



SVN-Ostrowski Type Inequalities for $\phi - \lambda$ -convex in 1^{st} kind

^{1,*}Ali Hassan, ²Asif Raza Khan and ³Sumbul Khatoon

¹Department of Mathematics, Shah Abdul Latif University Khairpur-66020, Pakistan.

²Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan.

³Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan.

alihassan.iiui.math@gmail.com¹, asifrk@uok.edu.pk², sumbulburo25@gmail.com³

Abstract

In this paper, we would like to state well-known Ostrowski inequality via $\phi - \lambda$ -convex in 1^{st} kind by using the SVN-Reimann integrals. In addition, we establish some SVN-Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\phi - \lambda$ -convex in 1^{st} kind by using different techniques including Hölder's inequality and power mean inequality. We are introducing very first time that the class of $\phi - \lambda$ -convex in 1^{st} kind, which is the generalization of many important classes including class of h -convex, Godunova-Levin s -convex, s -convex in the 2^{nd} kind and hence contains class of convex. It also contains class of P -convex and class of Godunova-Levin functions. In this way we also capture the results with respect to convexity of functions.

Keywords: Ostrowski inequality, $\phi - \lambda$ -convex in 1^{st} kind, Single valued Neutrosophic sets.

1 Introduction

In this section, from literature, we recall and introduce some definitions for various convex.

Definition 1.1. ³ A function $\eta : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex, if

$$\eta(tx + (1-t)y) \leq t\eta(x) + (1-t)\eta(y),$$

$\forall x, y \in I, t \in [0, 1]$.

Definition 1.2. ³ A function $\eta : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be MT -convex, if $\eta(x) \geq 0$ and

$$\eta(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}\eta(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}\eta(y),$$

$\forall x, y \in I, t \in [0, 1]$.

Definition 1.3. ¹⁸ The $\eta : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a P -convex, if $\eta(x) \geq 0$ and $\forall x, y \in I$ and $t \in [0, 1]$ we have

$$\eta(tx + (1-t)y) \leq \eta(x) + \eta(y).$$

Definition 1.4. ²¹ The $\eta : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a Godunova-Levin convex, if $\eta(x) \geq 0$ and $\forall x, y \in I$ and $t \in (0, 1)$ we have

$$\eta(tx + (1-t)y) \leq \frac{1}{t}\eta(x) + \frac{1}{1-t}\eta(y).$$

Definition 1.5. ⁵ Let $s \in [0, 1]$. A function $\eta : I \subset [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex in the 2^{nd} kind, if

$$\eta(tx + (1-t)y) \leq t^s\eta(x) + (1-t)^s\eta(y),$$

$\forall x, y \in I, t \in [0, 1]$.

Definition 1.6. ¹⁰ The the function $\eta : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ is of Godunova-Levin s -convex, with $s \in [0, 1]$, if

$$\eta(tx + (1-t)y) \leq \frac{1}{t^s} \eta(x) + \frac{1}{(1-t)^s} \eta(y),$$

$\forall t \in (0, 1)$ and $x, y \in I$.

Definition 1.7. ³⁴ Let $h : J \subseteq \mathbb{R} \rightarrow [0, \infty)$ with $h \neq 0$. The η is an h -convex if $\forall x, y \in I$, we have

$$\eta(tx + (1-t)y) \leq h(t)\eta(x) + h(1-t)\eta(y),$$

$\forall t \in [0, 1]$.

Now, we are introducing the very first time the new type of convexity of functions, named as ϕ -convexity.

Definition 1.8. ¹¹ Let $\phi : (0, 1) \rightarrow (0, \infty)$ be a measurable function. The the $\eta : I \rightarrow [0, \infty)$ is a ϕ -convex, if for all $x, y \in I$ we have

$$\eta(tx + (1-t)y) \leq t\phi(t)\eta(x) + (1-t)\phi(1-t)\eta(y), \tag{1}$$

$\forall t \in (0, 1)$.

Theorem 1.9. ³³ Let $\varphi : [\rho_a, \rho_b] \rightarrow \mathbb{R}$ be differentiable function on (ρ_a, ρ_b) with the property that $|\varphi'(t)| \leq M$ for all $t \in (\rho_a, \rho_b)$. Then

$$\left| \varphi(x) - \frac{1}{\rho_b - \rho_a} \int_{\rho_a}^{\rho_b} \varphi(t) dt \right| \leq M(\rho_b - \rho_a) \left[\frac{1}{4} + \left(\frac{x - \frac{\rho_a + \rho_b}{2}}{\rho_b - \rho_a} \right)^2 \right], \tag{2}$$

for all $x \in (\rho_a, \rho_b)$.

Now we present the extension of definitions of fuzzy numbers and their results as from the,^{7, 828} and.²⁰

Definition 1.10. ⁴ A SVN-Number is $\phi : \mathbb{R} \rightarrow [0, 1]$ can be defined as

1. $[\phi]^0 = \text{Closure}(\{r \in \mathbb{R} : T\phi(r) > 0, I\phi(r) > 0, F\phi(r) > 0\})$ is compact.
2. ϕ is Normal.(i.e, $\exists r_0 \in \mathbb{R}$ such that $T\phi(r_0) = 1, I\phi(r_0) = 0$ and $F\phi(r_0) = 0$).
3. ϕ is SVN-convex, i.e, $\forall r_1, r_2 \in \mathbb{R}, \eta \in [0, 1]$

$$\begin{aligned} T\phi(\eta r_1 + (1-\eta)r_2) &\geq \min\{T\phi(r_1), T\phi(r_2)\}, \\ I\phi(\eta r_1 + (1-\eta)r_2) &\leq \max\{I\phi(r_1), I\phi(r_2)\}, \\ F\phi(\eta r_1 + (1-\eta)r_2) &\leq \max\{F\phi(r_1), F\phi(r_2)\}. \end{aligned}$$

