



A Review on Recent Advantages in Algebraic Theory of Neutrosophic Matrices

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Abstract

This work is dedicated to give the reader a wide review for recent advantages in the algebraic study of neutrosophic matrices, refined neutrosophic matrices, and n-refined neutrosophic matrices.

Keywords: refined neutrosophic matrix, n-refined neutrosophic matrix, eigen value, eigen vector.

1. Introduction

Neutrosophy is a new kind of generalized logic proposed by Smarandache [10,21], to deal with indeterminacy in real life and science.

Neutrosophic logic found its way in many branches of human knowledge such as graph theory [33], number theory [3], topology [32], statistics [15], and equations [2].

The neutrosophic algebra was built over the idea of inserting the indeterminacy element I into classical algebraic structures [9]. This idea lead to many concepts such as neutrosophic spaces [11,16,31], neutrosophic modules [12,29,30], neutrosophic rings [18,22], groups [19,39], and functions [8,17,23].

By refining the indeterminacy I into many levels of indeterminacy I_1, \dots, I_n , we get refined and n-refined neutrosophic groups [24], rings [6,36], modules [12,27], and spaces [1,13].

In classical algebra, matrices are playing an important role in the theory of vector spaces. They were generalized to neutrosophic matrices [4,5,26], refined neutrosophic matrices [34], and n-refined neutrosophic matrices [40].

Recently, there is an increasing interest in the algebraic properties of these matrices such as diagonalization problem [37,38], invertibility [38], determinants [38,39,40], and algebraic representations by linear functions [35].

Through this chapter, we review the recent published developments in the algebraic study of neutrosophic matrices, refined neutrosophic matrices, and n -refined neutrosophic matrices. Also, we list some of the most important open questions about these matrices, which may represent the future of this branch of studies.

Elementary Properties of Neutrosophic Matrices

Definition 1: [28] Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $\{x, (\mu A(x), \delta A(x), \gamma A(x)) : x \in X\}$, where $\mu A(x)$, $\delta A(x)$ and $\gamma A(x)$ represent the degree of membership, the degree of indeterminacy, and the degree of non-membership respectively of each element $x \in X$ to the set A .

Definition 2: [9] Let K be a field, the neutrosophic field generated by $\langle K \cup I \rangle$ which is denoted by $K(I) = \langle K \cup I \rangle$.

Definition 3: [9] Classical neutrosophic number has the form $a + bI$ where a, b are real or complex numbers and I is the indeterminacy such that $0 \cdot I = 0$ and $I^2 = I$ which results that

$$I^n = I \text{ for all positive integers } n.$$

Definition 4: [4]

Let $M_{m \times n} = \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$ is a neutrosophic field. We call to be the neutrosophic matrix

Remark 5: [9]

The neutrosophic field $K(I)$ is not a field by classical meaning, since I is not invertible.

Definition 6: [38]

Let $M = A + BI$ a neutrosophic n square matrix, where A and B are two n squares matrices, then M is called an invertible neutrosophic n square matrix, if and only if there exists an n square matrix $S = S_1 + S_2I$, where S_1 and S_2 are two n square matrices such that

$$S \cdot M = M \cdot S = U_{n \times n}, \text{ where } U_{n \times n} \text{ denotes the } n \times n \text{ identity matrix.}$$

Definition 7: [38]

Let $M = A + BI$ be a neutrosophic n square matrix. The determinant of M is defined as

$$\det M = \det A + I[\det(A + B) - \det A].$$

Theorem 8: [38]

Let $M = A + BI$ a neutrosophic square $n \times n$ matrix, where A, B are two squares $n \times n$ matrices, then M is invertible if and only if A and $A + B$ are invertible matrices and

$$M^{-1} = A^{-1} + I[(A + B)^{-1} - A^{-1}].$$

Theorem 9: [38]

M is invertible matrix if and only if $\det M \neq 0$.

Proof:

From **Theorem 8** we find that M is invertible matrix if and only if $A + B, A$ are two invertible matrices, hence $\det[A + B] \neq 0, \det A \neq 0$ which means

$$\det M = \det A + I[\det(A + B) - \det A] \neq 0.$$

Example 10: [38]

Consider the following neutrosophic matrix

$$M = A + BI = \begin{pmatrix} 1 & -1 + I \\ I & 2 + I \end{pmatrix}. \text{ Where } A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(a) $\det A = 2, A + B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}, \det(A + B) = 3, \det M = 2 + I[3 - 2] = 2 + I \neq 0$, hence M is invertible.

(b) We have $A^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, (A + B)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$, thus $M^{-1} = (A^{-1}) + I[(A + B)^{-1} - A^{-1}]$

$$= \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} + I \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{3} & -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2}I \\ -\frac{1}{3}I & \frac{1}{2} - \frac{1}{6}I \end{pmatrix}.$$

(c) We can compute $MM^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = U_{2 \times 2}$.

