



## A Short Note on Some Novel Applications of Semi Module Homomorphisms

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### Abstract

In this paper, we study the algebraic relationships between  $n$ -refined neutrosophic modules by using semi-module homomorphisms. On the other hand, this work shows the relationship between neutrosophic geometrical AH-isometry and semi-module isomorphisms.

**Keywords;** AH-isometry; Semi homomorphism; semi isomorphism;  $n$ -refined neutrosophic module

### 1.Introduction

Neutrosophy is a new kind of philosophy founded by Smarandache [1] to study the uncertainty and the lack of information in many areas of science and life.

The indeterminacy element  $I$  and its refinements, were useful in algebra where we find some generalizations of classical algebraic structures such as neutrosophic modules,  $n$ -refined neutrosophic matrices [2-5].

The concept of semi-modules homomorphism /isomorphism was presented in [6] as a tool to study the relations between modules which are defined over different rings.

In this work, we show a new application of semi homomorphisms in the study of  $n$ -refined neutrosophic modules. Also, we present a novel application in neutrosophic Euclidean geometry [9] based on semi-isomorphisms, where we show that algebraic isometries used in the study of neutrosophic Euclidean geometrical shapes can be considered as semi-module isomorphisms.

### 2. Main discussion

First of all, we show that the concept of semi-homomorphism (semi-isomorphism) is essential in the study of neutrosophic Euclidean geometry.

Neutrosophic Euclidean geometry was built over the idea of AH-isometry, where the AH-isometry is a function that preserves distances between neutrosophic points in  $R(I) \times R(I)$ .

**Definition 1.**

(a). Let  $R(I) \times R(I)$  be the neutrosophic plane with two N-dimensions, then.

$$g: R(I) \times R(I) \rightarrow R^2 \times R^2$$

$$g(a + bI, c + dI) = ((a, a + b), (c, c + d))$$

is called the two dimensional AH-isometry.

(b). the function  $T: R(I) \rightarrow R \times R$ ;  $T(a + bI) = (a, a + b)$  is called the one dimensional AH-isometry.

**Theorem 2**

$g, T$  preserve operations and distances.

In the following, we prove that the two dimensional AH-isometry is a semi-module isomorphism.

**Definition 3:**

Let  $(M, +, \cdot)$  be a module over the ring  $R$ , then  $(M(I), +, \cdot)$  is called a weak neutrosophic module over the ring  $R$ , and it is called a strong neutrosophic module if it is a module over the neutrosophic ring  $R(I)$ .

Elements of  $M(I)$  have the form  $x + yI$ ;  $x, y \in M$ , i.e  $M(I)$  can be written as  $M(I) = M + MI$ .

**Definition 4:**

- (a) Let  $(M, +, \cdot)$  be a module over the ring  $R$ , we say that  $M_n(I) = M + MI_1 + \dots + MI_n = \{x_0 + x_1I_1 + \dots + x_nI_n$ ;  $x_i \in M\}$  is a weak n-refined neutrosophic module over the ring  $R$ . Elements of  $M_n(I)$  are called n-refined neutrosophic vectors, elements of  $R$  are called scalars.
- (b) If we take scalars from the n-refined neutrosophic ring  $R_n(I)$ , we say that  $M_n(I)$  is a strong n-refined neutrosophic module over the n-refined neutrosophic ring  $M_n(I)$ . Elements of  $M_n(I)$  are called n-refined neutrosophic scalars.

**Definition 5:**

Let  $M$  be a module over a ring  $R$ ,  $N$  be a module over a ring  $T$ ,  $\varphi: M \rightarrow N$  be a well defined map, we say that  $\varphi$  is an  $f$ -semi module homomorphism if and only if the following conditions are true:

- (a)  $\varphi(x + y) = \varphi(x) + \varphi(y)$  for all  $x, y \in M$ .
- (b) There is a ring homomorphism  $f: R \rightarrow T$  such  $\varphi(r \cdot x) = f(r) \cdot \varphi(x)$  for all  $r \in R, x \in M$ .

**Example 6:**

Let  $M$  be a module over a ring  $R$ ,  $M(I)$  be the corresponding strong neutrosophic module over  $R(I)$ ,  $M(I_1, I_2)$  be the corresponding strong refined neutrosophic module over  $R(I_1, I_2)$ . Then

$\varphi: M(I_1, I_2) \rightarrow M(I)$ ;  $\varphi(a, bI_1, cI_2) = a + (b + c)I$  is a semi homomorphism.

**Theorem 7.**

Let  $g: R(I) \times R(I) \rightarrow R^2 \times R^2$  be the two dimensional AH-isometry, and  $T: R(I) \rightarrow R \times R$  be the one dimensional AH-isometry, then,  $g$  is a semi-module isomorphism.