4. $\forall r_0 \in \mathbb{R}$ and $\epsilon > 0, \exists$ Neighborhood $V(r_0)$, such that $\forall r \in \mathbb{R}, T\phi(r) \leq T\phi(r_0) + \epsilon, I\phi(r) \geq I\phi(r_0) - \epsilon,$ and $F\phi(r) \geq \phi(r_0) - \epsilon,$

Definition 1.11. ^{4,22} For any $(\Gamma_1, \Gamma_2, \Gamma_3) \in [0, 1]^3$, and ϕ be any SVN-number, then Γ -level set $[\phi]^{(\Gamma_1, \Gamma_2, \Gamma_3)} = \{r \in \mathbb{R} : T\phi(r) \geq \Gamma_1, I\phi(r) \leq \Gamma_2, F\phi(r) \leq \Gamma_3\}$. Moreover $[\phi]^\Gamma = \left[\phi_-^{(\Gamma_1, \Gamma_2, \Gamma_3)}, \phi_+^{(\Gamma_1, \Gamma_2, \Gamma_3)} \right], \forall (\Gamma_1, \Gamma_2, \Gamma_3) \in [0, 1]^3$.

Proposition 1.12. ^{22,30} Let $\phi, \varphi \in SVN_{\mathbb{R}}$ (Set of all SVN-Numbers) and $\epsilon \in \mathbb{R}$, then the following properties holds:

1. $[\phi + \varphi]^{(\Gamma_1, \Gamma_2, \Gamma_3)} = [\phi]^{(\Gamma_1, \Gamma_2, \Gamma_3)} + [\varphi]^{(\Gamma_1, \Gamma_2, \Gamma_3)}$.
2. $[\epsilon \odot \phi]^{(\Gamma_1, \Gamma_2, \Gamma_3)} = \epsilon [\phi]^{(\Gamma_1, \Gamma_2, \Gamma_3)}$.
3. $\phi \oplus \varphi = \varphi \oplus \phi$.
4. $\epsilon \odot \phi = \phi \odot \epsilon$.
5. $1 \odot \phi = \phi$.

$\forall \Gamma \in [0, 1]$.

Definition 1.13. ³¹ Let $D : SVN_{\mathbb{R}} \times SVN_{\mathbb{R}} \rightarrow \mathbb{R}_+ \cup \{0\}$, defined as

$$D(\phi, \varphi) = \sup_{\Gamma \in [0,1]} \max \left\{ \left| T\phi_{-}^{(\Gamma)}, T\phi_{+}^{(\Gamma)} \right|, \left| T\varphi_{-}^{(\Gamma)}, T\varphi_{+}^{(\Gamma)} \right| \right\} + \inf_{\Gamma \in [0,1]} \min \left\{ \left| I\phi_{-}^{(\Gamma)}, I\phi_{+}^{(\Gamma)} \right|, \left| I\varphi_{-}^{(\Gamma)}, I\varphi_{+}^{(\Gamma)} \right| \right\} + \inf_{\Gamma \in [0,1]} \min \left\{ \left| F\phi_{-}^{(\Gamma)}, F\phi_{+}^{(\Gamma)} \right|, \left| F\varphi_{-}^{(\Gamma)}, F\varphi_{+}^{(\Gamma)} \right| \right\}.$$

$\forall \phi, \varphi \in SVN_{\mathbb{R}}$. Then D is metric on $SVN_{\mathbb{R}}$.

Proposition 1.14. ³¹ Let $\phi_1, \phi_2, \phi_3, \phi_4 \in SVN_{\mathbb{R}}$ and $\eta \in SVN_{\mathbb{R}}$, we have

1. $(SVN_{\mathbb{R}}, D)$ is complete.
2. $D(\phi_1 \oplus \phi_3, \phi_2 \oplus \phi_3) = D(\phi_1, \phi_2)$.
3. $D(\eta \odot \phi_1, \eta \odot \phi_2) = |\eta|D(\phi_1, \phi_2)$.
4. $D(\phi_1 \oplus \phi_2, \phi_3 \oplus \phi_4) = D(\phi_1, \phi_3) + D(\phi_2, \phi_4)$.
5. $D(\phi_1 \oplus \phi_2, \tilde{0}) = D(\phi_1, \tilde{0}) + D(\phi_2, \tilde{0})$.
6. $D(\phi_1 \oplus \phi_2, \phi_3) = D(\phi_1, \phi_3) + D(\phi_2, \tilde{0})$,

where $\tilde{0} \in SVN_{\mathbb{R}}$, defined by $\forall r \in \mathbb{R}, \tilde{0}(r) = (0, 0, 1)$.

Definition 1.15. ³⁰ Let $\phi, \varphi \in SVN_{\mathbb{R}}$, if $\exists \theta \in SVN_{\mathbb{R}}$, such that $\phi = \varphi \oplus \theta$, then θ is H -difference of ϕ and φ , denoted by $\theta = \phi \ominus \varphi$.