Theorem 11: [38]

Let $M = A + BI$ and $N = C + DI$ be two neutrosophic n square matrices, then

$$(3.7.1) \det(M \cdot N) = \det M \cdot \det N.$$

$$(3.7.2) \det(M^{-1}) = (\det M)^{-1}.$$

$$(3.7.3) \det M = 1 \text{ if and only if } \det A = \det(A + B) = 1.$$

Remark: The result in the section (c) can be generalized easily to the following fact:

$\det M = \det A$ if and only if $\det A = \det(A + B)$.

Definition 12: [38]

Let $M = A + BI$ be a neutrosophic n square matrix, where A and B are two n square matrices. M is satisfying the orthogonality property if and only if $M \cdot M^T = U_{n \times n}$.

Neutrosophic Eigen Values and Diagonalization Conditions

Definition 13: [38]

Let $M = A + BI$ be a square neutrosophic matrix, we say that M is diagonalizable if and only if there is an invertible neutrosophic matrix $S = C + DI$ such that $S^{-1}MS = D$. Where D is a diagonal neutrosophic matrix (i.e. $d_{ij} = 0 \ \forall i \neq j$, and $d_{ij} \neq 0 \ \forall i = j$).

Theorem 14: [3]

Let $M = A + BI$ be any square neutrosophic matrix. Then M is diagonalizable if and only if $A, A + B$ are diagonalizable.

Remark 15: [38]

If C is the diagonalization matrix of A , and D is the diagonalization matrix of $A + B$, then

$S = C + (D - C)I$ is the diagonalization matrix of $M = A + BI$.

Example 16: [38]

Consider the neutrosophic matrix defined in Example 10, we have:

(a) A is a diagonalizable matrix. Its diagonalization matrix is $C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, the corresponding diagonal matrix is $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, we can see that $C^{-1}AC = D_1$. Also, the diagonalization matrix of $A + B$ is $D =$

$\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. It is easy to check that

$$D^{-1}(A + B)D = D_2 .$$

(b) Since $A, A + B$ are diagonalizable, then M is diagonalizable. The neutrosophic diagonalization matrix of M is $S = C + (D - C)I = \begin{pmatrix} 1 & 1 - I \\ -\frac{1}{2}I & -1 + 2I \end{pmatrix}$. The corresponding diagonal matrix is

$$L = D_1 + I[D_2 - D_1] = \begin{pmatrix} 1 & 0 \\ 0 & 2 + I \end{pmatrix}.$$

(c) It is easy to see that $S^{-1} = C^{-1} + I[D^{-1} - C^{-1}] = \begin{pmatrix} 1 & 1 - I \\ \frac{1}{2}I & -1 + 2I \end{pmatrix}$.

(d) We can compute $S^{-1}MS = \begin{pmatrix} 1 & 1 - I \\ \frac{1}{2}I & -1 + 2I \end{pmatrix} \begin{pmatrix} 1 & -1 + I \\ I & 2 + I \end{pmatrix} \begin{pmatrix} 1 & 1 - I \\ -\frac{1}{2}I & -1 + 2I \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 + I \end{pmatrix} = L$.

Definition 17: [38]

Let $M = A + BI$ be a n square neutrosophic matrix over the neutrosophic field $F(I)$, we say that $Z = X + YI$ is a neutrosophic Eigen vector if and only if $MZ = (a + bI)Z$. The neutrosophic number $a + bI$ is called the Eigen value of the eigen vector Z .

Theorem 18: [38]

Let $M = A + BI$ be a n square neutrosophic matrix, then $a + bI$ is an eigen value of M if and only if a is an eigen value of A , and $a + b$ is an eigen value of $A + B$. As well as, the eigen vector of M is $Z = X + YI$ if and only if X is the corresponding eigen vector of A , and $X + Y$ is the corresponding eigen vector of $A + B$.

Proof:

We suppose that $Z = X + YI$ is an eigen vector of M with the corresponding eigen value $a + bI$, hence $MZ = (a + bI)Z$, this implies

$$(A + BI)(X + YI) = (a + bI)(X + YI), \text{ thus } AX + I[(A + B)(X + Y) - AX] = aX + I[(a + b)(X + Y) - aX].$$

We get:

$AX = aX, (A + B)(X + Y) = (a + b)(X + Y)$, so that X is an eigen vector of A , $X + Y$ is an eigen vector of $A + B$. The corresponding eigen value of X is a , and the corresponding eigen value of $X + Y$ is $a + b$.

For the converse, we assume that X is an eigen vector of A with a as the corresponding eigen value, and $X + Y$ is an eigen vector of $A + B$ with $a + b$ as the corresponding eigen value, so that we get $AX = aX, (A + B)(X + Y) = (a + b)(X + Y)$.