**Proof.**

we have  $R(I) \times R(I)$  is a module over the ring  $R(I)$ , and  $R^2 \times R^2$  is a module over the ring  $R \times R$ .

According to [14],  $T$  is a ring isomorphism.

On the other hand,  $g$  preserves addition between  $R(I) \times R(I)$  and  $R^2 \times R^2$ , and  $g$  is a bijection.

Now, we prove that  $g$  has the following property.

$$g[(m + nI). (a + bI, c + dI)] = T(m + nI)g(a + bI, c + dI)$$

$$l_1 = [(m + nI). (a + bI, c + dI)] = g[(m + nI)(a + bI), (m + nI)(c + dI)]$$

$$= g[(ma + I(ab + na + nb)), (mc + I(amd + nc + nd))]$$

$$= ((m, m + n). (a, a + b), (m, m + n). (c, c + d)) = (m, m + n)((a, a + b), (c, c + d))$$

$$= T(m + nI)g(a + bI, c + dI).$$

Thus  $g$  is a semi isomorphism.

**Result 8.**

According to the previous theorem, the neutrosophic Euclidean geometry can be understood as a semi isomorphic image of a neutrosophic module.

Now, we study semi homomorphisms between  $n$ -refined neutrosophic modules.

**Theorem 9**

Let  $R$  be a ring,  $R_n(I)$  be the corresponding  $n$ -refined neutrosophic ring.

Let  $M$  be a module over  $R$ ,  $M_n(I)$  be the corresponding  $n$ -refined neutrosophic module over  $R_n(I)$ , then.

(a).  $M_{n-1}(I)$  is a semi homomorphic image of  $M_n(I)$ .

(b).  $M$  is a semi homomorphic image of  $M_n(I)$ .

**Proof.**

$$(a). g: R_n(I) \rightarrow R_{n-1}(I); g(a_0 + a_1I_1 + \dots + a_nI_n) = a_0 + a_1I_1 + \dots + (a_{n-1} + a_n)I_{n-1}$$

is a ring homomorphism.

$$\text{We define } f: M_n(I) \rightarrow M_{n-1}(I); f(x_0 + x_1I_1 + \dots + x_nI_n) = x_0 + x_1I_1 + \dots + (x_{n-1} + x_n)I_{n-1}.$$

It is clear that  $f$  is well defined and preserves addition.

Now, Let's compute:

$$f[(a_0 + a_1I_1 + \dots + a_nI_n) \cdot (x_0 + x_1I_1 + \dots + x_nI_n)] = f\left[\sum_{i,j=0}^n a_i x_j I_i I_j\right]$$

The coefficient of  $I_n$  is  $a_0x_n + a_nx_0 + a_nx_n$ .

The coefficient of  $I_{n-1}$  is  $a_0x_{n-1} + a_{n-1}x_0 + a_nx_{n-1} + a_{n-1}x_n + a_{n-1}x_{n-1}$ .

This implies that

$$\begin{aligned} f\left[\sum_{i,j=0}^n a_i x_j I_i I_j\right] &= f[A + I_{n-1}(a_0x_{n-1} + a_{n-1}x_0 + a_nx_{n-1} + a_{n-1}x_n + a_{n-1}x_{n-1}) \\ &\quad + I_n(a_0x_n + a_nx_0 + a_nx_n)] \\ &= A + I_{n-1}(a_0x_{n-1} + a_{n-1}x_0 + a_nx_{n-1} + a_{n-1}x_n + a_{n-1}x_{n-1}) + I_n(a_0x_n + a_nx_0 + a_nx_n) \end{aligned}$$

Now, we compute.

$$\begin{aligned} &g(a_0 + a_1I_1 + \dots + a_nI_n) \cdot f(x_0 + x_1I_1 + \dots + x_nI_n) \\ &= (a_0 + \dots + a_{n-2}I_{n-2} + I_{n-1}(a_n + a_{n-1})) \cdot (x_0 + \dots + x_{n-2}I_{n-2} + I_{n-1}(x_n + x_{n-1})) \end{aligned}$$

Where  $B = a_0x_{n-1} + a_0x_n + a_nx_0 + a_{n-1}x_0 + a_nx_{n-1} + a_nx_n + a_{n-1}x_{n-1} + a_{n-1}x_n$

Which is exactly equal to the coefficient  $I_{n-1}$  in  $f[(\sum_{i=0}^n a_i I_i \cdot \sum_{i=0}^n x_i I_i)]$ .

This means that  $f[(\sum_{i=0}^n a_i I_i \cdot \sum_{i=0}^n x_i I_i)] = g(\sum_{i=0}^n a_i I_i) \cdot f(\sum_{i=0}^n x_i I_i)$ , hence  $f$  is a semi module homomorphism.