Definition 1.16. ³¹ A function $\phi : [r_0, r_0 + \epsilon] \rightarrow SVN_{\mathbb{R}}$ is H -differentiable at r , if $\exists \phi'(r) \in SVN_{\mathbb{R}}$, i.e both limits

$$\lim_{h \rightarrow 0^+} \frac{\phi(r+h) \ominus \phi(r)}{h}, \quad \lim_{h \rightarrow 0^+} \frac{\phi(r) \ominus \phi(r-h)}{h}$$

exists and are equal to $\phi'(r)$.

Definition 1.17. ^{30,31} Let $\phi : [\rho_a, \rho_b] \rightarrow SVN_{\mathbb{R}}$, if $\forall \Gamma > 0, \exists \eta > 0$, for any partition $P = \{[u, v] : \delta\}$ of $[\rho_a, \rho_b]$ with norm $\Delta(P) < \eta$, we have

$$D \left(\sum_P^* (v-u)\phi(\delta), \varphi \right) < \Gamma,$$

then The ϕ is SVN-Riemann integrable to $\varphi \in SVN_{\mathbb{R}}$, we write it as

$$\varphi = (SVNR) \int_{\rho_a}^{\rho_b} \phi(x) dx.$$

2 SVN-Ostrowski type inequalities via ϕ -convex in 1^{st} kind

In this section, we are introducing very first time the concept of $\phi - \lambda$ -convex in 1^{st} kind, which contain many classes of convex in literature.

Definition 2.1. Let $\lambda \in (0, 1], \phi : (0, 1) \rightarrow (0, \infty)$ be a measurable function. The the $\eta : I \rightarrow [0, \infty)$ is a $\phi - \lambda$ -convex in 1^{st} kind if $\forall x, y \in I$ we have

$$\eta(tx + (1-t)y) \leq t^\lambda \phi(t)\eta(x) + (1-t)^\lambda \phi(1-t)\eta(y), \tag{3}$$

$\forall t \in (0, 1)$.

Remark 2.2. In Definition 2.1,

1. If $\lambda = 1$, in (3), then we get the concept of ϕ -convex function.
2. If $l(t) = t$, and by taking $\lambda = 1, h = l\phi$ in (3), we get h -convex function.

3. If $\lambda = 1$, $\phi(t) = \frac{1}{t^{s+1}}$ with $s \in [0, 1]$ in (3), then we get the class of Godunova-Levin s -convex functions.
4. If $\lambda = 1$, $\phi(t) = \frac{1}{t^2}$ in (3), then we get the concept of Godunova-Levin convex function.
5. If $\lambda = 1$, $\phi(t) = t^{s-1}$ with $s \in [0, 1]$ in (3), then we get the concept of s -convex in 2^{nd} kind.
6. If $\lambda = 1$, $\phi(t) = \frac{1}{t}$ in (3), then we get the concept of P -convex function.
7. If $\lambda = 1$, $\phi(t) = 1$ in (3), then we get the concept of ordinary convex function.
8. If $\lambda = 1$, $\phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (3), then we get the concept of MT -convex function.

Lemma 2.3. Let $\varphi : [\rho_a, \rho_b] \rightarrow SVN_{\mathbb{R}}$ be an absolutely continuous mapping on (ρ_a, ρ_b) with $\rho_a < \rho_b$. If $\varphi' \in C_F[\rho_a, \rho_b] \cap L_F[\rho_a, \rho_b]$, then for $x \in (\rho_a, \rho_b)$ the following identity holds:

$$\begin{aligned} & \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt \oplus \frac{(x - \rho_a)^2}{\rho_b - \rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx + (1-t)\rho_a) dt \\ &= \varphi(x) \oplus \frac{(\rho_b - x)^2}{\rho_b - \rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx + (1-t)\rho_b) dt. \end{aligned} \tag{4}$$

We make use of the beta function of Euler type, which is for $x, y > 0$ defined as

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$.

Theorem 2.4. If Lemma 2.3 hold. Additionally, $\lambda \in (0, 1]$, $\phi : (0, 1) \rightarrow (0, \infty)$ be a measurable function with $\phi(t) \neq \frac{1}{t^2}$, $D(\varphi', \tilde{0})$ be a ϕ -convex on $[\rho_a, \rho_b]$ and $D(\varphi'(x), \tilde{0}) \leq M$. Then for each $x \in (\rho_a, \rho_b)$ the following inequality holds:

$$\begin{aligned} & D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq M \left(\int_0^1 (t^{1+\lambda}\phi(t) + t(1-t^\lambda)\phi(1-t)) dt \right) I(x), \end{aligned} \tag{5}$$

where $I(x) = \frac{(x-\rho_a)^2 + (\rho_b-x)^2}{\rho_b - \rho_a}$.

Proof. From the Lemma 2.3,

$$\begin{aligned} & D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq D\left(\frac{(x - \rho_a)^2}{\rho_b - \rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx + (1-t)\rho_a) dt, \right. \\ & \quad \left. \frac{(\rho_b - x)^2}{\rho_b - \rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx + (1-t)\rho_b) dt\right), \\ & \leq D\left(\frac{(x - \rho_a)^2}{\rho_b - \rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx + (1-t)\rho_a) dt, \tilde{0}\right) \\ & \quad + D\left(\frac{(\rho_b - x)^2}{\rho_b - \rho_a} \odot (SVNR) \int_0^1 t \odot \varphi'(tx + (1-t)\rho_b) dt, \tilde{0}\right), \\ & = \frac{(x - \rho_a)^2}{\rho_b - \rho_a} D\left((SVNR) \int_0^1 t \odot \varphi'(tx + (1-t)\rho_a) dt, \tilde{0}\right) \\ & \quad + \frac{(\rho_b - x)^2}{\rho_b - \rho_a} D\left((SVNR) \int_0^1 t \odot \varphi'(tx + (1-t)\rho_b) dt, \tilde{0}\right), \end{aligned}$$