Let us compute

$$MZ = (A + BI)(X + YI) = AX + I[(A + B)(X + Y) - AX]$$

$$= aX + I[(a + b)(X + Y) - aX] = (a + bI)(X + YI) = (a + bI)Z.$$

Thus $Z = X + YI$ is an eigen vector of M with $a + bI$ as a neutrosophic eigen value.

Theorem 19: [38]

The eigen values of a neutrosophic matrix $M = A + BI$ can be computed by solving the neutrosophic equation $\det(M - (a + bI)U_{n \times n}) = 0$.

Example 20: [38]

Consider M the neutrosophic matrix defined in Example 10, we have

(a) The eigen values of the matrix A are $\{1,2\}$, and $\{1,3\}$ for the matrix $A + B$. This implies that the eigen values of the neutrosophic matrix M are

$$\{1 + (3 - 1)I, 1 + (1 - 1)I, 2 + (3 - 2)I, 2 + (1 - 2)I\} = \{1 + 2I, 1, 2 + I, 2 - I\}.$$

(b) If we solved the equation $\det(M - (a + bI)U_{n \times n}) = 0$ has been solved, the same values will be gotten.

(c) The eigen vectors of A are $\{(1,0), (1, -1)\}$, the eigen vectors of $A + B$ are $\{(1, -1/2), (0,1)\}$. Thus, the neutrosophic eigen vectors of M are

$$\left\{ (1,0) + I[(0,1) - (1,0)], (1,0) + I\left[\left(1, -\frac{1}{2}\right) - (1,0)\right], (1, -1) + I[(0,1) - (1, -1)], (1, -1) + I\left[\left(1, -\frac{1}{2}\right) - (1, -1)\right] \right\} = \{(1,0) + I(-1,1), (1,0) + I(0, -1/2), (1, -1) + I(-1,2), (1, -1) + I(0,1/2)\} = \{(1 - I, I), (1, -1/2I), (1 - I, -1 + 2I), (1, -1 + 1/2I)\}.$$

Neutrosophic matrices Represented as linear transformations

Theorem 21: [35]

Let V, W be two vector spaces over the field F with $\dim(V) = n, \dim(W) = m, V(I), W(I)$ be the corresponding neutrosophic vector spaces over the corresponding neutrosophic field $F(I)$. Let $g, h: V \rightarrow W$ be two linear transformations, then there exists a neutrosophic linear transformation $f = g + hI: V(I) \rightarrow W(I)$, where f is defined as follows: $f(x + yI) = g(x) + [(g + h)(x + y) - g(x)]I$.

Definition 22: [35]

The neutrosophic linear transformation f defined in Theorem 21 is called a full AH- linear transformation.

Definition 23: [35]

Let $f = g + hI: V(I) \rightarrow W(I)$ be a full AH- linear transformation, $M = A + BI$ be an $n \times m$ neutrosophic matrix over $F(I)$, we call M the neutrosophic matrix of f if and only if $f(x + yI) = M(x + yI)$ for every $x + yI \in V(I)$.

Theorem 24: [35]

Let $f = g + hI: V(I) \rightarrow W(I)$ be any full AH- linear transformation, then $M = A + BI$ is the corresponding neutrosophic matrix if and only if A is the matrix of g , B is the matrix of h .

Example 25: [35]

(a) Let $V(I) = R^2(I) = \{(a, b) + (c, d)I = (a + cI, b + dI); a, b, c, d \in R\}$, consider the following neutrosophic matrix

$M = \begin{pmatrix} 1+I & I \\ -I & 2-I \end{pmatrix}$. The corresponding neutrosophic linear transformation is defined as follows:

$$f(x + yI) = M \cdot \begin{pmatrix} a + cI \\ b + dI \end{pmatrix} = (a + I[c + a + c + b + d], -aI - cI + 2b + 2dI - bI - dI) = (a + I[a + 2c + b + d], 2b + I[-a - c - b + d]) = (a, 2b) + I(a + 2c + b + d, -a - c - b + d).$$

(b) $f = g + hI; g(x, y) = (x, 2y), h(x, y) = (x + y, -x - y)$. Where $g, h: V \rightarrow V$.

Theorem 26: [35]

Let V, W be two vector spaces over the field F , with $\dim(V) = n, \dim(W) = m$, let $M = A + BI$ be any $n \times m$ neutrosophic matrix over $F(I)$. Then M can be represented by a unique full AH-linear transformation $f = g + hI$, where A is the matrix of g and B is the matrix of h .

Proof:

According to Theorem 24, the neutrosophic matrix $M = A + BI$ can be represented by a neutrosophic full AH-linear transformation $f = g + hI$, where A is the matrix of g and B is the matrix of h . For the uniqueness condition, we suppose that $F = G + HI$ is another linear AH-transformation with the property

$M(x + yI) = F(x + yI)$. We have:

$M.(x + yI) = F(x + yI) = f(x + yI)$ for all $x + yI \in V(I)$. Thus $F = f$ and f is unique.

