An Introduction to Neutro-Fine Topology with Separation Axioms and Decision Making

V. Chinnadurai[1] and M. P. Sindhu[2]

[1]Department of Mathematics, Annamalai University, Annamalainagar, TN, India; kv.chinnaduraimaths@gmail.com
[2]Department of Mathematics, Karpagam College of Engineering, Coimbatore, TN, India; sindhu.mp@kce.ac.in

Correspondence: kv.chinnaduraimaths@gmail.com

Abstract

The set which describes the uncertainty incident with three levels of attributes is entitled as a neutrosophic set. The unique collection of open sets which contains all types of open sets is termed as fine-open sets. The current study introduces a topology on merging these two sets, called neutro-fine topological space. Additionally, the approach of separation axioms is implemented in such space. Furthermore, the real-life application is examined as a decision-making problem in this space. The problem is to make an unfavorable query into a favorable one by determining the complement and absolute complement of such issued neutro-fine open sets. This problem desires to find a positive solution. The solving stepwise mechanism reveals in the algorithm, also formulae provide to compute the outcome with explanatory examples.

Keywords: Subset of neutrosophic sets, neutro-fine sets (NFSs), neutro-fine topological space (NFTS), neutro-fine open sets (NFOSs), neutro-fine interior and closure, neutro-fine $T_{i=0,1,2}$ -spaces, absolute complement, decision making (DM).

1. Introduction

The fuzzy set (FS) is an advanced version of the classical set. FS was introduced in (1965) by Zadeh [32], whose elements describe vague features as true and false membership functions. The FS theory applied in the boundless area of a domain, while in (1986) this theory was extended as an intuitionistic fuzzy set (IFS) theory by Atanassov [6]. Later, in 1998 Smarandache [26] explored a neutrosophic set (NS) that contains one more membership function called indeterminacy degrees. Also, he widespreaded the NS on IFS [27] and newly projected his work on features valued set, called plithogenic set (PS) [28]. Nowadays, these sets made an outstanding impact on many applications [1, 2, 11, 18, 29] and plays vital role in COVID-19 [25], decision making (DM) problems [3, 4, 5] and multi-criteria DM (MCDM) problems [10, 21].

Topology is a study of flexible objects under frequent damages without splitting. In recent times, topological space (TS) is performing a lead character in the enormous branch of applied sciences and numerous categories of mathematics. The topological structure developed on NS as a generalization of IFTS which was originated in (2012) by Salama & Alblowi [23, 24], named as neutrosophic topological space (NTS). Few typical sets, open sets, and other TS explored [8, 9, 14, 15, 20, 30], and extended to separation axioms [7, 31] on such TS.

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The most general class of sets which contains few open sets termed as fine-open sets (FOSs), in (2012) by Powar & Rajak [16], and investigated the special case of generalized TS, called fine- topological space (FTS). Many researchers studied this concept on some sets like FS [13, 22] and others [12, 19, 17].

This paper desires to initiate the new form of TS to put together NS and FS, through defining the concept of the subset of NSs on these TS. The notion of the interior and closure are launched and certain theorems are stated with proof, also disproved in counter examples. The idea of separation axioms is also executed in NFTS. Moreover, the DM problem describes the negative state of the problem into a positive solution by determining the absolute complement of each NFOS. The procedure of this problem-solving method is listed in the algorithm and a unique decision is computed with the specified formula.

The layout of this study are arranged as follows. In Section 2, essential definitions of NS and FOS are recollected. In Section 3, the subset of NS, NFS, NFTS, interior, and closure of NFTS are defined and investigated its properties with illustrative examples. In Section 4, the correlation of neutro-fine $T_{i=0,1,2}$-spaces are explored via perfect examples. In Section 5, the real-life application is intimated to take a correct decision on DM problems and an example is investigated in two different manners. At the end of Section 6, the conclusions are conveyed with future works.

2. Fundamental concepts

In this section, some essential definitions associated with this work are pointed.

