The Kernel Of Fuzzy and Anti-Fuzzy Groups

Katy D. Ahmad[1,*], Mikail Bal[2] , Malath Aswad[3]

[1]Islamic University Of Gaza, Palestine

[2] Gaziantep University, Turkey

[3]Malath Aswad, Albaath university, Syria

Emails : katyon765@gmail.com[1,*], mikailbal46@hotmail.com[2], malaz.aswad@yahoo.com[3]

Abstract: The aim of this paper is to define the concept of kernel subgroup of a fuzzy group and anti-fuzzy group respectively. Also, we prove that these kernels are groups in the ordinary algebraic meaning, as well as presenting many results about fuzzy groups and anti-fuzzy groups.

Keywords: Fuzzy group, anti-fuzzy group, fuzzy kernel, anti-fuzzy kernel

1.Introduction

Fuzzy set theory began with the work of Zadeh [1], were he has defined fuzzy subsets and relations.

These ideas have been used by many authors to study the algebra of fuzzy sets such as fuzzy groups [2,3], anti-fuzzy groups [20], intuitionistic fuzzy algebras [11] and some other interesting generalizations such as neutrosophic structures [8-9, 12-14].

The concept of neutrosophic group was firstly defined in [2], and studied on a wide range in [4-7,15-19], as well as anti-fuzzy group theory [20], where we find concepts such as fuzzy abelian subgroups, fuzzy nilpotency, anti fuzzy normality, and many other algebraic concepts applied to fuzzy set theory.

In this work, we use the definition of fuzzy and anti fuzzy groups to derive a new subgroup of fuzzy and anti fuzzy group which we have called the fuzzy\anti fuzzy kernel. Also, we define and study the closed normal factors in these groups.

Main discussion

Definition 1

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Let G be a group, $A: G \to [0,1]$, then G is called a fuzzy group if:

1. $A(xy) \geq min(A(x), A(y))$.
2. $A(x^{-1}) = A(x)$ for all $x, y \in G$

Definition 2

Let G be a group, $B: G \to [0,1]$, then G is called anti-fuzzy group if:

1. $B(xy) \leq max(B(x), B(y))$.
2. $B(x^{-1}) = B(x)$ for all $x, y \in G$

Definition 3

Let G be a group, $A, B: G \to [0,1]$, then G is called an intuitionstic-fuzzy group if:

1. $A(xy) \geq min(A(x), A(y))$, $A(x^{-1}) = A(x)$.
2. $B(xy) \leq max(B(x), B(y))$, $B(x^{-1}) = B(x)$.

For all $x, y \in G$

Remark 4:

The intuitionstic-fuzzy group is fuzzy and anti fuzzy 49ogether.

Example 5:

Let $G = (z_5^*, .)$ Be the group of integers modulo 5 with multiplication modulo 5, we define:

$A: G \to [0,1]$; $A(x) = 1$.

$B: G \to [0,1]$; $B(x) = \frac{1}{3}$.

For all $x \in G$.

(G, A, B) is an intuitionstic fuzzy group.

Theorem 6:

1. Let G be a fuzzy group with $A: G \to [0,1]$, then:

   $A(e) \geq A(x)$; $\forall x \in G$.

2. Let G be an anti-fuzzy group with $B: G \to [0,1]$, then:

   $B(e) \leq B(x)$; $\forall x \in G$.

Theorem 7:

Let G be a fuzzy group with $A: G \to [0,1]$, H be a normal subgroup of G with the property $A(x) = A(e)$; $\forall x \in H$, then there exists a function $A_H: G / H \to [0,1]$, such that $(G/H , A_H)$ is a fuzzy group.

Proof.

By the normality of H, we get that G/H is a group.

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Define $A_H: G/H \to [0,1]$; $\begin{cases} A_H(xH) = A(e)\,; x \in H \\ A_H(xH) = A(x)\,; x \in H \end{cases}$

$A_H$is well define mapping.

Assume that $xH = yH$, then $xy^{-1} \in H$.

On the other hand, we have $A(xy^{-1}) = A(e)$, this implies that $A(x) = A(y)$, thus $A_H(xH) = A_H(yH)$.

Now, we check the conditions of a fuzzy group:

$A_H(xH)^{-1} = A_H(x^{-1}H) = A(x^{-1}) = A(x) = A_H(xH);\ x \in H.$

Also, $A_H(xH.yH) = A_H(xyH) = A(xy) \geq min\big(A(x), A(y)\big) = min\big(A_H(xH), A_H(yH)\big)\ if\ xy\ is\ not\ in\ H.$

If $xy \in H$, we have:

$A_H(xH.yH) = A_H(xyH) = A(e) \geq min\big(A(x), A(y)\big) = min\big(A_H(xH), A_H(yH)\big).$

Thus,$(G/H\,, A_H)$ is a fuzzy group.

Definition 8:

Let (G, A) be a fuzzy group, H be normal subgroup with $A(x) = A(e)$ for all $x \in H$.

H is called a fuzzy closed normal factor of G with respect to$A$.

Example 9:

Consider the group $G = (z_5^*, .) = \{1,2,3,4\}$.

Define $A: G \to [0,1]$ such that $A(1) = A(3) = 1, A(2) = A(4) = \frac{1}{2}$.

We have $H = \{1,3\}$ is a normal subgroup of G, and $A(x) = A(1) = 1$ for all $x \in H$, thus H is closed normal factor.

Definition 10:

Let (G, A) be a fuzzy group, we define the fuzzy kernel of $G$with respect to A as follows:

$K_A = \{x \in G;\ A(x) = A(e)\}$.

Theorem 11:

K_A is a subgroup of G.

Proof.

K_A is not empty, that is because $e \in K_A$.

Let x, y be two arbitrary elements of G, we have.