The following theorem shows an algorithm to find a basis for the neutrosophic vector space $V(I)$ from any basis of the corresponding classical vector space V .

Theorem 27: [35]

Let $V(I)$ be any neutrosophic vector space over the neutrosophic field $F(I)$, V be its corresponding classical vector space over the field F . Let $S = \{v_1, v_2, \dots, v_n\}$ be a basis of V over F , then

$L = \{l_{ij} = v_i + (v_j - v_i)I; 1 \leq i, j \leq n\}$ is a basis of $V(I)$ over $F(I)$.

Example 28: [35]

It is well known that $\{x=(1,0), y=(0,1)\}$ is a basis of $V=R^2$. The corresponding basis of $V(I)=R^2(I)$ is

$\{x, y, x + (y - x)I, y + (x - y)I\} = \{(1,0), (0,1), (1,0) + (-1,1)I, (0,1) + (1, -1)I\}$.

The following theorem shows that every linear transformation between $V(I)$ and $W(I)$ must be a full AH-linear transformation.

Theorem 29: [35]

Let V, W be two vector spaces over the field F , with $\dim(V) = n, \dim(W) = m$, let $V(I), W(I)$ be the corresponding neutrosophic vector spaces over $F(I)$. Let $f: V(I) \rightarrow W(I)$ be any linear transformation, then f is a full AH-linear transformation.

Refined neutrosophic Matrices

Definition 31: [34]

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}$ be an $n \times m$ matrix, if $a_{ij} = x + yI_1 + zI_2 \in R_2(I)$, then it is called an refined neutrosophic matrix. Where $R_2(I)$ is an refined neutrosophic field.

Example 32: [34]

$X = \begin{pmatrix} I_1 & I_1 + I_2 \\ 3 - I_1 & 2I_2 \end{pmatrix}$, is a 2×2 refined neutrosophic matrix.

Remark 33: [34]

(a) If A is an $m \times n$ matrix, then it can be represented as an element of the refined neutrosophic ring of matrices like the following: $A = B + CI_1 + DI_2$. Where D, B, C are classical matrices with elements from the ring R and from size $m \times n$.

For example $A = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}I_2$.

(b) The addition operation can be defined by using the representation in Remark 3.2 as follows:

$$(A + BI_1 + CI_2) + (X + YI_1 + ZI_2) = (A + X) + (B + Y)I_1 + (C + Z)I_2.$$

(c) Multiplication can be defined by using the same representation as a special case of multiplication on refined neutrosophic rings as follows:

$$(A + BI_1 + CI_2)(X + YI_1 + ZI_2) = (AX) + (AY + BX + BY + BZ + CY)I_1 + (AZ + CZ + CX)I_2.$$

This method of multiplication is exactly equivalent to the normal multiplication between matrices; but it is easier to deal with in this way.

Example 34: [34]

Let $X = \begin{pmatrix} I_1 & I_1 + I_2 \\ 3 - I_1 & 2I_2 \end{pmatrix}, Y = \begin{pmatrix} -1 & I_1 \\ 1 + I_2 & 3I_1 \end{pmatrix}$ be two refined neutrosophic matrices over the refined neutrosophic field of reals. We have:

$$(a) X = A + BI_1 + CI_2; A = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

$$(b) Y = M + NI_1 + SI_2; M = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$(c) X + Y = \begin{pmatrix} -1 + I_1 & 2I_1 + I_2 \\ 4 - I_1 + I_2 & 3I_1 + 2I_2 \end{pmatrix}.$$

$$(d) XY = \begin{pmatrix} -I_1 + (I_1 + I_2)(1 + I_2) & I_1I_1 + (I_1 + I_2)(3I_1) \\ -3 + I_1 + (2I_2)(1 + I_2) & (3 - I_1)(I_1) + (2I_2)(3I_1) \end{pmatrix} = \begin{pmatrix} I_1 + 2I_2 & 7I_1 \\ -3 + I_1 + 4I_2 & 8I_1 \end{pmatrix}.$$

(e) If we computed the multiplication using the previous representation, we get:

$$AM = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix}, AN = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, BM = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, BN = \begin{pmatrix} 0 & 4 \\ 0 & -1 \end{pmatrix}, BS = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, CN = \begin{pmatrix} 0 & 3 \\ 0 & 6 \end{pmatrix}, AS = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, CS = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, CM = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}.$$

$$\text{Hence, } XY = AM + I_1(AN + BN + BM + BS + CN) + I_2(AS + CS + CM) = \begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix} + I_1 \begin{pmatrix} 1 & 7 \\ 1 & 8 \end{pmatrix} + I_2 \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} I_1 + 2I_2 & 7I_1 \\ -3 + I_1 + 4I_2 & 8I_1 \end{pmatrix}.$$

Theorem 35: [34]

The set of all square $n \times n$ refined neutrosophic matrices together make a ring.

