A Review On Some Neutrosophic Algebraic Linear Structures

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Abstract

This paper is dedicated to review some of basic concepts in neutrosophic linear algebra and its generalizations, especially neutrosophic vector spaces, refined neutrosophic and n-refined neutrosophic vector spaces. Also, this work gives the interested reader a strong background in the study of neutrosophic matrix theory and n-refined neutrosophic matrix theory. We study elementary properties of these cocepts such as Kernel, AH-Quotient, and dimension.

Keywords: Neutrosophic matrix, neutrosophic vector space, refined neutrosophic vector space, n-refined neutrosophic matrix

1.Introduction

Neutrosophy is a new branch of philosophy founded by Smarandache [6,36], to study the indeterminacy in the real world problems and science. It has a huge effect in many areas such as topology [7,27,29], equations [3,30], decision making [8], abstract algebra [25,26,39,41], and number theory [35].

Neutrosophic algebra began with the definitions of neutrosophic groups [9,17], and rings [13]. The neutrosophic rings and their generalizations such as refined neutrosophic rings [19], and n-refined neutrosophic rings [11,12], were very useful in the study of linear structures.

Neutrosophic linear structures were defined as new generalizations of classical ones based on neutrosophic rings and fields, where we find many concepts from linear algebra were generalized into neutrosophic systems such as neutrosophic matrices and spaces over neutrosophic fields [1,42], refined neutrosophic spaces and matrices over refined neutrosophic fields [24], n-refined neutrosophic spaces over n-refined neutrosophic fields [21,32], linear modules and ideals [4,5,20,22].

Through this paper, we give the interested reader a good background to deal with algebraic neutrosophic linear structures such as AH-spaces [2], n-refined matrices [32], and other related concepts [13,14,15].

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2. Neutrosophic vector spaces

Definition 2.1.

Let $(V,+,.)$ be any vector space over a field K, let $V(I) = \langle V \cup I \rangle$ be a neutrosophic set generated by $V$and I. The triple $(V(I),+,.)$ is called a weak neutrosophic vector space over a field K. If $V(I)$ is a neutrosophic vector space over a neutrosophic field $K(I)$, then $V(I)$ is called a strong neutrosophic vector space. The elements of $K(I)$are called neutrosophic vectors and the elements of $K(I)$ are called neutrosophic scallars.

If $u = a + bI$, $v = c + dI \in V(I)$ where a, b, c and d are vectors in vectors $V$and $\alpha = k + mI \in K(I)$ where k and $m$are scallars in K,we define:

$u + v = (a + bI) + (c + dI) = (a + c) + (b + d)I$, and $\alpha u = (k + mI)(a + bI) = k.a + (k.b + m.a + m.b)I$

Definition 2.2.

A linearly independent subset $B[I] = \{v_1, v_2, \dots, v_n\}$ of $V(I)$ is called a basis for $V(I)$ if $B[I]$spans $V(I)$.

Definition 2.3.

Let V, Wbe two vector spaces over the field F, with $dim(V) = n, dim(W) = m$, and $V(I), W(I)$ be the corresponding neutrosophic vector spaces over the corresponding neutrosophic field $F(I)$. Let $g, h: V \to W$ be two linear transformations, then there exists a neutrosophic linear transformation $f = g + hI: V(I) \to W(I)$, where $f$is defined as follows:

$$f(x + yI) = g(x) + [(g + h)(x + y) - g(x)]I$$

The neutrosophic linear transformation f is called a full AH-linear transformation.

Definition 2.4.

Let $V(I) = V + VI$be a strong/weak neutrosophic vector space, the set $S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and $Q$are subspaces of $V$is called an AH-subspace of $V(I)$.

If $P = Q$, then S is called an AHS-subspace of $V(I)$.

Definition 2.5.

Let $f = g + hI: V(I) \to W(I)$ be a full AH-linear transformationand $M = A + BI$ be an $n \times m$ neutrosophic matrix over $F(I)$, and we call M the neutrosophic matrix of f if and only if $f(x + yI) = M(x + yI)$ for every $x + yI \in V(I)$.

Example 2.1.

