A Short Note On The Solutions Of Fermat's Diohantine Equation In Some Neutrosophic Rings

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Abstract: This paper is dedicated to study the concept of Fermat's triples in rings. Also, it determines the possible Fermat's triples in the neutrosophic ring of integers Z(I). Also, it discussed these triples in several finite commutative rings such as Z_n.

Keywords: Neutrosophic Ring, Fermat's Diophantine equation, Neutrosophic Integer

1.Introduction

Number Theory is always a rich material for pure mathematical ideas. Where concepts such as ideals, and cyclic groups were derived from number theory [16].

The equation $X^n + Y^n = Z^n; n = 2$, appeared firstly from Pythagoras famous theorem, and then it was generalized to the integers.

In the literature, Fermat presented his famous conjecture that the Diophantine equation $X^n + Y^n = Z^n; n \geq 3$ has only trivial solutions in the ring of integers. This conjecture was solved completely in 1993 by Andrew Wiles [16].

Neutrosophy is a new kind of generalized logic founded by Smarandache [1]. It represents a useful tool in the study of spaces [2], rings [6,7,9], number theory [15], and geometry [10].

In this work, we extend the Fermat's equation $X^n + Y^n = Z^n; n \geq 3$ to Neutrosophic algebraic rings, where we define the concept of Fermat's triples in any ring as the solutions of the previous equation, and

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we handle some special cases in several finite rings such as Z_n (integers modulo n) and neutrosophic rings of integers.

This work is motivated by finding novel applications and connections between number theory and algebraic structures. Also, it will provide some new open questions, which will represent the future of this theory.

Main discussion

Definition: [5]

Let (R,+,×) be a ring, R(I)={a+bI ; a,b∈ R} is called the neutrosophic ring where I is a neutrosophic element with condition $I^2 = I$.

Definition 2:

Let R be a ring, $F = (X, Y, Z)$ be a triple, where $X, Y, Z \in R$. F is called a general Fermat's triple if and only if $X^n + Y^n = Z^n; for\ all\ \ integers\ n \geq 3$.

Definition 3:

Let R be a ring, $F = (X, Y, Z)$ be a triple, where $X, Y, Z \in R$. F is called an n-Fermat's triple if and only if $X^n + Y^n = Z^n; for\ a\ fixed\ \ integer\ n \geq 3$.

Example 4:

Let Z_3 be the ring of integers modulo 3, then (0,1,1) is a general Fermat's triple, that is because for all integers $n \geq 3$, $0^n + 1^n = 1^n$

Example 5:

Let Z_5 be the ring of integers modulo 5, then (0,2,1) is a 4-Fermat's triple, that is because:

$0^4 + 2^4 = 1^4$.

Theorem 6: [16]

Let Z be the ring of integers, then:

(a) The set of general Fermat's triples is {(0,1,1),(1,0,1),(0,0,0)}.

(b) For every fixed integer $n \geq 3$, the set of n-Fermat's triples is {(0,1,1),(1,0,1),(0,0,0)}.

Theorem 7:

Let R be any ring, F be the set of all general Fermat's triples in R, F_n be the set of all n-Fermat's triples in R, then $F = \bigcap_{n=1}^{\infty} F_n$.

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Proof:

The proof holds directly from the definition.

Theorem 8:

Let Z(I)={$a + bI; a, b \in Z\ and\ I^2 = I$} be the ring of neutrosophic integers. The equation $X^n + Y^n = Z^n; n \geq 3$, has nine solutions where $X, Y, Z \in Z(I)$.

Proof.

$$X^n + Y^n = Z^n \Leftrightarrow {x_0}^n + I[(x_0 + x_1)^n - {x_0}^n] + {y_0}^n + I[(y_0 + y_1)^n - {y_0}^n]$$
$$= {z_0}^n + I[(z_0 + z_1)^n - {z_0}^n]$$

$$X^n + Y^n = Z^n \Leftrightarrow \begin{cases} {x_0}^n + {y_0}^n = {z_0}^n \dots (1) \\ (x_0 + x_1)^n + (y_0 + y_1)^n = (z_0 + z_1)^n \dots (2) \end{cases}$$

Now, solutions of (1) is.

$$\begin{cases} x_0 = y_0 = z_0 = 0 \dots (a) \\ x_0 = z_0 = 1, y_0 = 0 \dots (b) \\ y_0 = z_0 = 1, x_0 = 0 \dots (c) \end{cases}$$

And solutions of (2) is.

$$\begin{cases} x_0 + x_1 = y_0 + y_1 = z_0 + z_1 = 0 \dots (d) \\ x_0 + x_1 = z_0 + z_1 = 1, y_0 + y_1 = 0 \dots (e) \\ y_0 + y_1 = z_0 + z_1 = 1, x_0 + x_1 = 0 \dots (f) \end{cases}$$

We discuss possible cases.