Proof:

The proof holds directly from the definition of n-refined neutrosophic rings by taking $n = 2$.

Remark 36: [34]

The identity with respect to multiplication is the normal unitary matrix.

Definition 37: [34]

Let A be a square $n \times n$ refined neutrosophic matrix, then it is called invertible if there exists a refined square $n \times n$ neutrosophic matrix B such that $AB = U_{n \times n}$. Where $U_{n \times n}$ is the unitary classical matrix.

Theorem 38: [34]

Let $X = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix, then it is invertible if and only if $A, A + C, A + B + C$ are invertible. The inverse of X is $X^{-1} = A^{-1} + ((A + B + C)^{-1} - (A + C)^{-1})I_1 + ((A + C)^{-1} - A^{-1})I_2$.

Definition 39: [34]

We defined the determinant of a square $n \times n$ refined neutrosophic matrix as $\det X = \det A + [\det(A + B + C) - \det(A + C)]I_1 + [\det(A + C) - \det A]I_2$.

This definition is supported by the condition of invertibility.

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Theorem 40: [34]

Let $X=A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix, we have:

- (a) X is invertible if and only if $\det X \neq 0$.
- (b) If $Y = M + NI_1 + SI_2$ is a square $n \times n$ refined neutrosophic matrix, then $\det XY = \det X \det Y$.
- (c) $\det X^{-1} = (\det X)^{-1}$.

Proof:

(a) If $\det X \neq 0$, this will be equivalent to $\det A \neq 0, \det(A + C) \neq 0, \det(A + B + C) \neq 0$, i.e. $A, A + C, A + B + C$ are invertible, thus X is invertible.

(b) $XY = AM + I_1[(A + B + C)(M + N + S) - (A + C)(M + S)] + I_2[(A + C)(M + S) - AM]$. Hence $\det XY = \det(AM) + I_1[\det((A + B + C)(M + N + S))] + I_2[\det((A + C)(M + S))]=$

$\det A \det M + I_1[\det(A + B + C) \det(M + N + S)] + I_2[\det(A + C) \det(M + S)]=$

$(\det A + I_1[\det(A + B + C) - \det(A + C)] + I_2[\det(A + C) - \det A])(\det M + I_1[\det(M + N + S) - \det(M + S)] + I_2[\det(M + S) - \det M]) = \det X \det Y$.

(c) It holds directly from (b).

Theorem 41: [34]

Let $X=A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix, we have:

- (a) X is nilpotent if and only if $A, A + C, A + B + C$ are nilpotent.
- (b) X is idempotent if and only if $A, A + C, A + B + C$ are idempotent.

Proof:

(a) First of all we will prove that $X^r = A^r + I_1[(A + B + C)^r - (A + C)^r] + I_2[(A + C)^r - A^r]$.

We use the induction, for $r = 1$ it is clear. Suppose that it is true for $r = k$, we prove it for $k + 1$.

$X^{k+1} = X^k X = (A^k + I_1[(A + B + C)^k - (A + C)^k] + I_2[(A + C)^k - A^k])(A + BI_1 + CI_2)=$

$A^{k+1} + I_1[A^k B + (A + B + C)^k A + (A + B + C)^k B + (A + B + C)^k C - (A + C)^k A - (A + C)^k B - (A + C)^k C + (A + C)^k B - A^k B] + I_2[A^k C + (A + C)^k C - A^k C + (A + C)^k A - A^k A]=$

$A^{k+1} + I_1[(A + B + C)^{k+1} - (A + C)^{k+1}] + I_2[(A + C)^{k+1} - A^{k+1}]$.

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X is nilpotent if there is a positive integer r such that $X^r = O_{n \times n}$. This is equivalent to

$$A^r = (A + C)^r = (A + B + C)^r = O_{n \times n}, \text{ which implies the proof.}$$

(b) The proof is similar to (a).

Example 42: [34]

Consider the following refined neutrosophic matrix $A = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix}$, we have:

$$(a) A = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2$$

$$\text{Where } B = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix}. B + D = \begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}, B + C + D = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}.$$

$$(b) B^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}, (B + D)^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ -\frac{7}{5} & 1 \end{pmatrix}, (B + C + D)^{-1} = \begin{pmatrix} \frac{2}{19} & \frac{1}{19} \\ -\frac{7}{19} & \frac{6}{19} \end{pmatrix}.$$

$$(c) A^{-1} = B^{-1} + I_1[(B + C + D)^{-1} - (B + D)^{-1}] + I_2[(B + D)^{-1} - B^{-1}] = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} + I_1 \begin{pmatrix} -\frac{9}{95} & \frac{1}{19} \\ \frac{98}{95} & -\frac{13}{19} \end{pmatrix} +$$

$$I_2 \begin{pmatrix} \frac{6}{5} & -1 \\ -\frac{22}{5} & 3 \end{pmatrix} = \begin{pmatrix} -1 - \frac{9}{95}I_1 + \frac{6}{5}I_2 & 1 + \frac{1}{19}I_1 - I_2 \\ 3 + \frac{98}{95}I_1 - \frac{22}{5}I_2 & -2 - \frac{13}{19}I_1 + 3I_2 \end{pmatrix}.$$

It is easy to find that $A^{-1}A = AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$(d) \det B = -1, \det(B + D) = 5, \det(B + C + D) = 19, \det A = -1 + I_1[19 - 5] + I_2[5 - (-1)] = -1 + 14I_1 + 6I_2.$$

If we computed the determinant of A by using the classical way, we will get the same result.