(a). Let $V(I) = R^2(I) = \{(a, b) + (c, d)I = a + cI , b + dI ; a, b, c, d \in R\}$, consider the following neutrosophic matrix $M = \begin{pmatrix} 1+I & I \\ -I & 2-I \end{pmatrix}$.The corresponding neutrosophic linear transformation is defined as follows:

$$f(x + yI) = g + hI = M(x + yI) = \begin{pmatrix} 1+I & I \\ -I & 2-I \end{pmatrix}(x + yI)$$

$$= \begin{pmatrix} 1+I & I \\ -I & 2-I \end{pmatrix}\begin{pmatrix} a+cI \\ b+dI \end{pmatrix} = (a + I[c + a + c + b + d], -aI - cI + 2b + 2dI - bI - dI)$$

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$= (a + I[a + 2c + b + d], 2b + I[-a - c - b + d]) = (a, 2b) + I(a + 2c + b + d, -a - c - b + d)$

$= (a, 2b) + I\big(a + 2c + b + d, -(a + c) - (b + d)\big)$

$(b). f = g + hI; g(x, y) = (x, 2y), h(x, y) = (x + y, -x - y)$,where $g, h: V \to V$.

Definition 2.6.

Let $V$and W be two spaces, $L_v: V \to W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

$L: V(I) \to W(I); L(a + bI) = L_v(a) + L_v(b)I.$

Definition 2.7:

(a) Let V and W be two vector spaces, $L_V: V \to W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

$L: V(I) \to W(I); L(a + bI) = L_V(a) + L_V(b)I.$

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S) = L_V(P) + L_V(Q)I$.

(c) If $S = P + QI$ is an AH-subspace of W(I), $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$.

(d) $AH - Ker\ L = KerL_V + KerL_V\ I = \{x + yI; x, y \in KerL_V\}$.

Theorem 2.1:

Let W(I) and V(I) be two neutrosophic strong/weak vector spaces, and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have:

(a) $AH - Ker\ L$ is an AHS-subspace of V(I).

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S)$ is an AH-subspace of W(I).

(c) If $S = P + QI$ is an AH-subspace of W(I), $L^{-1}(S)$ is an AH-subspace of V(I).

Theorem 2.2:

Let W(I) and V(I) be two neutrosophic strong vector spaces over a neutrosophic field K(I), and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have:

$L(x + y) = L(x) + L(y), L(m.x) = m.L(x)$, for all $x, y \in V(I), m \in K(I)$.

Theorem 2.3:

Let $S = P + QI$ be an AH-subspace of a neutrosophic weak vector space V(I) over a field K, suppose that

$X = \{x_i; 1 \le i \le n\}$ is a bases of P and $Y = \{y_j; 1 \le j \le m\}$ is a bases of Q then $X \cup YI$ is a bases of S.

Result 2.1: Let $S = P + QI$ be an AH-subspace of a neutrosophic weak vector space V(I) with finite dimension over a field K, from theorem 2.3 and the fact that $X \cup YI = \emptyset$, we find $dim(S) = dim(P) + dim(Q)$.

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3. n-Refined neutrosophic vector space

Definition 3.1.

Let $(V,+,.)$ be any vector space over a field K. Then we say that $V_n(I)=V+VI_1+\cdots+VI_n=\{x_0+x_1I_1+\cdots+x_nI_n; x_i\in V\}$is a weak n-refined neutrosophic vector space over the field K. Elements of $V_n(I)$ are called n-refined neutrosophic vectors, elements of $K$are called scalars.

Example 3.1.

Let $\boldsymbol{V}=\boldsymbol{R}^2$ be a vector space over the field $\boldsymbol{R}$, $\boldsymbol{W}=\langle \boldsymbol{0},\boldsymbol{1}\rangle$ is a subspace of $\boldsymbol{V}$, $\boldsymbol{R}^2(\boldsymbol{I})=\{(\boldsymbol{a},\boldsymbol{b})+(\boldsymbol{m},\boldsymbol{s})\boldsymbol{I_1}+(\boldsymbol{k},\boldsymbol{t})\boldsymbol{I_2}; \boldsymbol{a},\boldsymbol{b},\boldsymbol{m},\boldsymbol{s},\boldsymbol{k},\boldsymbol{t}\in\boldsymbol{R}\}$ is a corresponding weak/strong 2-refined neutrosophic vector space.