$$\begin{aligned} &\leq \frac{(x - \rho_a)^2}{\rho_b - \rho_a} \int_0^1 tD \left(\varphi'(tx + (1 - t)\rho_a), \tilde{0} \right) dt \\ &+ \frac{(\rho_b - x)^2}{\rho_b - \rho_a} \int_0^1 tD \left(\varphi'(tx + (1 - t)\rho_b), \tilde{0} \right) dt, \end{aligned} \tag{6}$$

Since $D(\varphi', \tilde{0})$ be $\phi - \lambda$ -convex in 1st kind and $D(\varphi'(x), \tilde{0}) \leq M$, we have

$$\begin{aligned} D \left(\varphi'(tx + (1 - t)\rho_a), \tilde{0} \right) &\leq t^\lambda \phi(t)D \left(\varphi'(x), \tilde{0} \right) + (1 - t^\lambda)\phi(1 - t)D \left(\varphi'(\rho_a), \tilde{0} \right) \\ &\leq M [t^\lambda \phi(t) + (1 - t^\lambda)\phi(1 - t)] \end{aligned} \tag{7}$$

$$\begin{aligned} D \left(\varphi'(tx + (1 - t)\rho_b), \tilde{0} \right) &\leq t^\lambda \phi(t)D \left(\varphi'(x), \tilde{0} \right) + (1 - t^\lambda)\phi(1 - t)D \left(\varphi'(\rho_b), \tilde{0} \right) \\ &\leq M [t^\lambda \phi(t) + (1 - t^\lambda)\phi(1 - t)]. \end{aligned} \tag{8}$$

Now using (7) and (8) in (6) we get (5). □

Corollary 2.5. *In Theorem 2.4,*

1. *If $\lambda = 1$ in (5), one has the SVN–Ostrowski inequality for ϕ -convex:*

$$D \left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt \right) \leq M \left(\int_0^1 (t^2 \phi(t) + t(1 - t)\phi(1 - t)) dt \right) I(x).$$

2. *If $\lambda = 1, l(t) = t$, then by taking $h = l\phi$, in (5), one has the SVN–Ostrowski inequality for h -convex:*

$$D \left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt \right) \leq M \left(\int_0^1 (t h(t) + t h(1 - t)) dt \right) I(x).$$

3. *If $\lambda = 1, \phi(t) = t^{-(s+1)}$ in (5), then the SVN–Ostrowski inequality for Godunova-Levin s -convex:*

$$D \left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt \right) \leq M \left(\frac{1}{1 - s} \right) I(x).$$

4. *If $\lambda = 1, \phi(t) = t^{s-1}$ where $s \in (0, 1]$ in (5), then the SVN–Ostrowski inequality for s -convex in 2nd kind:*

$$D \left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt \right) \leq M \left(\frac{1}{1 + s} \right) I(x).$$

5. *If $\lambda = 1, \phi(t) = t^{-1}$ in (5), then the SVN–Ostrowski inequality for P -convex:*

$$D \left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt \right) \leq MI(x).$$

6. *If $\lambda = \phi(t) = 1$ in (5), then the SVN–Ostrowski inequality for convex:*

$$D \left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt \right) \leq \frac{M}{2} I(x).$$

7. *If $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (5), then the SVN–Ostrowski inequality for MT-convex:*

$$D \left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t)dt \right) \leq \frac{M\pi}{4} I(x).$$

Theorem 2.6. *If Lemma 2.3 hold. Additionally, $\lambda \in (0, 1]$, $\phi : (0, 1) \rightarrow (0, \infty)$ be a measurable function with $\phi(t) \neq \frac{1}{t^2}$, $[D(\varphi', \tilde{0})]^q$ for $q \geq 1$ be $\phi - \lambda$ -convex in 1st kind on $[\rho_a, \rho_b]$ and $D(\varphi'(x), \tilde{0}) \leq M$. Then $\forall x \in (\rho_a, \rho_b)$ the following inequality holds:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} \left(\int_0^1 (t^{\lambda+1}\phi(t) + t(1-t^\lambda)\phi(1-t)) dt\right)^{\frac{1}{q}} I(x). \tag{9}$$

Proof. From the inequality (6) and power mean inequality³⁵

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{(x - \rho_a)^2}{\rho_b - \rho_a} \left(\int_0^1 t dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t [D(\varphi'(tx + (1-t)\rho_a), \tilde{0})]^q dt\right)^{\frac{1}{q}} + \frac{(\rho_b - x)^2}{\rho_b - \rho_a} \left(\int_0^1 t dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t [D(\varphi'(tx + (1-t)\rho_b), \tilde{0})]^q dt\right)^{\frac{1}{q}}. \tag{10}$$