Now, we illustrate an example to clarify the application of refined neutrosophic matrices in solving refined neutrosophic algebraic equations defined in [2].

Example 43: [34]

Consider the following system of refined neutrosophic linear equations:

$$(2 + I_1 + 3I_2)X + (1 - I_1 - I_2)Y = -I_1 (*), (3 + 4I_2)X + (1 + I_1)Y = I_2 (**).$$

The corresponding refined neutrosophic matrix is $A = \begin{pmatrix} 2 + I_1 + 3I_2 & 1 - I_1 - I_2 \\ 3 + 4I_2 & 1 + I_1 \end{pmatrix}$.

Since A is invertible, we get the solution of the previous system by computing the product:

$$A^{-1} \begin{pmatrix} -I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} -1 - \frac{9}{95}I_1 + \frac{6}{5}I_2 & 1 + \frac{1}{19}I_1 - I_2 \\ 3 + \frac{98}{95}I_1 - \frac{22}{5}I_2 & -2 - \frac{13}{19}I_1 + 3I_2 \end{pmatrix} \begin{pmatrix} -I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} I_1 \left[1 + \frac{9}{95} - \frac{6}{5} + \frac{1}{19} \right] \\ I_1 \left[-3 - \frac{98}{95} + \frac{22}{5} - \frac{13}{19} \right] + I_2[-2 + 3] \end{pmatrix} = \begin{pmatrix} -I_1 \frac{1}{19} \\ -\frac{6}{19}I_1 + I_2 \end{pmatrix}. \text{ Thus } X = -\frac{1}{19}I_1, Y = -\frac{6}{19}I_1 + I_2$$

n-Refined Neutrosophic Matrices

Definition 44: [40]

Let $A = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{pmatrix}$ be an $m \times n$ matrix, if $a_{ij} = x + yI_1 + zI_2 + \dots + tI_n \in R_n(I)$, then it is called an n-refined neutrosophic matrix. Where $R_n(I)$ is an n-refined neutrosophic ring.

Remark 45: [40]

If A is an $m \times n$ matrix, then it can be represented as an element of the n-refined neutrosophic ring of matrices like the following: $A = B + CI_1 + DI_2 + \dots + KI_n$. Where D, B, C, \dots, K are classical matrices with elements from the ring R and from size $m \times n$.

For example $A = \begin{pmatrix} 2 + I_1 + 3I_2 - I_3 & 1 - I_1 - I_2 \\ 3 + 4I_2 + 2I_3 & 1 + I_1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} I_1 + \begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} I_2 + \begin{pmatrix} -1 & 0 \\ 2 & 0 \end{pmatrix} I_3$ is a 3-refined neutrosophic matrix.

Remark 47: [40]

The identity with respect to multiplication is the normal unitary matrix.

Definition 48: [40]

Let A be a square $m \times m$ n-refined neutrosophic matrix, then it is called invertible if there exists an n-refined square $m \times m$ neutrosophic matrix B such that $AB = U_{m \times m}$. Where $U_{m \times m}$ is the unitary classical matrix.

Definition 49: [40]

Let $X = A_0 + A_1I_1 + \dots + A_nI_n$ be an n-refined neutrosophic element, we define its canonical sequence as follows:

$$M_0 = A_0, M_j = A_0 + A_j + A_{j+1} + \dots + A_n; 1 \leq j \leq n. \text{ For example } M_3 = A_0 + A_3 + A_4 + \dots + A_n.$$

Remark 50: [40]

The multiplication operation between two n-refined neutrosophic elements can be represented by the following equation:

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$(A_0 + A_1I_1 + \dots + A_nI_n)(B_0 + B_1I_1 + \dots + B_nI_n) = M_0N_0 + (M_nN_n - M_0N_0)I_n + \sum_{i=1}^{n-1}(M_iN_i - M_{i+1}N_{i+1})I_i$,
where M_i, N_i are the canonical sequences of $A_0 + A_1I_1 + \dots + A_nI_n, B_0 + B_1I_1 + \dots + B_nI_n$ respectively.