$\boldsymbol{W_2}(\boldsymbol{I})=\{\boldsymbol{a_0}+\boldsymbol{a_1}\boldsymbol{I_1}+\boldsymbol{a_2}\boldsymbol{I_2}\}=\{(\boldsymbol{0},\boldsymbol{x})+(\boldsymbol{0},\boldsymbol{y})\boldsymbol{I_1}+(\boldsymbol{0},\boldsymbol{z})\boldsymbol{I_2}; \boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\boldsymbol{R}\}$ is a weak 2-refined neutrosophic subspace of the weak 2-refined sneutrosophic vector space $\boldsymbol{R^2}_{\boldsymbol{2}}(\boldsymbol{I})$ over the field $\boldsymbol{R}$.

$\boldsymbol{W_2}(\boldsymbol{I})=\{\boldsymbol{a_0}+\boldsymbol{a_1}\boldsymbol{I_1}+\boldsymbol{a_2}\boldsymbol{I_2}\}=\{(\boldsymbol{0},\boldsymbol{x})+(\boldsymbol{0},\boldsymbol{y})\boldsymbol{I_1}+(\boldsymbol{0},\boldsymbol{z})\boldsymbol{I_2}; \boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\boldsymbol{R}\}$ is a strong 2-refined neutrosophic subspace of the strong 2-refined sneutrosophic vector space $\boldsymbol{R^2}_{\boldsymbol{2}}(\boldsymbol{I})$ over the n-refined neutrosophic field $\boldsymbol{R_2}(\boldsymbol{I})$.

Remark 3.1: If $\boldsymbol{n}=\boldsymbol{2}$ then the n-refined neutrosophicvector space is called a refined neutrosophicvector space.

Definition 3.2:

Let $\boldsymbol{V_n}(\boldsymbol{I}),\boldsymbol{W_n}(\boldsymbol{I})$ be two strong n-refined neutrosophic vector space over the n-refined neutrosophic field $\boldsymbol{K_n}(\boldsymbol{I})$, let $\boldsymbol{f}:\boldsymbol{V_n}(\boldsymbol{I})\to\boldsymbol{U_n}(\boldsymbol{I})$ be a well defined map. It is called a strong n-refined neutrosophic homomorphism if:

$\boldsymbol{f}(\boldsymbol{a}.\boldsymbol{x}+\boldsymbol{b}.\boldsymbol{y})=\boldsymbol{a}.\boldsymbol{f}(\boldsymbol{x})+\boldsymbol{b}.\boldsymbol{f}(\boldsymbol{y})$ for all $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{V_n}(\boldsymbol{I}),\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{K_n}(\boldsymbol{I})$.

A weak n-refined neutrosophic homomorphism can be defined as the same.

We can understand the strong n-refined homomorphism as a module homomorphism, weak n-refined neutrosophic homomorphism can be understood as a vector space homomorphism.

Definition 3.3:

Let $\boldsymbol{V_n}(\boldsymbol{I})$ be a strong n-refined neutrosophic vector space over an n-refined neutrosophic field $\boldsymbol{K_n}(\boldsymbol{I})$, $\boldsymbol{x}$ be an arbitrary element of $\boldsymbol{V_n}(\boldsymbol{I})$, we say that $\boldsymbol{x}$ is a linear combination of $\{\boldsymbol{x_1},\boldsymbol{x_2},\dots,\boldsymbol{x_m}\}\subseteq\boldsymbol{V_n}(\boldsymbol{I})$ is $\boldsymbol{x}=\boldsymbol{a_1}\boldsymbol{x_1}+\boldsymbol{a_2}\boldsymbol{x_2}+\cdots+\boldsymbol{a_m}\boldsymbol{x_m}$: $\boldsymbol{a_i}\in\boldsymbol{K_n}(\boldsymbol{I}),\boldsymbol{x_i}\in\boldsymbol{V_n}(\boldsymbol{I})$.