Since $[D(\varphi', \tilde{0})]^q$ be $\phi - \lambda$ -convex in 1st kind and $D(\varphi'(x), \tilde{0}) \leq M$, we have

$$[D(\varphi'(tx + (1-t)\rho_a), \tilde{0})]^q \leq t^\lambda \phi(t) [D(\varphi'(x), \tilde{0})]^q + (1-t^\lambda)\phi(1-t) [D(\varphi'(\rho_a), \tilde{0})]^q \leq M^q [t^\lambda \phi(t) + (1-t^\lambda)\phi(1-t)], \tag{11}$$

$$[D(\varphi'(tx + (1-t)\rho_b), \tilde{0})]^q \leq t^\lambda \phi(t) [D(\varphi'(x), \tilde{0})]^q + (1-t^\lambda)\phi(1-t) [D(\varphi'(\rho_b), \tilde{0})]^q \leq M^q [t^\lambda \phi(t) + (1-t^\lambda)\phi(1-t)], \tag{12}$$

Now using (11) and (12) in (10) we get (9). □

Corollary 2.7. *In Theorem 2.6,*

1. *If $q = 1$, one has the Theorem 2.4.*
2. *If $\lambda = 1$ in (9), one has the SVN-Ostrowski inequality for ϕ -convex:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} \left(\int_0^1 (t^2\phi(t) + t(1-t)\phi(1-t)) dt\right)^{\frac{1}{q}} I(x).$$

3. *If $\lambda = 1$, $l(t) = t$, then by taking $h = l\phi$, in (9), one has the SVN-Ostrowski inequality for h -convex:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} \left(\int_0^1 (th(t) + th(1-t)) dt\right)^{\frac{1}{q}} I(x).$$

4. *If $\lambda = 1$, $\phi(t) = t^{-(s+1)}$ in (9), then one has SVN-Ostrowski inequality for Godunova-Levin s -convex:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} \left(\frac{1}{1-s}\right)^{\frac{1}{q}} I(x).$$

5. *If $\lambda = 1$, $\phi(t) = t^{s-1}$ where $s \in [0, 1]$ in (9), then one has SVN-Ostrowski inequality for s -convex in 2nd kind:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} \left(\frac{1}{1+s}\right)^{\frac{1}{q}} I(x).$$

6. *If $\lambda = 1$, $\phi(t) = t^{-1}$, in (9), then the SVN-Ostrowski inequality for P -convex:*

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{1-\frac{1}{q}}} I(x).$$

7. If $\lambda = \phi(t) = 1$, in (9), then the SVN–Ostrowski inequality for convex:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2} I(x).$$

8. If $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (9), then the SVN–Ostrowski inequality for MT–convex:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}} I(x).$$

Theorem 2.8. If Lemma 2.3 hold. Additionally, assume that $\lambda \in (0, 1], \phi : (0, 1) \rightarrow (0, \infty)$ be a measurable function with $\phi(t) \neq \frac{1}{t^2}, [D(\varphi', \tilde{0})]^q$ be a $\phi - \lambda$ -convex in 1st kind on $[\rho_a, \rho_b], q > 1$ and $D(\varphi'(x), \tilde{0}) \leq M$. Then for each $x \in (\rho_a, \rho_b)$, the following inequality holds:

$$\begin{aligned} & D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{(p+1)^{\frac{1}{p}}} \left(\int_0^1 (t^\lambda \phi(t) + (1-t^\lambda)\phi(1-t)) dt\right)^{\frac{1}{q}} I(x), \end{aligned} \tag{13}$$

where $p^{-1} + q^{-1} = 1$.

Proof. From the inequality (6) and Hölder’s inequality³⁶

$$\begin{aligned} & D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{(x - \rho_a)^2}{\rho_b - \rho_a} \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 [D(\varphi'(tx + (1-t)\rho_a), \tilde{0})]^q dt\right)^{\frac{1}{q}} \\ & \quad + \frac{(\rho_b - x)^2}{\rho_b - \rho_a} \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 [D(\varphi'(tx + (1-t)\rho_b), \tilde{0})]^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{14}$$

Since $[D(\varphi', \tilde{0})]^q$ be $\phi - \lambda$ -convex in 1st kind and $D(\varphi'(x), \tilde{0}) \leq M$, we have

$$\begin{aligned} [D(\varphi'(tx + (1-t)\rho_a), \tilde{0})]^q & \leq t^\lambda \phi(t) [D(\varphi'(x), \tilde{0})]^q + (1-t^\lambda)\phi(1-t) [D(\varphi'(\rho_a), \tilde{0})]^q \\ & \leq M^q [t^\lambda \phi(t) + (1-t^\lambda)\phi(1-t)], \end{aligned} \tag{15}$$

$$\begin{aligned} [D(\varphi'(tx + (1-t)\rho_b), \tilde{0})]^q & \leq t^\lambda \phi(t) [D(\varphi'(x), \tilde{0})]^q + (1-t^\lambda)\phi(1-t) [D(\varphi'(\rho_b), \tilde{0})]^q \\ & \leq M^q [t^\lambda \phi(t) + (1-t^\lambda)\phi(1-t)], \end{aligned} \tag{16}$$

Now using (15) and (16) in (14) we get (13). □

Corollary 2.9. In Theorem 2.8,

1. If $\lambda = 1$ in (13), one has the SVN–Ostrowski inequality for ϕ -convex:

$$\begin{aligned} & D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{(p+1)^{\frac{1}{p}}} \left(\int_0^1 (t\phi(t) + (1-t)\phi(1-t)) dt\right)^{\frac{1}{q}} I(x). \end{aligned}$$

2. If $l(t) = t$, then by taking $h = l\phi$, in (13), one has the SVN–Ostrowski inequality for h -convex:

$$\begin{aligned} & D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{(p+1)^{\frac{1}{p}}} \left(\int_0^1 (h(t) + h(1-t)) dt\right)^{\frac{1}{q}} I(x). \end{aligned}$$

