \frac{1}{a_0} - b_1 = \frac{-a_1-a_2}{a_0(a_0+a_1+a_2)} - b_1$.

On the order hand, we have $A.A^{-1} = 1$, thus $(a_0 + a_1I_1 + a_2I_2)\left(\frac{1}{a_0} + b_1I_1 + \left[\frac{-a_1-a_2}{a_0(a_0+a_1+a_2)} - b_1\right]I_2\right) = 1$

$\Longrightarrow 1 + a_0b_1I_1 + I_2\left(\frac{-a_1-a_2}{a_0(a_0+a_1+a_2)} - a_0b_1\right) + \frac{a_1}{a_0}I_1 + a_1b_1I_2 + I_1\left(\frac{-a_1{}^2-a_1a_2}{a_0(a_0+a_1+a_2)} - a_1b_1\right) + \frac{a_2}{a_0}I_2 + a_2b_1I_1 + I_2\left(\frac{-a_1a_2-a_2{}^2}{a_0(a_0+a_1+a_2)} - a_2b_2\right) = 1,$

$\Longrightarrow I_1\left(a_0b_1 + \frac{a_1}{a_0} - a_1b_1\frac{-a_1{}^2-a_1a_2}{a_0(a_0+a_1+a_2)} + a_2b_1\right) + I_2\left(\frac{-a_1-a_2}{a_0(a_0+a_1+a_2)} - a_0b_1 + \frac{a_2}{a_0} - a_2b_2 + \frac{-a_1a_2-a_2{}^2}{a_0(a_0+a_1+a_2)}\right) = 0,$

We get the following equations:

(*) $a_0b_1 + \frac{a_1}{a_0} - a_1b_1\frac{-a_1{}^2-a_1a_2}{a_0(a_0+a_1+a_2)} + a_2b_1 = 0$

(**) $\frac{-a_1-a_2}{a_0(a_0+a_1+a_2)} - a_0b_1 + \frac{a_2}{a_0} - a_2b_2 + \frac{-a_1a_2-a_2{}^2}{a_0(a_0+a_1+a_2)} = 0$

We simplify (*) as follows:

$$b_1(a_0 - a_1 + a_2) + \frac{a_1(a_0 + a_1 + a_2)}{a_0(a_0 + a_1 + a_2)} + \frac{-a_1{}^2 - a_1a_2}{a_0(a_0 + a_1 + a_2)} = 0$$

Hence

$$b_1(a_0 - a_1 + a_2) + \frac{a_0a_1}{a_0(a_0 + a_1 + a_2)} = 0$$

So

$$b_1 = \frac{-a_1}{(a_0 - a_1 + a_2)(a_0 + a_1 + a_2)}$$

By solving (**), we get the same solution of (*).

This implies that the units have the form $A = a_0 + a_1I_1 + a_2I_2, with\ the\ condition\ a_0 + a_1 + a_2 \neq 0, a_0 \neq 0, a_0 - a_1 + a_2 \neq 0$. The converse of A is $A^{-1} = \frac{1}{a_0} + I_1\left[\frac{-a_1}{(a_0-a_1+a_2)(a_0+a_1+a_2)}\right] + I_2\left[\frac{-a_1-a_2}{a_0(a_0+a_1+a_2)} + \frac{a_1}{(a_0+a_1+a_2)(a_0-a_1+a_2)}\right]$.

Doi : https://doi.org/10.54216/IJNS.180304
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Example:

Consider the following 2-cyclic refined neutrosophic number $A = 1 + 2I_1 - I_2$. It is invertible, that is because

$a_0 + a_1 + a_2 = 2 \neq 0, a_0 = 1 \neq 0, a_0 - a_1 + a_2 = -2 \neq 0.$

The converse of A is $A^{-1} = 1 + I_1\left[\frac{-2}{(-2)(2)}\right] + I_2\left[\frac{-1}{1(2)} + \frac{2}{(2)(-2)}\right] = 1 + \frac{1}{2}I_1 - I_2.$

Example:

Now, we can find the inverse of any square 2-cyclic refined real neutrosophic matrix, as the following example shows.

Consider the following 2-cyclic refined neutrosophic matrix $M = \begin{pmatrix} 1 & -2I_2 \\ I_1 & 1 - I_2 \end{pmatrix}$, we have

$\det(M) = 1 + 2I_1 - I_2$, which is invertible according to the previous example. This implies that M is invertible.

$M^{-1} = \frac{1}{\det(M)}\left(adj(M)\right) = \left(1 + \frac{1}{2}I_1 - I_2\right)\begin{pmatrix} 1 - I_2 & 2I_2 \\ -I_1 & 1 \end{pmatrix}.$

4. Conclusion

In this paper, we have provided a novel solution of the group of units problem in some 2-cyclic refined neutrosophic rings, such as 2-cyclic refined neutrosophic ring of integers $Z_2(I)$, 2-cyclic refined neutrosophic ring of integers modulo 2 and 3 $(Z_2)_2(I), (Z_3)_2(I)$ and 2-cyclic refined neutrosophic ring of real numbers $R_2(I)$. Also, we have illustrated many examples especially for the invertibility of any square 2-cyclic refined neutrosophic real matrix.

According to this research, many open questions are coming to light.

Open problem 1: If the ring R has no zero divisors, then is the group of units of its corresponding 2-cyclic refined neutrosophic ring $R_2(I)$ isomorphic to $U(R) \times U(R) \times U(R)$. (Remark that the answer is yes if R is the ring of integers, or the ring of integers modulo 3).

Open problem 2: Is there a ring homomorphism between the 2-cyclic refined neutrosophic ring $R_2(I)$ and the direct product $R \times R \times R$.

Open problem 3: Is the group of units of the 2-cyclic refined neutrosophic ring or real numbers isomorphic to $R^* \times R^* \times R^*$.

Open problem 4: Determine the structure of the group of units of previous rings in the case of n=3, n=4, n=5.

Open Problem 5: Answer open problems 1, 2, 3 for the case n is greater than 2.

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