Example 3.2:

Consider the strong 2-refined neutrosophic vector space $\boldsymbol{R^2_2}(\boldsymbol{I})=\{(\boldsymbol{a},\boldsymbol{b})+(\boldsymbol{m},\boldsymbol{s})\boldsymbol{I_1}+(\boldsymbol{k},\boldsymbol{t})\boldsymbol{I_2}; \boldsymbol{a},\boldsymbol{b},\boldsymbol{m},\boldsymbol{s},\boldsymbol{k},\boldsymbol{t}\in\boldsymbol{R}\}$ over the 2-refined neutrosophic field $\boldsymbol{R_2}(\boldsymbol{I})$,

$\boldsymbol{x}=(\boldsymbol{0},\boldsymbol{2})+(\boldsymbol{3},\boldsymbol{3})\boldsymbol{I_1}+(-\boldsymbol{1},\boldsymbol{0})\boldsymbol{I_2}=(\boldsymbol{2}+\boldsymbol{I_1})\cdot(\boldsymbol{0},\boldsymbol{1})+(\boldsymbol{1}+\boldsymbol{I_2})\cdot(\boldsymbol{1},\boldsymbol{1})\boldsymbol{I_1}+(\boldsymbol{I_1}-\boldsymbol{I_2})\cdot(\boldsymbol{1},\boldsymbol{0})\boldsymbol{I_2}$, hence $\boldsymbol{x}$ is a linear combination of the set $\{(\boldsymbol{0},\boldsymbol{1}),(\boldsymbol{1},\boldsymbol{1})\boldsymbol{I_1},(\boldsymbol{1},\boldsymbol{0})\boldsymbol{I_2}\}$ over the 2-refined neutrosophic field $\boldsymbol{R_2}(\boldsymbol{I})$.

Definition 3.4:

Let $\boldsymbol{f}:\boldsymbol{V_n}(\boldsymbol{I})\to\boldsymbol{U_n}(\boldsymbol{I})$ be a weak/strong n-refined neutrosophic homomorphism, we define:

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(a) $\boldsymbol{Ker(f) = \{x \in V_n(I); f(x) = 0\}}$.

(b) $\boldsymbol{Im(f) = \{y \in U_n(I);\ \exists x \in V_n(I) \text{and} y = f(x)\}}$.

Theorem 3.1:

Let $\boldsymbol{f: V_n(I) \to U_n(I)}$ be a weak n-refined neutrosophic homomorphism. Then

(a) $\boldsymbol{Ker(f)}$ is a weak n-refined neutrosophic subspace of $\boldsymbol{V_n}$(I).

(b) $\boldsymbol{Im(f)}$ is a weak n-refined neutrosophic subspace of $\boldsymbol{U_n(I)}$.

Theorem 3.2:

Let $\boldsymbol{f: V_n(I) \to U_n(I)}$ be a strong n-refined neutrosophic homomorphism. Then

(a) $\boldsymbol{Ker(f)}$ is a strong n-refined neutrosophic subspace of $\boldsymbol{V_n}$(I).

(b) $\boldsymbol{Im(f)}$ is a strong n-refined neutrosophic subspace of $\boldsymbol{U_n(I)}$.

Example 3.3:

Let $\boldsymbol{W_2(I) =< (0,0,1)I_1 >= \{q.(0,0,a)I_1; a \in R, q \in R_2(I)\}}$, $\boldsymbol{U_2(I) =< (0,1,0)I_1 >= \{q.(0,a,0)I_1; a \in R; q \in R_2(I)\}}$ be two strong 2-refined neutrosophic subspaces of the strong 2-refined neutrosophic vector space $\boldsymbol{R_2^3(I)}$ over 2-refined neutrosophic field $\boldsymbol{R_2}$(I). Define $\boldsymbol{f: W_2(I) \to U_2(I); f[q(0,0,a)I_1] = q(0,a,0)I_1\ ; q \in R_2(I)}$.

$\boldsymbol{f}$ is a strong 2-refined neutrosophic homomorphism:

Let $\boldsymbol{A = q_1(0,0,a)I_1, B = q_2(0,0,b)I_1 \in W_2(I);\ q_1, q_2 \in R_2(I)}$, we have $\boldsymbol{A + B = (q_1 + q_2)(0,0,a + b)I_1, f(A + B) = (q_1 + q_2).(0, a + b, 0)I_1 = f(A) + f(B)}$.