3. If $\phi(t) = t^{-(s+1)}$ where $s \in [0, 1]$ in (13), then the SVN–Ostrowski inequality for Godunova-Levin s -convex:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}} \left(\frac{2}{1-s}\right)^{\frac{1}{q}} I(x).$$

4. If $\phi(t) = t^{s-1}$, where $s \in (0, 1]$ in (13), then the SVN–Ostrowski inequality for s -convex in 2nd kind:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}} \left(\frac{2}{1+s}\right)^{\frac{1}{q}} I(x).$$

5. If $\phi(t) = t^{-1}$, in (13), then the SVN–Ostrowski inequality for P -convex:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{2^{\frac{1}{q}} M}{(p+1)^{\frac{1}{p}}} I(x).$$

6. If $\phi(t) = 1$, in (13), then the SVN–Ostrowski inequality for convex:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{(p+1)^{\frac{1}{p}}} I(x).$$

7. If $\phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (13), then the SVN–Ostrowski inequality for MT-convex:

$$D\left(\varphi(x), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M \left(\frac{\pi}{2}\right)^{\frac{1}{q}}}{(1+p)^{\frac{1}{p}}} I(x).$$

2.1 SVN-Ostrowski type midpoint inequalities via $\phi - \lambda$ -convex in 1st kind

Remark 2.10. In Theorem 2.6,

1. If $x = \frac{\rho_a + \rho_b}{2}$ in (9), one has the SVN–Ostrowsli midpoint inequality for $\phi - \lambda$ -convex in 1st kind:

$$\begin{aligned} & D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{2^{2-\frac{1}{q}}} \left(\int_0^1 (t^{1+\lambda}\phi(t) + t(1-t^\lambda)\phi(1-t)) dt\right)^{\frac{1}{q}} (\rho_b - \rho_a). \end{aligned}$$

2. If $x = \frac{\rho_a + \rho_b}{2}, \lambda = 1$ in (9), one has the SVN–Ostrowsli midpoint inequality for ϕ -convex:

$$\begin{aligned} & D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{2^{2-\frac{1}{q}}} \left(\int_0^1 (t^2\phi(t) + t(1-t)\phi(1-t)) dt\right)^{\frac{1}{q}} (\rho_b - \rho_a). \end{aligned}$$

3. If $x = \frac{\rho_a + \rho_b}{2}, \lambda = 1, l(t) = t$ and $h = l\phi$ in (9), then the SVN–Ostrowsli midpoint inequality for h -convex:

$$\begin{aligned} & D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{2^{2-\frac{1}{q}}} \left(\int_0^1 (th(t) + th(1-t)) dt\right)^{\frac{1}{q}} (\rho_b - \rho_a). \end{aligned}$$

4. If $x = \frac{\rho_a + \rho_b}{2}$ and $\lambda = 1, \phi(t) = t^{-(s+1)}$ in (9), then one has *SVN*–Ostrowsli midpoint inequality for Godunova-Levin s –convex:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{2-\frac{1}{q}}} \left(\frac{1}{1-s}\right)^{\frac{1}{q}} (\rho_b - \rho_a).$$

5. If $x = \frac{\rho_a + \rho_b}{2}$ and $\lambda = 1, \phi(t) = t^{s-1}$ where $s \in [0, 1]$ in (9), then one has *SVN*–Ostrowsli midpoint inequality for s –convex in 2nd kind:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{2-\frac{1}{q}}} \left(\frac{1}{1+s}\right)^{\frac{1}{q}} (\rho_b - \rho_a).$$

6. If $x = \frac{\rho_a + \rho_b}{2}$ and $\lambda = 1, \phi(t) = t^{-1}$ in (9), then the *SVN*–Ostrowsli midpoint inequality for P –convex:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2^{2-\frac{1}{q}}} (\rho_b - \rho_a).$$

7. If $x = \frac{\rho_a + \rho_b}{2}$ and $\lambda = \phi(t) = 1$ in (9), then the *SVN*–Ostrowsli midpoint inequality for convex:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{4} (\rho_b - \rho_a).$$

8. If $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (9), then the *SVN*–Ostrowski inequality for MT –convex:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M\pi^{\frac{1}{q}}}{2^{1+\frac{1}{q}}} (\rho_b - \rho_a).$$

Remark 2.11. In Theorem 2.8,

1. If $x = \frac{\rho_a + \rho_b}{2}$ in (13), one has the *SVN*–Ostrowsli midpoint inequality for $\phi - \lambda$ –convex in 1st kind:

$$\begin{aligned} & D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{2(p+1)^{\frac{1}{p}}} \left(\int_0^1 (t^\lambda \phi(t) + (1-t^\lambda) \phi(1-t)) dt\right)^{\frac{1}{q}} (\rho_b - \rho_a), \end{aligned}$$

2. If $\lambda = 1, x = \frac{\rho_a + \rho_b}{2}$ in (13), one has the *SVN*–Ostrowsli midpoint inequality for ϕ –convex:

$$\begin{aligned} & D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{2(p+1)^{\frac{1}{p}}} \left(\int_0^1 (t\phi(t) + (1-t)\phi(1-t)) dt\right)^{\frac{1}{q}} (\rho_b - \rho_a), \end{aligned}$$