$A(x^{-1}) = A(x) = A(e)$, thus $x^{-1} \in K_A$.

$A(xy) \geq min\big(A(x), A(y)\big) = min\big(A(e), A(e)\big) = A(e)$.

So that $xy \in K_A$ and K_A is a subgroup of G.

Remark 12:

The fuzzy kernel of (G, A)containts any closed normal factor.

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Example 13:

Let $(z_7^*,.) = \{1,2,3,4,5,6\}$ be the group of untegers modulo 7with multiplication.

Define $A: G \to [0,1]$; $\begin{cases} A(1) = A(2) = A(4) = \frac{1}{2} \\ A(3) = A(5) = A(6) = \frac{1}{4} \end{cases}$

$K_A = \{1,2,4\}$ which is a subgroup of G.

Theorem 14:

Let (G, A) be a fuzzy group, K_A be its fuzzy kernel, then.

1. $\forall g \in G, x \in K_A: A(gxg^{-1}) \geq A(g)$.
2. If K_A is normal, then $A(gx) = A(g)$ for all $g \in G, x \in K_A$.

Proof.

1. $A(gxg^{-1}) \geq min\big(A(g), A(x), A(g^{-1})\big) = min\big(A(g), A(e)\big) = A(g)$.
2. Assume that K_A is normal, hence $gxg^{-1} \in K_A$ for all $x \in K_A$ and $g \in G$.

This implies $A(gxg^{-1}) = A(e)$, thus $A(gx) = A(g)$.

Theorem 15:

Let G be an anti fuzzy group with $B: G \to [0,1]$, H be a normal subgroup of G with the property $B(x) = B(e)$; $\forall x \in H$, then there exists a function $B_H: G / H \to [0,1]$, such that $(G/H, B_H)$ is anti fuzzy group.

Proof.

By the normality of H, we get that G/H is a group.

Define $B_H: G/H \to [0,1]$; $\begin{cases} B_H(xH) = B(e)\,; x \in H \\ B_H(xH) = B(x)\,; x \in H \end{cases}$

$B_H$is well define mapping.

Assume that $xH = yH$, then $xy^{-1} \in H$.

On the other hand, we have $B(xy^{-1}) = B(e)$, this implies that $B(x) = B(y)$, thus $B_H(xH) = B_H(yH)$.

Now, we check the conditions of anti fuzzy group:

$B_H(xH)^{-1} = B_H(x^{-1}H) = B(x^{-1}) = B(x) = B_H(xH);\ x \in H.$

Also, $B_H(xH.yH) = B_H(xyH) = B(xy) \leq max\big(B(x), B(y)\big) = max\big(B_H(xH), B_H(yH)\big)\ if\ xy\ is\ not\ in\ H.$

If $xy \in H$, we have:

$B_H(xH.yH) = B_H(xyH) = B(e) \leq max\big(B(x), B(y)\big) = max\big(B_H(xH), B_H(yH)\big).$

Thus,$(G/H, B_H)$ is anti fuzzy group.

Definition 16:

Let (G, B) be anti fuzzy group, H be normal subgroup with $B(x) = B(e)$ for all $x \in H$.

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H is called anti fuzzy closed normal factor of G with respect to$B$.

Example 17:

Consider the group $G = (z_5^*, .) = \{1,2,3,4\}$.

Define $B: G \to [0,1]$ such that $B(1) = B(3) = \frac{1}{2}, B(2) = B(4) = 1$.

We have $H = \{1,3\}$ is a normal subgroup of G, and $B(x) = B(1) = 1$ for all $x \in H$, thus H is closed normal factor.

Definition 18:

Let (G, B) be anti fuzzy group, we define the anti fuzzy kernel of G with respect to B as follows:

$K_B = \{x \in G;\ B(x) = B(e)\}$.

Theorem 19:

K_B is a subgroup of G.

Proof.

K_B is not empty, that is because $e \in K_B$.

Let x, y be two arbitrary elements of G, we have.

$B(x^{-1}) = B(x) = B(e)$, thus $x^{-1} \in K_B$.

$B(xy) \le max\big(B(x), B(y)\big) = max\big(B(e), B(e)\big) = B(e)$.

So that $xy \in K_B$ and K_B is a subgroup of G.

Remark 20:

The anti fuzzy kernel of (G, B)containts any closed normal factor.

Example 21:

Let $(z_7^*, .) = \{1,2,3,4,5,6\}$ be the group of untegers modulo 7with multiplication.

Define $B: G \to [0,1]$; $\begin{cases} B(1) = B(2) = B(4) = \frac{1}{4} \\ B(3) = B(5) = B(6) = \frac{1}{2} \end{cases}$

$K_B = \{1,2,4\}$ which is a subgroup of G.

Theorem 22:

Let (G, B) be anti fuzzy group, K_B be its anti fuzzy kernel, then.

1. $\forall g \in G, x \in K_B: B(gxg^{-1}) \le B(g)$.
2. If K_B is normal, then $B(gx) = B(g)$ for all $g \in G, x \in K_B$.

Proof.

1. $B(gxg^{-1}) \le max\big(B(g), B(x), B(g^{-1})\big) = max\big(B(g), B(e)\big) = B(g)$.
2. Assume that K_B is normal, hence $gxg^{-1} \in K_B$ for all $x \in K_B$ and $g \in G$.

This implies $B(gxg^{-1}) = B(e)$, thus $B(gx) = B(g)$.

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Conclusion

In this paper, we have introduced the concept of fuzzy kernel of a fuzzy group and anti-fuzzy kernel of an anti-fuzzy group. Also, we have proved that these kernels are subgroups by classical algebraic meaning, as well as, we have presented many other properties of fuzzy and anti-fuzzy groups with many examples to clarify the validity of our work.

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