Let $\boldsymbol{m = c + dI_1 + eI_2 \in R_2(I)}$ be a 2-refined neutrosophic scalar, we have

$$\boldsymbol{m \cdot A = c \cdot q_1(0,0,a)I_1 + d \cdot q_1(0,0,a)I_1I_1 + e \cdot q_1(0,0,a)I_2I_1 = q_1(0,0,c.a + d.a + e.a)I_1},$$

$\boldsymbol{f(m.A) = q_1(0, c.a + d.a + e.a, 0)I_1 = m \cdot f(A)}$, hence $\boldsymbol{f}$ is a strong 2-refined neutrosophic homomorphism.

$\boldsymbol{Ker(f) = (0,0,0) + (0,0,0)I_1 + (0,0,0)I_2}$.

$\boldsymbol{Im(f) = U_2(I)}$.

4. Refined neutrosophic matrix.

Definition 4.1. Let $\boldsymbol{A = \left(\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}\right)}$ be an $\boldsymbol{n \times m}$ matrix; if $\boldsymbol{a_{ij} = x + yI_1 + zI_2 \in R_2(I)}$, then it is called an refined neutrosophic matrix, where $\boldsymbol{R_2(I)}$ is an refined neutrosophic field.

Definition 4.2. A refined neutrosophic matrix can be defined as follow as:

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$\boldsymbol{M = A + BI_1 + CI_2}$, where $\boldsymbol{A, B, C}$ is a classical matrices.

Example 4.1: Let $M = A + BI_1 + CI_2 = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix} I_1 + \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix} I_2$ is a 2×2refined neutrosophic matrix.

Definition 4.3. Let $\boldsymbol{M = A + BI_1 + CI_2}$ be an square refined neutrosophic matrix. then the determinate of $\boldsymbol{M}$ defined as follows:

$$\boldsymbol{detM = det(A + BI_1 + CI_2) = detA + [det(A + B + C) - det(A + C)]I_1 + [det(A + C) - detA]I_2}$$

Example 4.2. Consider the following neutrosophic matrix

$\boldsymbol{M = A + BI_1 + CI_2}$. Where $= \begin{pmatrix} \boldsymbol{2} & \boldsymbol{1} \\ \boldsymbol{3} & \boldsymbol{1} \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{0} & \boldsymbol{1} \end{pmatrix}$, $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{3} & \boldsymbol{-1} \\ \boldsymbol{4} & \boldsymbol{0} \end{pmatrix}$. Then.

$A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $+C = \begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}$, $A + B + C = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}$ Then.

$detA = -1$, $det(A + B + C) = 19$, $det(A + C) = 5$

$$detM = detA + [det(A + B + C) - det(A + C)]I_1 + [det(A + C) - detA]I_2 = -1 + [19 - 5]I_1 + [5 + 1]I_2 = \\ = -1 + 4I_1 + 6I_2$$

Definition 4.4: Let $M = A + BI_1 + CI_2$ be an square refined neutrosophic matrix. then the invertible of M defined as follows:

$$\boldsymbol{M^{-1} = A^{-1} + ((A + B + C)^{-1} - (A + C)^{-1})I_1 + ((A + C)^{-1} - A^{-1})I_2}$$

Theorem 4.1: Let $M = A + BI_1 + CI_2$ be a square $n \times n$ refined neutrosophic matrix; then it is invertible if only of $A, A + C$ and $A + B + C$ are invertible. The inverse of M is

$$M^{-1} = A^{-1} + ((A + B + C)^{-1} - (A + C)^{-1})I_1 + ((A + C)^{-1} - A^{-1})I_2$$

Example 4.3: Consider the following refined neutrosophic matrix

$\boldsymbol{M = A + BI_1 + CI_2}$. Where $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{2} & \boldsymbol{1} \\ \boldsymbol{3} & \boldsymbol{1} \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{1} & \boldsymbol{-1} \\ \boldsymbol{0} & \boldsymbol{1} \end{pmatrix}$, $\boldsymbol{C} = \begin{pmatrix} \boldsymbol{3} & \boldsymbol{-1} \\ \boldsymbol{4} & \boldsymbol{0} \end{pmatrix}$. Then