Definition 2.1 [27] Let W be a non-empty set and $w \in W$. A NS R in W is characterized by a truth-membership function T_R, an indeterminacy-membership function I_R, and a false-membership function F_R which are subsets of $]^-0,1^+[$ and is defined as

$$R = \{\langle w, T_R(w), I_R(w), F_R(w)\rangle : w \in W\},$$

where

$$0 \le \sup T_R(w) + \sup I_R(w) + \sup F_R(w) \le 3 .$$

Definition 2.2 [31] Let $NS(W)$ be the family of all NSs over the universe W and $w \in W$. Then NS is said to be a neutrosophic point (NP), for $0 \le \alpha, \beta, \gamma \le 1$ and is defined as follows:

$$w^{\langle \alpha,\beta,\gamma\rangle}(v) = \begin{cases} (\alpha,\beta,\gamma), & if \quad w = v \\ (0,0,1), & if \quad w \ne v \end{cases}.$$

Every NS is the union of its NPs.

Example 2.3 [31] Let $W = \{w_1, w_2, w_3\}$. Then NS

$$R = \{\langle w_1,.2,.4,.7\rangle, \langle w_2,.6,.3,.1\rangle, \langle w_3,.4,.5,.6\rangle\}$$

is the union of NPs $w_1^{\langle .2,.4,.7\rangle}$, $w_2^{\langle .6,.3,.1\rangle}$ and $w_3^{\langle .4,.5,.6\rangle}$.

Definition 2.4 [23] Let $NS(W)$ denote the family of all NSs over W and $\tau_n \subset NS(W)$. Then τ_η is called a neutrosophic topology (NT) on W if it satisfies the following conditions

(i) $0_n, 1_n \in \tau_n$, where null NS $0_n = \{\langle w,0,0,1)\rangle : w \in W\}$ and an absolute NS $1_n = \{\langle w,1,1,0)\rangle : w \in W\}$.

(ii) the intersection of any finite number of members of τ_n belongs to τ_n.

(iii) the union of any collection of members of τ_n belongs to τ_n.

Then the pair (W, τ_n) is called a NTS.

Every member of τ_n is called τ_n-open neutrosophic set (τ_n - ONS). An NS is called τ_n-closed (τ_n - CNS) if and only if its complement is τ_n - ONS.

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Definition 2.5 [23] Let R be a NS over W. Then the complement of R is denoted by R' and defined by

$$R' = \{\langle w, F_R(w), 1 - I_R(w), T_R(w)\rangle : w \in W\}.$$

Clearly, $(R')' = R$.

Example 2.6 [23] Let $W = \{w_1, w_2, w_3\}$ and $\tau_n = \{0_n, 1_n, R, S, T, U\}$ where R, S, T, and U are NSs over W and are defined as follows

$$R = \{\langle w_1, .2, .4, .7\rangle, \langle w_2, .6, .3, .1\rangle, \langle w_3, .4, .5, .6\rangle\},$$
$$S = \{\langle w_1, .9, .3, .6\rangle, \langle w_2, .6, .5, .4\rangle, \langle w_3, .7, .8, .1\rangle\},$$
$$T = \{\langle w_1, .9, .4, .6\rangle, \langle w_2, .6, .5, .1\rangle, \langle w_3, .7, .8, .1\rangle\} \text{ and}$$
$$U = \{\langle w_1, .2, .3, .7\rangle, \langle w_2, .6, .3, .4\rangle, \langle w_3, .4, .5, .6\rangle\}.$$

Here $R \bigcup S = T$, $R \bigcup T = T$, $R \bigcup U = R$, $S \bigcup T = T$, $S \bigcup U = S$, $T \bigcup U = T$ and $R \bigcap S = U$, $R \bigcap T = R$, $R \bigcap U = U$, $S \bigcap T = S$, $S \bigcap U = U$, $T \bigcap U = U$.

Then R, S, T and U are τ_n - ONSs over W.

Thus (W, τ_n) is a NTS over W.