Case1. If $(x_0 = y_0 = z_0 = 0)\ and\ (x_0 + x_1 = y_0 + y_1 = z_0 + z_1 = 0)$, then $X = Y = Z = 0$.

Case2. If $(x_0 = y_0 = z_0 = 0)\ and\ (x_0 + x_1 = z_0 + z_1 = 1, y_0 + y_1 = 0)$, then $X = Z = 1, Y = 0$.

Case3. If $(x_0 = y_0 = z_0 = 0)\ and\ (y_0 + y_1 = z_0 + z_1 = 1, x_0 + x_1 = 0)$, then $Y = Z = 1, X = 0$.

Case4. If $(x_0 = z_0 = 1, y_0 = 0)\ and\ (x_0 + x_1 = z_0 + z_1 = y_0 + y_1 = 0)$, then $X = 1 - I, Y = 0, Z = 1 - I$.

Case5. If $(x_0 = z_0 = 1, y_0 = 0)\ and\ (x_0 + x_1 = z_0 + z_1 = 1, y_0 + y_1 = 0)$, then $X = 1, Y = 0, Z = 1$.

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Case6. If $(x_0 = z_0 = 1, y_0 = 0)$ and $(z_0 + z_1 = y_0 + y_1 = 1, x_0 + x_1 = 0)$, then $X = 1 - I, Y = I, Z = 1$.

Case7. If $(z_0 = y_0 = 1, x_0 = 0)$ and $(z_0 + z_1 = y_0 + y_1 = x_0 + x_1 = 0)$, then $X = 0, Y = 1 - I, Z = 1 - I$.

Case8. If $(z_0 = y_0 = 1, x_0 = 0)$ and $(x_0 + x_1 = z_0 + z_1 = 1, y_0 + y_1 = 0)$, then $X = 0, Y = 1 - I, Z = 1 - I$.

Case9. If $(x_0 = z_0 = 1, y_0 = 0)$ and $(y_0 + y_1 = z_0 + z_1 = 1, x_0 + x_1 = 0)$, then $X = 0, Y = 1, Z = 1$.

Theorem 9:

Let Z_2 be the ring of integers modulo 2, hence it has exactly 4 general Fermat's triples.

Proof:

Consider the equation $X^n + Y^n = Z^n; n \geq 3$, where X,Y,Z∈ Z_2 . We have the following solutions:

(1,1,0), (0,1,1),(1,0,1),(0,0,0).

Theorem 10:

Let Z_3 be the ring of integers modulo 3. If n is odd then it has exactly 9 n-Fermat's triples.

Proof:

Consider the equation $X^n + Y^n = Z^n; n \geq 3$

Where n is odd. It has the following solutions:

(0,0,0), (1,0,1), (0,1,1), (2,0,2), (0,2,2), (2,2,1), (2,1,0), (1,2,0), (1,1,2).

Theorem 11:

Let Z_3 be the ring of integers modulo 3. If n is even, then it has exactly 5 n-Fermat's triples.

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Proof:

Consider the equation $X^n + Y^n = Z^n; n \geq 3$

Where n is even. It has the following solutions:

(0,0,0), (1,0,1), (0,1,1),(0,2,1),(2,0,1).

Remark 12:

According to the previous two theorems, the set of general Fermat's triples in Z_3 is

{(0,0,0),(1,0,1),(0,1,1)}.

Conclusion

In this paper, we have presented an algorithm to solve the Fermat's Diophantine equation in neutrosophic rings. Also, we solved this Diohantine equation in some numerical rings modulo n.

As a future research direction, we aim to solve the Fermat's Diohantine equation in refined neutrosophic rings and n-refined neutrosophic rings respectively.

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