A Contribution To Kothe's Conjecture In Refined Neutrosophic Rings

Arwa A. Hajjari, Rozina Ali,

Cairo University, Egypt

Correspondance: rozyyy123n@gmail.com

Abstract

The objective of this paper is to answer an open question asked in [42], about the equivalence between Kothe's conjecture in a ring R and its corresponding refined neutrosophic ring $R(I_1, I_2)$. Where it proves that Kothe's conjecture is true in R if and only if it is true in $R(I_1, I_2)$.

Keywords: refined neutrosophic ring, Kothe Conjecture, Nilpotent Ideal.

1.Introduction

Neutrosophic algebra began with the work of Smarandache and Kandasamy in [6], where they presented concepts such as neutrosophic groups, neutrosophic rings and loops.

This motivates researchers to study a lot of extensions of classical algebraic structures by using neutrosophic ideas. See [2,3,4,17,46].

In [28,30], we find an interesting generalization of neutrosophic rings, where the concept of refined neutrosophic rings was invented. Many interesting properties of these rings were discussed such as congruencies [10], partial ordering [44], AH-homomorphisms [15], and idempotency [45].

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Refined neutrosophic rings were applicable in other algebraic structures, where refined neutrosophic modules [19], refined matrices [31,35], can be built over these rings in a similar way to classical cases.

In [9], Abobala has proved that every neutrosophic ring is a homomorphic image of the corresponding refined neutrosophic ring.

Recently, the structure of maximal and minimal full ideals of refined neutrosophic rings was presented [5]. This structure implies an inclusion between the classical parts of this ideal.

In this article, we will show that inclusion's condition is not true in the case of rings with no unity. In particular, we determine the condition of any full ideal to be nil in any refined neutrosophic ring $R(I_1, I_2)$. Also, an equivalence between Kothe's conjecture in classical ring R and the corresponding refined neutrosophic ring $R(I_1, I_2)$ will be presented.

The motivation of our work is the open question listed in [42].

2. Preliminaries.

For the concepts of neutrosophic rings, refined neutrosophic rings, and ideals. See [30,32,42,45].

Kothe Conjecture:

If R is a ring, then the sum of any two left nil ideals is nil again.

Theorem [5]:

Consider the following:

$R(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in R\}$ be a refined neutrosophic ring, $M = (P, QI_1, SI_2)$ be a subset of $R(I_1, I_2)$

. M is an ideal of $R(I_1, I_2)$ if and only if

(a) P, Q, S are ideals on R

(b)$P \leq S \leq Q$.

Open problem [42]:

If Kothe's conjecture is true in the ring R, then is it true in the corresponding refined neutrosophic ring R(I_1, I_2).

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3. Main discussion.

In [5], authors have proved that $P = (P_0, P_1 I_1, P_2 I_2)$ is a full ideal of $R(I_1, I_2)$ if and only if P_0, P_1, P_2 *are ideals in R with* $P_0 \leq P_2 \leq P_1$, under the condition that R has unity. First of all, we give an example to show that this statement is not true if R has not a unity.

Example 3.1:

Consider $2Z(I_1, I_2) = \{(2a, 2bI_1, 2cI_2); a, b, c \in Z\}$ the refined neutrosophic ring of even integers, let $P = (2Z, 4ZI_1, 4ZI_2) = \{(2a, 4bI_1, 4cI_2); a, b, c \in Z\}$ be a subset of it. First of all, we show that P is a full ideal of $2Z(I_1, I_2)$. It is easy to see that (P,+) is a subgroup. Let $x = (2m, 4nI_1, 4tI_2)$ be any element of P, $r = (2a, 2bI_1, 2cI_2)$ be any element of $2Z(I_1, I_2)$, we have:

$rx = (4am, I_1[8an + 4bm + 8bn + 8bt + 8cn], I_2[8at + 8ct + 4cm]) \in P$. Thus P is a full ideal and the inclusion's condition is not available, that is because 2Z is not contained in 4Z.

The following theorem describes the structure of nil ideals in $R(I_1, I_2)$.

Theorem 3.2:

Let $R(I_1, I_2)$ be any neutrosophic ring, we have:

(a) (x, yI_1, zI_2) is nilpotent in $R(I_1, I_2)$ if and only if $x, x + z, x + y + z$ are nilpotent elements in R.

(b) If $P = (Q, MI_1, NI_2)$ is an ideal of $R(I_1, I_2)$, then P is nilpotent if and only if $Q, M, N, Q + N, Q + M + N$ are nilpotent.

(c) If $P = (Q, MI_1, NI_2)$ is a right/left ideal of $R(I_1, I_2)$, then P is nil if and only if $Q, M, N, Q + N, Q + M + N$ are nil.

Proof:

(a) In the beginning, we show that $(x, yI_1, zI_2)^n = (x^n, I_1[(x + y + z)^n - (x + z)^n], I_2[(x + z)^n - x^n])$, where n is any positive integer.

For n=1 it is clear. We suppose that it is true for $n = k$, we shall prove it for k+1.