The complement of τ_n - ONSs are

$$R' = \{\langle w_1, .7, .6, .2\rangle, \langle w_2, .1, .7, .6\rangle, \langle w_3, .6, .5, .4\rangle\},$$
$$S' = \{\langle w_1, .6, .7, .9\rangle, \langle w_2, .4, .5, .6\rangle, \langle w_3, .1, .2, .7\rangle\},$$
$$T' = \{\langle w_1, .6, .6, .9\rangle, \langle w_2, .1, .5, .6\rangle, \langle w_3, .1, .2, .7\rangle\} \text{ and}$$
$$U' = \{\langle w_1, .7, .7, .2\rangle, \langle w_2, .4, .7, .6\rangle, \langle w_3, .6, .5, .4\rangle\}.$$

Then R', S', T' and U' are τ_n - CNSs over W.

Definition 2.7 [16] Let (W, τ) be a topological space and define

$$\varsigma(R_i) = \varsigma_i = \{K_i (\neq W) : K_i \subset W, R_i \cap K_i \neq \varphi, \text{for } R_i \in \tau \text{ and } R_i \neq \varphi, W, \text{for some } i \in I, \text{where I is the index set}\}.$$

Now, define $\tau_f = \left\{\phi, W, \bigcup_{i \in I} \varsigma_i\right\}$.

This collection τ_f of subsets of W is called the fine collection of subsets of W and (W, τ, τ_f) is said to be the fine space W generated by the topology τ on W.

Definition 2.8 [16] A subset U of a fine space W is said to be fine-open sets of W if U belongs to the collection τ_f and the complement of every fine-open set of W is called the fine-closed sets of W and denote the collection by F_f.

Example 2.9 [16] Let $W = \{w_1, w_2, w_3\}$ with topology $\tau = \{\phi, W, \{w_1\}\}$.

Clearly, (W, τ) is a topological space over W.

Then (W, τ, τ_f) is a fine-topological space over W,

where the members in

$$\tau_f = \{\phi, W, \{w_1\}, \{w_1, w_2\}, \{w_1, w_3\}\}$$

are fine-open sets, and in

$$F_f = \{\phi, W, \{w_2, w_3\}, \{w_3\}, \{w_2\}\}$$

are fine-closed sets.

Example 2.10 [16] Let $W = \{w_1, w_2, w_3\}$ with topology $\tau = \{\phi, W, \{w_1\}, \{w_1, w_3\}\}$.

Clearly, (W, τ) is a topological space over W.

Then the collection of fine-open sets is

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$$\tau_f = \{\phi, W, \{w_1\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_3\}, \{w_3, w_2\}\}$$

and the collection of fine-closed sets is

$$F_f = \{\phi, W, \{w_2, w_3\}, \{w_3\}, \{w_2\}, \{w_1, w_2\}, \{w_1\}\}.$$

Thus (W, τ, τ_f) is a fine-topological space over W.

Here $\{w_1, w_2\}, \{w_3, w_2\} \in \tau_f$ but $\{w_1, w_2\} \cap \{w_3, w_2\} = \{w_2\} \notin \tau_f$.

Also, $\{w_1\}, \{w_3\} \in F_f$ but $\{w_1\} \cup \{w_3\} = \{w_1, w_3\} \notin F_f$.

Hence (W, τ_f) and (W, F_f) are not topological space over W.

3. Neutro-Fine Topology

In this section, the conception of NFTS is defined and probable results are carried by some major expressive examples.

Definition 3.1 Let W be a set of universe and $w_i \in W$ where $i \in I$. Let R be a NS over W. Then the subset of NS (sub-NS) R is denoted as $\varsigma_R(W^*)$ and defined as

$$\varsigma_R(W^*) = \{\langle w_i, T_R(w_i), I_R(w_i), F_R(w_i)\rangle, \langle w_{i,j}, \max(T_R(w_i), T_R(w_j)), \max(I_R(w_i), I_R(w_j)), \min(F_R(w_i), F_R(w_j))\rangle\}.$$

where $i, j \in I$ and $i \neq j$.