Theorem 51: [40]

Let $X=A_0 + A_1I_1 + \dots + A_nI_n$ be an n -refined neutrosophic element, then it is invertible if and only if $M_j; 0 \leq j \leq n$ are invertible. The inverse of X is $X^{-1} = (M_0)^{-1} + (M_n^{-1} - M_0^{-1})I_n + \sum_{j=1}^{n-1}(M_j^{-1} - M_{j+1}^{-1})I_j = (A_0)^{-1} + ((A_0 + A_1 + \dots + A_n)^{-1} - ((A_0 + A_2 + \dots + A_n)^{-1})I_1 + ((A_0 + A_2 + \dots + A_n)^{-1} - ((A_0 + A_3 + \dots + A_n)^{-1})I_2 + ((A_0 + A_3 + \dots + A_n)^{-1} - (A_0 + A_4 + \dots + A_n)^{-1})I_3 + \dots + ((A_0 + A_n)^{-1} - (A_0)^{-1})I_n$.

Example 52: [40]

Consider $Z(I) = \{a + bI_1 + cI_2; a, b, c \in Z_2\}$ the 2-refined neutrosophic ring of integers, the set of invertible elements in Z_2 is $\{-1, 1\}$. Hence the set of all invertible elements in the corresponding 2-refined neutrosophic ring is $\{1, -1, 1 - 2I_2, -1 + 2I_2, 1 - 2I_1, -1 + 2I_1, 1 + 2I_1 - 2I_2, -1 - 2I_1 + 2I_2\}$.

Remark 53: [40]

Let $X=A_0 + A_1I_1 + \dots + A_nI_n$ be a square $m \times m$ n -refined neutrosophic matrix, then it is invertible if and only if $M_j; 0 \leq j \leq n$ are invertible. The inverse of X is $X^{-1} = (M_0)^{-1} + (M_n^{-1} - M_0^{-1})I_n + \sum_{j=1}^{n-1}(M_j^{-1} - M_{j+1}^{-1})I_j = (A_0)^{-1} + (((A_0 + A_1 + \dots + A_n)^{-1} - ((A_0 + A_2 + \dots + A_n)^{-1})I_1 + ((A_0 + A_2 + \dots + A_n)^{-1} - ((A_0 + A_3 + \dots + A_n)^{-1})I_2 + ((A_0 + A_3 + \dots + A_n)^{-1} - (A_0 + A_4 + \dots + A_n)^{-1})I_3 + \dots + ((A_0 + A_n)^{-1} - (A_0)^{-1})I_n$.

Definition 54: [40]

We defined the determinant of a square $m \times m$ n -refined neutrosophic matrix as $\det X = \det A_0 + [\det((A_0 + A_1 + \dots + A_n) - \det((A_0 + A_2 + \dots + A_n))]I_1 + [\det((A_0 + A_2 + \dots + A_n) - \det((A_0 + A_3 + \dots + A_n))]I_2 + \dots + [\det(A_0 + A_n) - \det(A_0)]I_n = \det(M_0) + (\det(M_n) - \det(M_0))I_n + \sum_{i=1}^{n-1}(\det(M_i) - \det(M_{i+1}))I_i$.

Theorem 55: [40]

Let $X=A_0 + A_1I_1 + \dots + A_nI_n$ be a square $m \times m$ n -refined neutrosophic matrix, we have:

- X is invertible if and only if $\det X \neq 0$.
- If $Y = B_0 + B_1I_1 + \dots + B_nI_n$ is a square $m \times m$ n -refined neutrosophic matrix, then $\det XY = \det X \det Y$.
- $\det X^{-1} = (\det X)^{-1}$.

The Diagonalization Algorithm of Refined Neutrosophic Matrices**Definition 56: [43]**

Let $L = A + BI_1 + CI_2$ be a refined neutrosophic matrix, and $M = X + YI_1 + ZI_2$ is a strong refined neutrosophic vector, then it is called a refined neutrosophic eigen vector of L if and only if

$$LM = (a + bI_1 + cI_2)M.$$

The refined neutrosophic number $a + bI_1 + cI_2$ is called a refined neutrosophic eigen value.

Theorem 57: [43]

Let $L = A + BI_1 + CI_2$ be a refined neutrosophic matrix, then $M = X + YI_1 + ZI_2$ is a refined neutrosophic eigen vector with $a + bI_1 + cI_2$ as the corresponding eigen value if and only if $X, X + Z, X + Y + Z$ are eigen vectors of $A, A + C, A + B + C$ respectively. As well as, $a, a + c, a + b + c$ are the corresponding eigen values respectively.

Theorem 58: [43]

Eigen values can be gotten by solving the refined neutrosophic equation $\det(L - (a + bI_1 + cI_2)U_{n \times n}) = 0$.

Example 59: [43]

Consider the following refined neutrosophic matrix $L = \begin{pmatrix} 1 - I_1 + I_2 & 1 - I_2 \\ -I_1 + I_2 & -1 - I_1 + 4I_2 \end{pmatrix}$.