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$(x, yI_1, zI_2)^{k+1} = (x, yI_1, zI_2)^k (x, yI_1, zI_2) = \left(x^k, I_1[(x+y+z)^k - (x+z)^k], I_2[(x+z)^k - x^k]\right).(x, yI_1, zI_2) = (x^{k+1}, I_1[x(x+y+z)^k - x(x+z)^k + y(x+y+z)^k - y(x+z)^k + yx^k - yx^k + y(x+z)^k + z(x+y+z)^k - z(x+z)^k], I_2[x(x+z)^k - xx^k + z(x+z)^k + zx^k - zx^k]).$

Hence $(x, yI_1, zI_2)^{k+1} = (x^{k+1}, I_1[(x+y+z)^{k+1} - (x+z)^{k+1}], I_2[(x+z)^{k+1} - x^{k+1}])$

Thus it is true by induction.

Now, we suppose that (x, yI_1, zI_2) is nilpotent in $R(I_1, I_2)$, hence there is a positive integer n such that

$(x, yI_1, zI_2)^n = 0$. By the previous statement, we get $x^n = 0$ *and* $(x+z)^n - x^n = 0$, *and* $(x+y+z)^n - (x+z)^n$, *thus* $(x+y+z)^n = (x+z)^n = 0$. Thus $x, x+y+z, x+z$ are nilpotent elements in R. The converse is easy.

(b) Let $P = (Q, MI_1, NI_2)$ be a nilpotent ideal of $R(I_1, I_2)$, there exists a positive integer n such that $P^n = \{0\}$.

For any $x \in Q$ we have $(x, 0,0) \in P$, hence $x^n = 0$, and Q is nilpotent.

On the other hand for any $y \in M$ we have $(0, yI_1, 0) \in (0, MI_1, 0) \leq P$, hence $(0, yI_1, 0)^n = \{0\}$, *thus* $y^n = 0$, and M is nilpotent . By the same, we get that N is nilpotent.

Now, for every $x \in Q, y \in M, z \in N$, we have $A = (x, yI_1, zI_2) \in P$, by the assumption of the nilpotency of P, we get $A^n = (x^n, I_1[(x+y+z)^n - (x+z)^n], I_2[(x+z)^n - x^n]) = 0$, hence $x^n = 0$ *and* $(x+z)^n - x^n = 0$, *and* $(x+y+z)^n - (x+z)^n$, *thus* $(x+y+z)^n = (x+z)^n = 0$, which implies that $Q+N, Q+M+N$ are nilpotent.

The converse is easy and clear.

(c) Let $P = (Q, MI_1, NI_2)$ be a nil ideal of $R(I_1, I_2)$, and $A = (x, yI_1, zI_2)$ be an arbitrary element of P, then there exists a positive integer n such that $A^n = (x^n, I_1[(x+y+z)^n - (x+z)^n], I_2[(x+z)^n - x^n]) = 0$, thus $x^n = 0, (x+y+z)^n = (x+z)^n = 0$, so that $Q, Q+M+N, Q+M$ are nil.

Also, N, M are nil ideals, that is because $M, N \leq Q+M+N$.

For the converse, we assume that $Q, M, N, Q+N, Q+M+N$ are nil ideals in the classical ring R, we must prove that $P = (Q, MI_1, NI_2)$ is a nil ideal of $R(I_1, I_2)$.

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Let $A = (x, yI_1, zI_2)$ be an arbitrary element of P, we have $x \in Q, y \in M, z \in N$. There exists three positive integers m, n, k such that $x^n = (x + y + z)^m = (x + z)^k = 0$. Now, we compute

$A^{m+n+K} = \left(x^{m+n+k}, I_1[(x + y + z)^{m+n+k} - (x + z)^{m+n+k}], I_2[(x + z)^{m+n+k} - x^{m+n+k}]\right) =$ $(0,0,0) = 0$. This means that P is a nil ideal of the refined neutrosophic ring $R(I_1, I_2)$.

The following theorem shows the equivalence between Kothe's conjecture in the classical ring R and the corresponding refined neutrosophic ring $R(I_1, I_2)$.

Theorem 3.4:

Kothe's conjecture is true in the refined neutrosophic ring $R(I_1, I_2)$ if and only if it is true in the corresponding classical ring R.

Proof:

If Kothe's conjecture is true in $R(I_1, I_2)$, then it is true in R, that is because R is a homomorphic image of $R(I_1, I_2)$. See [9].

Now, suppose that Kothe's conjecture is true in R. If $P = (Q, MI_1, NI_2), L = (S, TI_1, GI_2)$ are two nil ideals of $R(I_1, I_2)$, then by theorem , we get $Q, Q + N, Q + M + N, S, S + G, S + T + G$ are nil in R, hence $P + L = (Q + S, [M + T]I_1, [N + G]I_2)$ is a nil ideal in $R(I_1, I_2)$, that is because

$\boldsymbol{Q + S, Q + S + N + G, Q + S + N + G + M + T}$ are nil in R (Since the Kothe's conjecture is true in the ring R by the assumption). This implies that Kothe's conjecture is true in the refined neutrosophic ring $\boldsymbol{R(I_1, I_2)}$.

4. Conclusion

In this article, we have proved the equivalence between Kothe's conjecture in classical rings and refined neutrosophic rings. Also, we have determined the necessary and sufficient condition for any ideal in $R(I_1, I_2)$ to be a nil ideal. On the other hand, we give a counter example to show that inclusion's condition between the parts any ideal in $R(I_1, I_2)$ is not available if R has not a unity.

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