Clearly, $(w_{i,j}) = (w_{j,i})$.

Example 3.2 Let $W = \{w_1, w_2, w_3\}$ be a set of features of the refrigerator, where w_1 = energy efficiency, w_2 = capacity, w_3 =price. Let R be a NS over W, defined as

$$R = \{\langle w_1, .7, .5, .4\rangle, \langle w_2, .2, .7, .9\rangle, \langle w_3, .4, .1, .3\rangle\}.$$

Then the sub-NS R is

$$\varsigma_R(W^*) = \{\langle w_1, .7, .5, .4\rangle, \langle w_2, .2, .7, .9\rangle, \langle w_3, .4, .1, .3\rangle, \langle w_{1,2}, .7, .7, .4\rangle, \langle w_{1,3}, .7, .5, .3\rangle, \langle w_{2,3}, .4, .7, .3\rangle\}.$$

Definition 3.3 Let W be a set of universe and $w_i \in W$ where $i \in I$. Let R be a NS over W. Then the subset of NS R with respect to w_i (sub-NS R_{w_i}) and w_i, w_j (sub-NS R_{w_i, w_j}) are denoted as $\varsigma_R(w_i)$ and $\varsigma_R(w_i, w_j)$, and defined as

$$\varsigma_R(w_i) = \{\langle w_i, T_R(w_i), I_R(w_i), F_R(w_i)\rangle, \langle w_{i,j}, \max(T_R(w_i), T_R(w_j)), \max(I_R(w_i), I_R(w_j)), \min(F_R(w_i), F_R(w_j))\rangle,$$
$$\langle w_k, T_R(0_n), I_R(0_n), F_R(0_n)\rangle, \langle w_{k,l}, T_R(0_n), I_R(0_n), F_R(0_n)\rangle\}$$

where $i \in I$, $j \in I - \{i\}$, $k, l \in I - \{i, j\}$ and $k \neq l$

and

$$\varsigma_R(w_i, w_j) = \{\langle w_i, T_R(w_i), I_R(w_i), F_R(w_i)\rangle, \langle w_j, T_R(w_j), I_R(w_j), F_R(w_j)\rangle, \langle w_k, T_R(0_n), I_R(0_n), F_R(0_n)\rangle$$
$$\langle w_{i,j}, \max(T_R(w_i), T_R(w_j)), \max(I_R(w_i), I_R(w_j)), \min(F_R(w_i), F_R(w_j))\rangle,$$
$$\langle w_{i,k}, \max(T_R(w_i), T_R(w_k)), \max(I_R(w_i), I_R(w_k)), \min(F_R(w_i), F_R(w_k))\rangle,$$
$$\langle w_{j,k}, \max(T_R(w_j), T_R(w_k)), \max(I_R(w_j), I_R(w_k)), \min(F_R(w_j), F_R(w_k))\rangle\}$$

where $i, j, k \in I$ and $i \neq j \neq k$, respectively.

Definition 3.4 Let W be a set of universe and $w \in W$. Let R be a NS over W and V be any proper non- empty subset of W. Then $\varsigma_R(V)$ is said to be neutro-fine set (NFS) over W.

Example 3.5 Consider Example 3.2.

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Then NFS $\varsigma_R(w_3)$ is defined as

$$\varsigma_R(w_3)=\{\langle w_3,.4,.1,.3\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.4,.7,.3\rangle\}.$$

That is,

$$\varsigma_R(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.4,.1,.3\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.4,.7,.3\rangle\}.$$

and NFS $\varsigma_R(w_1,w_2)$ is defined as

$$\varsigma_R(w_1,w_2)=\{\langle w_1,.7,.5,.4\rangle,\langle w_2,.2,.7,.9\rangle,\langle w_{1,2},.7,.7,.4\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.4,.7,.3\rangle\}.$$

That is,

$$\varsigma_R(w_1,w_2)=\{\langle w_1,.7,.5,.4\rangle,\langle w_2,.2,.7,.9\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.7,.7,.4\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.4,.7,.3\rangle\}.$$