$A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$, $A + C = \begin{pmatrix} 5 & 0 \\ 7 & 1 \end{pmatrix}$, $A + B + C = \begin{pmatrix} 6 & -1 \\ 7 & 2 \end{pmatrix}$. Then

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix}, (A + C)^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{-7}{5} & 1 \end{pmatrix}, (A + B + C)^{-1} = \begin{pmatrix} \frac{2}{19} & \frac{1}{19} \\ \frac{-7}{19} & \frac{6}{19} \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} + \left(\begin{pmatrix} \frac{2}{19} & \frac{1}{19} \\ \frac{-7}{19} & \frac{6}{19} \end{pmatrix} - \begin{pmatrix} \frac{1}{5} & 0 \\ \frac{-7}{5} & 1 \end{pmatrix} \right) I_1 + \left(\begin{pmatrix} \frac{1}{5} & 0 \\ \frac{-7}{5} & 1 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \right) I_2$$

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$$M^{-1}=\begin{pmatrix}-1 & 1\\ 3 & -2\end{pmatrix}+\begin{pmatrix}\frac{-9}{95} & \frac{1}{19}\\ \frac{98}{95} & \frac{-13}{19}\end{pmatrix}I_1+\begin{pmatrix}\frac{6}{5} & -1\\ \frac{-22}{5} & 3\end{pmatrix}I_2$$

$$M^{-1}=\begin{pmatrix}-1-\frac{9}{95}I_1+\frac{6}{5}I_2 & 1+\frac{1}{19}I_1-I_2\\ 3+\frac{98}{95}I_1-\frac{22}{5}I_2 & -2-\frac{13}{19}I_1+3I_2\end{pmatrix}$$

5. n-refined neutrosophic matrix.

Definition 5.1: An n-refined neutrosophic matrix can be defined as follow as:

$\boldsymbol{A=A_0+A_1I_1+A_2I_2+\cdots+A_nI_n}$, where $\boldsymbol{A_0,A_1,A_2,\dots,A_n}$ are classical matrices.

We can put:

$$\boldsymbol{N_0=A_0,\ N_j=A_0+A_j+A_{j+1}+\cdots+A_n;\ 1\le j\le n}$$

Example 5.1: Let $A=A_0+A_1I_1+A_2I_2+A_3I_3$ where $A_0=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, $A_1=\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}$, $A_2=\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix}$,

$A_3=\begin{pmatrix}-2 & 1\\ 1 & 3\end{pmatrix}$ is a 2×2 3-refined neutrosophic matrix.

Definition 5.2. Let $\boldsymbol{A=A_0+A_1I_1+A_2I_2+A_3I_3}$ be an square n-refined neutrosophic matrix, and $\boldsymbol{N_0=A_0,\ N_j=A_0+A_j+A_{j+1}+\cdots+A_n;\ 1\le j\le n}$ then the determinate of $\boldsymbol{A}$ defined as follows:

$$detA=detA_0+[det(A_0+A_1+A_2+\cdots+A_n)-det(A_0+A_2+\cdots+A_n)]I_1 \\ +[det(A_0+A_2+\cdots+A_n)-det(A_0+A_3+\cdots+A_n)]I_2+\cdots+[det(A_0+A_n)-detA_0]I_n$$

$$detA=detA_0+\sum_{i=1}^{n-1}[det(N_i)-det(N_{i+1})]I_i+[det(N_n)-det(N_0)]I_n$$

Example 4.2: Consider the following 3-refined neutrosophic matrix

$A=A_0+A_1I_1+A_2I_2+A_3I_3$ where $A_0=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, $A_1=\begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix}$, $A_2=\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix}$, $A_3=\begin{pmatrix}-2 & 1\\ 1 & 3\end{pmatrix}$

Then.