3. If $\lambda = 1, x = \frac{\rho_a + \rho_b}{2}, l(t) = t$ and $h = l\phi$ in (13), one has the *SVN*–Ostrowsli midpoint inequality for h –convex:

$$\begin{aligned} & D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \\ & \leq \frac{M}{2(p+1)^{\frac{1}{p}}} \left(\int_0^1 (h(t) + h(1-t)) dt\right)^{\frac{1}{q}} (\rho_b - \rho_a). \end{aligned}$$

4. If $\lambda = 1, x = \frac{\rho_a + \rho_b}{2}$ and $\phi(t) = t^{-(s+1)}$ where $s \in [0, 1]$ in (13), then the *SVN*–Ostrowsli midpoint inequality for Godunova-Levin s –convex:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{2^{\frac{1}{q}-1} M}{(p+1)^{\frac{1}{p}}} \left(\frac{1}{1-s}\right)^{\frac{1}{q}} (\rho_b - \rho_a).$$

5. If $\lambda = 1, x = \frac{\rho_a + \rho_b}{2}$ and $\phi(t) = t^{s-1}$, where $s \in (0, 1]$ in (13), then the SVN–Ostrowsli midpoint inequality for s -convex in 2^{nd} kind:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{2^{\frac{1}{q}-1} M}{(p+1)^{\frac{1}{p}}} \left(\frac{1}{1+s}\right)^{\frac{1}{q}} (\rho_b - \rho_a).$$

6. If $\lambda = 1, x = \frac{\rho_a + \rho_b}{2}$ and $\phi(t) = t^{-1}$ in (13), then the SVN–Ostrowsli midpoint inequality for P -convex:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{2^{\frac{1}{q}-1} M}{(p+1)^{\frac{1}{p}}} (\rho_b - \rho_a).$$

7. If $x = \frac{\rho_a + \rho_b}{2}$ and $\lambda = \phi(t) = 1$ in (13), then the SVN–Ostrowsli midpoint inequality for convex:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M}{2(p+1)^{\frac{1}{p}}} (\rho_b - \rho_a).$$

8. If $\lambda = 1, \phi(t) = \frac{1}{2\sqrt{t(1-t)}}$ in (13), then the SVN–Ostrowski inequality for MT -convex:

$$D\left(\varphi\left(\frac{\rho_a + \rho_b}{2}\right), \frac{1}{\rho_b - \rho_a} \odot (SVNR) \int_{\rho_a}^{\rho_b} \varphi(t) dt\right) \leq \frac{M\pi^{\frac{1}{q}}}{2^{\frac{1}{q}+1}(1+p)^{\frac{1}{p}}} (\rho_b - \rho_a).$$

3 Conclusion

Ostrowski inequality is one of the most celebrated inequalities, we can find its various generalizations and variants in literature. In this paper, we presented the generalized notion of $\phi - \lambda$ -convex in 1^{st} kind which is the generalization of many important classes including class of h -convex,¹¹ h -convex,³⁴ Godunova-Levin s -convex,¹⁰ s -convex in the 2^{nd} kind⁵ and hence contains class of convex.³ It also contains class of P -convex¹⁸ and class of Godunova-Levin functions.²¹ We would like to state the SVN-Ostrowski inequality via $\phi - \lambda$ -convex in 1^{st} kind. In addition, we establish some SVN-Ostrowski type inequalities for the class of functions whose derivatives in absolute values at certain powers are $\phi - \lambda$ -convex in 1^{st} kind by using different techniques including Hölder’s inequality³⁶ and power mean inequality.³⁵

References

1. M. Alomari, M. Darus, S.S. Dragomir and P. Cerone, “Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense”, *Appl. Math. Lett.*, **23**(1), (2010), pp. 1071–1076.
2. A. Arshad and A. R. Khan, “Hermite–Hadamard–Fejer Type Integral Inequality for $s - p$ -convex of Several Kinds”, *TJMM.*, **11** (2), (2019), pp. 25–40.
3. E. F. Beckenbach, “convex”, *Bull. Amer. Math. Soc.*, **54**(1), (1948), pp. 439–460.
4. P. Biswas, S. Pramanik, and B. C. Giri, “TOPSIS Strategy for Multi-Attribute Decision Making with Trapezoidal Neutrosophic Numbers”, *Neutrosophic Sets and Systems*, **19** (1), (2018), pp. 29-39.
5. W. W. Breckner, “Stetigkeitsaussagen Fur Eine Klasse Verallgemeinerter Konvexer Funktionen in Topologischen Linearen Raumen. (German)”, *Publ. Inst. Math.*, **23** (37), (1978), pp. 13–20.
6. P. Cerone and S.S. Dragomir, “Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions”, *Demonstratio Math*, **37** (2), (2004), pp. 299–308.
7. W. Congxin and M. Ming, “On embedding problem of fuzzy number space: Part 1”, *Fuzzy Sets and Systems*, **44** (1), (1991), pp. 33–38.
8. W.Congxin and G. Zengtai, “On Henstock integral of fuzzy-number-valued functions (I)”, *Fuzzy Sets and Systems*, **120** (1), (2001) pp. 523–532.