(a) L is written as $L = A + BI_1 + CI_2$. Where $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}$.

(b) We have $A + C = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}, A + B + C = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. The set of eigen values of A is $\{1, -1\}$, for $A + C$ it is $\{2, 3\}$, and for $A + B + C$ it is $\{1, 2\}$.

(c) The set of refined neutrosophic eigen values of L is

$\{1 + I_1[1 - 2] + I_2[2 - 1], 1 + I_1[2 - 2] + I_2[2 - 1], 1 + I_1[1 - 3] + I_2[3 - 1], 1 + I_1[2 - 3] + I_2[3 - 1], -1 + I_1[1 - 2] + I_2[2 + 1], -1 + I_1[2 - 2] + I_2[2 + 1], -1 + I_1[1 - 3] + I_2[3 + 1], -1 + I_1[2 - 3] + I_2[3 + 1]\} = \{1 - I_1 + I_2, 1 + I_2, 1 - 2I_1 + 2I_2, 1 - I_1 + 2I_2, -1 - I_1 + 3I_2, -1 + 3I_2, -1 - 2I_1 + 4I_2, -1 - I_1 + 4I_2\}$.

(d) We get the same values by solving the following refined neutrosophic equation:

$\det(L - (a + bI_1 + cI_2)U_{n \times n}) = (a + bI_1 + cI_2)^2 + (a + bI_1 + cI_2)(2I_1 - 5I_2) + (-1 - 4I_1 + 7I_2) = 0$. For the solution, we can use the algebraic algorithm which was introduced in [3].

Definition 60: [43]

Let $X = A + BI_1 + CI_2$ be any refined neutrosophic matrix, then it is called diagonalizable if there exists an invertible refined neutrosophic matrix $Y = F + GI_1 + HI_2$ such that $Y^{-1}XY = D$. Where D is a refined neutrosophic diagonal matrix.

Theorem 61: [43]

Let $X = A + BI_1 + CI_2$ be any refined neutrosophic matrix, then it is diagonalizable if and only if $A, A + C, A + B + C$ are diagonalizable.

Remark 62: [43]

If F, G, H are the diagonalization matrices of $A, A + B + C, A + C$ respectively, then $F + (G - H)I_1 + (H - F)I_2$ is the diagonalization matrix of X . Also, the corresponding diagonal matrix of X is $D = D_0 + (D_1 - D_2)I_1 + (D_2 - D_0)I_2$, where D_0, D_1, D_2 are the corresponding diagonal matrices of $A, A + B + C, A + C$ respectively.

Example 63: [43]

Consider the refined neutrosophic matrix defined in the previous Example, we have:

(a) The diagonalization matrix of A is $F = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$, the corresponding diagonal matrix is $D_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It is clear that $F^{-1}AF = D_0$.

(b) The diagonalization matrix of $A + B + C$ is $G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. It is clear that $G^{-1}(A + B + C)G = D_1$.

(c) The diagonalization matrix of $A + C$ is $H = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, the corresponding diagonal matrix is $D_2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. It is clear that $H^{-1}(A + C)H = D_2$.

(d) The refined neutrosophic diagonalization matrix of L is :

$Y = F + (G - H)I_1 + (H - F)I_2 = \begin{pmatrix} 1 & 1 - I_2 \\ -2 + I_1 + I_2 & I_2 \end{pmatrix}$. The corresponding diagonal matrix is

$D = D_0 + (D_1 - D_2)I_1 + (D_2 - D_0)I_2 = \begin{pmatrix} -1 - I_1 + 3I_2 & 0 \\ 0 & 1 - I_1 + 2I_2 \end{pmatrix}$.

(e) $F^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$, $G^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $Y^{-1} = F^{-1} + (G^{-1} - H^{-1})I_1 + (H^{-1} - F^{-1})I_2$

$= \begin{pmatrix} I_2 & -\frac{1}{2} + \frac{1}{2}I_2 \\ 1 - I_1 & \frac{1}{2} + \frac{1}{2}I_2 \end{pmatrix}$, $Y^{-1}LY = D$.

Conclusion

In this review, we gave the interested reader a good background about the algebraic theory of refined neutrosophic matrices, neutrosophic matrices, and n-refined neutrosophic matrices. We discussed the recent advantages in the field of finding eigen vectors and values and diagonalizing matrices. Also, some elementary properties about invertibility and determinants were revised.

According to this review, we list the following open questions, which may represent the future of research in the algebra of neutrosophic matrix theory.

Future Research Directions

1-) How refined neutrosophic matrices can be represented by AH-linear transformations? Describe these transformations.

2-) How n-refined neutrosophic matrices can be represented by AH-linear transformations?

Describe these transformations.

3-) Find an algorithm to compute the eigen values/vectors of n-refined neutrosophic matrices.

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4-) Determine the necessary and sufficient conditions for the diagonalization of n-refined neutrosophic matrices.

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