$N_0=A_0=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\Rightarrow detN_0=1$

$N_j=A_0+A_j+A_{j+1}+\cdots+A_n\ ;1\le j\le n$

$N_1=A_0+A_1+A_2+A_3=\begin{pmatrix}0 & 2\\ 3 & 7\end{pmatrix}\Rightarrow detN_1=-6$

$N_2=A_0+A_2+A_3=\begin{pmatrix}-1 & 2\\ 2 & 6\end{pmatrix}\Rightarrow detN_2=-10$

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$N_3 = A_0 + A_3 = \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix} \Rightarrow detN_1 = -6$

$detA = detN_0 + [det(N_1) - det(N_2)]I_1 + [det(N_2) - det(N_3)]I_2 + \cdots + [det(N_3) - det(N_0)]I_3$

$detA = 1 + [-6 + 10]I_1 + [-10 + 6]I_2 + [-6 - 1]I_3$

$detA = 1 + 4I_1 - 4I_2 - 7I_3$

Definition 5.3: Let $A = A_0 + A_1I_1 + A_2I_2 + A_3I_3$ be an square n-refined neutrosophic matrix, and $N_0 = A_0,\ N_j = A_0 + A_j + A_{j+1} + \cdots + A_n;\ 1 \le j \le n$. then the invertible of A defined as follows:

$$\boldsymbol{A^{-1} = A_0{}^{-1} + \sum_{i=1}^{n-1}[(N_i)^{-1} - (N_{i+1})^{-1}]I_i + [(N_n)^{-1} - (N_0)^{-1}]I_n}$$

Example 5.3: Consider the following neutrosophic matrix

$A = A_0 + A_1I_1 + A_2I_2 + A_3I_3$ .Where$A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$, $A_3 = \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix}$

Then.

$N_0 = A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow (N_0)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$N_j = A_0 + A_j + A_{j+1} + \cdots + A_n\ ;\ 1 \le j \le n$

$N_1 = A_0 + A_1 + A_2 + A_3 = \begin{pmatrix} 0 & 2 \\ 3 & 7 \end{pmatrix} \Rightarrow (N_1)^{-1} = \begin{pmatrix} 7 & \frac{-1}{2} \\ \frac{-1}{3} & 0 \end{pmatrix}$

$N_2 = A_0 + A_2 + A_3 = \begin{pmatrix} -1 & 2 \\ 2 & 6 \end{pmatrix} \Rightarrow (N_2)^{-1} = \begin{pmatrix} \frac{6}{10} & \frac{2}{10} \\ \frac{2}{10} & \frac{-1}{10} \end{pmatrix}$

$N_3 = A_0 + A_3 = \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix} \Rightarrow (N_3)^{-1} = \begin{pmatrix} \frac{-4}{6} & \frac{2}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}$

$A^{-1} = (N_0)^{-1} + [(N_1)^{-1} - (N_2)^{-1}]I_1 + [(N_2)^{-1} - (N_3)^{-1}]I_2 + [(N_3)^{-1} - (N_0)^{-1}]I_3.$

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$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \left[\begin{pmatrix} 7 & \frac{-1}{2} \\ \frac{-1}{3} & 0 \end{pmatrix} - \begin{pmatrix} \frac{6}{10} & \frac{2}{10} \\ \frac{2}{10} & \frac{-1}{10} \end{pmatrix}\right] I_1 + \left[\begin{pmatrix} \frac{6}{10} & \frac{2}{10} \\ \frac{2}{10} & \frac{-1}{10} \end{pmatrix} - \begin{pmatrix} \frac{-4}{6} & \frac{2}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}\right] I_2 + \left[\begin{pmatrix} \frac{-4}{6} & \frac{2}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right] I_3$$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{63}{10} & \frac{-7}{10} \\ \frac{-13}{10} & \frac{1}{10} \end{pmatrix} I_1 + \begin{pmatrix} \frac{76}{60} & \frac{-8}{60} \\ \frac{-2}{60} & \frac{-16}{60} \end{pmatrix} I_2 + \begin{pmatrix} \frac{-10}{6} & \frac{8}{6} \\ \frac{1}{6} & \frac{-5}{6} \end{pmatrix} I_3$$

Conclusion

In this article, we have given a review study for many interesting algebraic neutrosophic linear structures such as neutrosophic vector spaces, AH-subspaces, n-refined neutrosophic vector spaces, and AH-linear transformations. On the other hand, many related concepts were shown and discussed such as refined neutrosophic matrices, and n-refined neutrosophic matrices.

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