9. S. S. Dragomir, "A Functional Generalization of Ostrowski Inequality via Montgomery identity", *Acta Math. Univ. Comenianae, LXXXIV*, **1**(2), (2015), pp. 63–78.
10. S. S. Dragomir, "Integral inequalities of Jensen type for λ -convex", *In Proceedings of RGMIA, Res. Rep. Coll.*, **1** (17), (2014).
11. S. S. Dragomir, "Inequalities of Jensen Type for ϕ -convex", *Fasciculi Mathematici*, **5**(1), (2015), pp. 35–52.
12. S. S. Dragomir, "A Companion of Ostrowski's Inequality for Functions of Bounded Variation and Applications", *Int. J. Nonlinear Anal. Appl.*, **5** (2), (2014), pp. 89–97.
13. S. S. Dragomir, "On the Ostrowski's Integral Inequality for Mappings with Bounded Variation and Applications", *Math. Inequal. Appl.*, **4** (1), (2001), pp. 59–66.
14. S. S. Dragomir, "Refinements of the Generalised Trapezoid and Ostrowski Inequalities for Functions of Bounded Variation", *Arch. Math.*, **91** (5), (2008), pp. 450–460.
15. S. S. Dragomir and N. S. Barnett, "An Ostrowski Type Inequality for Mappings whose Second Derivatives are Bounded and Applications", *J. Indian Math. Soc. (N.S.)*, **66** (4), (1999), pp. 237–245.
16. S. S. Dragomir, P. Cerone, N. S. Barnett and J. Roumeliotis, "An Inequality of the Ostrowski Type for Double Integrals and Applications for Cubature Formulae", *Tamsui Oxf. J. Math. Sci.*, **16**(1), (2000), pp. 1–16.
17. S. S. Dragomir, P. Cerone and J. Roumeliotis, "A new Generalization of Ostrowski Integral Inequality for Mappings whose Derivatives are Bounded and Applications in Numerical Integration and for Special Means", *Appl. Math. Lett.*, **13**(1), (2000), pp. 19–25.
18. S. S. Dragomir, J. Pečarić and L. Persson, "Some inequalities of Hadamard type", *Soochow J. Math.*, **21** (3), (1995), pp. 335–341.
19. A. Ekinci, "Klasik Eşitsizlikler Yoluyla Konveks Fonksiyonlar için Integral Eşitsizlikler", *Ph.D. Thesis*, Thesis ID: 361162 in tez2.yok.gov.tr Atatürk University, 2014.
20. S. Gal, "Approximation theory in fuzzy setting, Chapter 13 in Handbook of Analytic Computational Methods in Applied Mathematics (edited by G. Anastassiou)", *Chapman and Hall, CRC Press, Boca Raton, New York*, (2000), pp. 617–666.
21. E. K. Godunova, V. I. Levin, "Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions", *Numerical Mathematics and Mathematical Physics, (Russian)*, **166**(1), (1985), pp. 138–142.
22. G. Á. Gómez, J. V. Moya, and J. E. Ricardo, "Method to measure the formation of pedagogical skills through neutrosophic numbers of unique value," *Revista Asociación Latinoamericana de Ciencias Neutrosóficas*, ISSN 2574-1101, **11** (1), (2020), pp. 41-48.
23. N. Irshad, A. R. Khan and A. Nazir, "Extension of Ostrowki Type Inequality Via Moment Generating Function", *Adv. Inequal. Appl.*, **2**(1), (2020), pp. 1–15.
24. N. Irshad, A. R. Khan and M. A. Shaikh, "Generalization of Weighted Ostrowski Inequality with Applications in Numerical Integration", *Adv. Ineq. Appl.*, **7**(1), (2019), pp. 1–14.
25. N. Irshad, A. R. Khan, and Muhammad Awais Shaikh, "Generalized Weighted Ostrowski-Gruss Type Inequality with Applications", *Global J. Pure Appl. Math.*, **15** (5), (2019), pp. 675–692.
26. N. Irshad and A. R. Khan, "On Weighted Ostrowski Gruss Inequality with Applications", *TJMM*, **10** (1), (2018), pp. 15–22.
27. N. Irshad and A. R. Khan, "Generalization of Ostrowski Inequality for Differentiable functions and its applications to numerical quadrature rules", *J. Math. Anal.*, **8**(1), (2017), pp. 79–102.
28. O. Kaleva, "Fuzzy differential equations", *Fuzzy Sets and Systems*, **24**(1) (1987) pp. 301–317.
29. D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, "Inequalities Involving Functions and Their Integrals and Derivatives", *Kluwer Academic, Dordrecht*, (1991).

30. A. S. Molina, W. A. C. Calle, and J. D. B. Remache, “The application of Microsoft Solution Framework Software Testing using Neutrosophic Numbers,” *Neutrosophic Sets and Systems*, **37**(1), (2020), pp. 267-276.
31. M. Mullai and R. Surya, “Neutrosophic Inventory Backorder Problem Using Triangular Neutrosophic Numbers”, *Neutrosophic Sets and Systems*, **31**(1), (2020), pp. 148-155.
32. M. A. Noor and M. U. Awan, “Some integral inequalities for two kinds of convexities via fractional integrals”, *TJMM*, **5**(1), (2013), pp. 129 – 136.
33. A. M. Ostrowski, “Über die absolutabweichung einer differentiebaren funktion von ihrem integralmittelwert”, *Comment. Math. Helv.*, **10**(1), (1938), pp. 226–227.
34. S. Varošanec, “On h-convexity”, *J. Math. Anal. Appl.*, **326**(1), (2007), pp. 303–311.
35. Z. G. Xiao, and A. H. Zhang, “Mixed power mean inequalities”, *Research Communication on Inequalities*, **8** (1), (2002), pp. 15—17.
36. X. Yang, “A note on Hölder inequality”, *Appl. Math. Comput.*, **134**(1), (2003), pp. 319–322