(b). since  $M_{n-i}(I)$  is a semi homomorphic image of  $M_{n-i+1}(I)$ , we get the following sequence:

$$M_n(I) \rightarrow M_{n-1}(I) \rightarrow \dots \rightarrow M_2(I) \rightarrow M_1(I) \rightarrow M(I)$$

Thus  $M$  is a semi homomorphic image of  $M_n(I)$ .

**Example 10**

Let  $R = Z, M = Z_3$ .  $M$  is a module over  $R$ .

Let  $M_3(I) = \{a_0 + a_1I_1 + a_2I_2 + a_3I_3; a_i \in Z_3\}$  be the corresponding 3-refined neutrosophic module over  $R_3(I)$

$$Z_3(I) = \{b_0 + b_1I_1 + b_2I_2 + b_3I_3; b_i \in Z\}.$$

$$(a). g: Z_3(I) \rightarrow Z_2(I); g(a_0 + a_1I_1 + a_2I_2 + a_3I_3) = a_0 + a_1I_1 + (a_2 + a_3)I_2$$

Is a ring homomorphism.

$$(b). f: M_3(I) \rightarrow M_2(I); f(x_0 + x_1I_1 + x_2I_2 + x_3I_3) = x_0 + x_1I_1 + (x_2 + x_3)I_2$$

is a semi homomorphism, that is because:

$f$  preserves addition clearly.

$$f[(a_0 + a_1I_1 + a_2I_2 + a_3I_3) \cdot (x_0 + x_1I_1 + x_2I_2 + x_3I_3)]$$

$$\begin{aligned}
&= f[a_0x_0 + I_1(a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + a_1x_3 + a_2x_1 + a_3x_1) + I_2(a_0x_2 + a_2x_2 + a_2x_3 + a_3x_2) \\
&\quad + I_3(a_0x_3 + a_3x_0 + a_3x_3)] + a_2x_0 \\
&= a_0x_0 + I_1(a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + a_1x_3 + a_2x_1 + a_3x_1) \\
&\quad + I_2(a_0x_2 + a_2x_2 + a_2x_3 + a_3x_2 + a_2x_0 + a_0x_2 + a_3x_0 + a_3x_3)
\end{aligned}$$

On the other hand.

$$\begin{aligned}
&g(a_0 + a_1I_1 + a_2I_2 + a_3I_3).f(x_0 + x_1I_1 + x_2I_2 + x_3I_3) \\
&\quad = (a_0 + a_1I_1 + I_2(a_3 + a_2))(x_0 + x_1I_1 + I_2(x_3 + x_2)) \\
&= a_0x_0 + I_1(a_0x_1 + a_1x_0 + a_1x_1 + a_1x_2 + a_1x_3 + a_2x_1 + a_3x_1) \\
&\quad + I_2(a_2x_0 + a_3x_0 + a_2x_2 + a_2x_3 + a_3x_2 + a_0x_2 + a_3x_3 + a_0x_3) \\
&\quad = f[(a_0 + a_1I_1 + a_2I_2 + a_3I_3).(x_0 + x_1I_1 + x_2I_2 + x_3I_3)]
\end{aligned}$$

$$(c).kerf = \{a_2I_2 - a_2I_3 \in M_3(I)\} = \{a_2(I_2 - I_3); a_2 \in M\}$$

$$M_3(I)/kerf \cong_s f(M_3(I)) = M_2(I).$$

The symbol  $\cong_s$  means semi isomorphic property.

### 3. Conclusion

In this paper, we have presented some novel applications of semi module homomorphisms/isomorphisms, where we proved that every module is a semi homomorphic image of it corresponding n-refined neutrosophic module. Also, we have shown that the AH-isometry used in the theory of neutrosophic Euclidean geometry is a semi-module isomorphism.

### References

- [1] Smarandache, F., " A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press. Rehoboth, 2003.
- [2] Abobala, M., Hatip, A., and Bal, M., " A Review On Recent Advantages In Algebraic Theory Of Neutrosophic Matrices", IJNS, Vol.17, 2021.
- [3] Agboola, A.A.A., "On Refined Neutrosophic Algebraic Structures", NSS, Vol.10, pp. 99-101. 2015.
- [4] Abobala, M., " A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences", Hindawi, 2021
- [5] Olgun, N., Hatip, A., Bal, M., and Abobala, M., "A novel approach to necessary and sufficient conditions for the diagonalization of refined neutrosophic matrices", IJNS, 2021.
- [6] Abobala, M., " Semi Homomorphisms and Algebraic Relations Between Strong Refined Neutrosophic Modules and Strong Neutrosophic Modules", Neutrosophic Sets and Systems, Vol. 39, 2021.
- [7] Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021
- [8] Abobala, M., "Neutrosophic Real Inner Product Spaces", NSS, Vol. 43, 2021.

[9] Abobala, M., and Hatip, A., "An Algebraic Approach To Neutrosophic Euclidean Geometry", NSS, Vol. 43, 2021.

[10] Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", NSS , Vol. 45, 2021.