Definition 3.6 Let *V* be any proper non- empty subset of *W*. Then the null NFS is denoted as 0_{nf} and defined as

$$0_{nf}=\{\langle V,T_R(V)=0,I_R(V)=0,F_R(V)=1\rangle:\forall V\}.$$

Definition 3.7 Let *V* be any proper non- empty subset of *W*. Then the absolute NFS is denoted as 1_{nf} and defined as

$$1_{nf}=\{\langle V,T_R(V)=1,I_R(V)=1,F_R(V)=0\rangle:\forall V\}.$$

Definition 3.8 Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then their union is denoted as $\varsigma_R(V_1)\cup\varsigma_R(V_2)=\varsigma_R(V_{1\vee 2})$ and is defined as

$$\varsigma_R(V_{1\vee 2})=\{\langle V_{1\vee 2},\max(T_R(V_1),T_R(V_2)),\max(I_R(V_1),I_R(V_2)),\min(F_R(V_1),F_R(V_2))\rangle:\forall V\subset W,V\neq\varphi\}.$$

Definition 3.9 Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then their intersection is denoted as $\varsigma_R(V_1)\cap\varsigma_R(V_2)=\varsigma_R(V_{1\wedge 2})$ and is defined as

$$\varsigma_R(V_{1\wedge 2})=\{\langle V_{1\wedge 2},\min(T_R(V_1),T_R(V_2)),\min(I_R(V_1),I_R(V_2)),\max(F_R(V_1),F_R(V_2))\rangle:\forall V\subset W,V\neq\varphi\}.$$

Definition 3.10 Let $\varsigma_R(V)$ be a NFS over *W*. Then its complement is denoted as $\varsigma_R(V)'$ and is defined as

$$\varsigma_R(V)'=\{\langle V,T_R(V),1-I_R(V),F_R(V)\rangle:\forall V\subset W,V\neq\varphi\}.$$

Clearly, $(\varsigma_R(V)')'=\varsigma_R(V)$.

Definition 3.11 Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then $\varsigma_R(V_1)$ is said to be a neutro-fine subset of $\varsigma_R(V_2)$ if

$$T_R(V_1)\leq T_R(V_2),\ T_R(I_1)\leq T_R(I_2),\ F_R(V_1)\geq F_R(V_2),\ \forall V\subset W,V\neq\varphi.$$

It is denoted by $\varsigma_R(V_1)\subseteq\varsigma_R(V_2)$.

Also $\varsigma_R(V_1)$ is said to be neutro-fine equal to $\varsigma_R(V_2)$ if $\varsigma_R(V_1)$ is a neutro-fine subset of $\varsigma_R(V_2)$ and $\varsigma_R(V_2)$ is a neutro-fine subset of $\varsigma_R(V_1)$. It is denoted by $\varsigma_R(V_1)=\varsigma_R(V_2)$.

Proposition 3.12 Let $\varsigma_R(V_1)$, $\varsigma_R(V_2)$ and $\varsigma_R(V_3)$ be NFSs over *W*. Then,

(i) $\varsigma_R(V_1)\cup 0_{nf}=\varsigma_R(V_1)$.

(ii) $\varsigma_R(V_1)\cup 1_{nf}=1_{nf}$.

(iii) $\varsigma_R(V_1)\cup[\varsigma_R(V_2)\cup\varsigma_R(V_3)]=[\varsigma_R(V_1)\cup\varsigma_R(V_2)]\cup\varsigma_R(V_3)$.

(iv) $\varsigma_R(V_1)\cup[\varsigma_R(V_2)\cap\varsigma_R(V_3)]=[\varsigma_R(V_1)\cup\varsigma_R(V_2)]\cap[\varsigma_R(V_1)\cup\varsigma_R(V_3)]$.

Proof. Straightforward.

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Proposition 3.13 Let $\varsigma_R(V_1)$, $\varsigma_R(V_2)$ and $\varsigma_R(V_3)$ be NFSs over *W*. Then,

(i) $\varsigma_R(V_1) \cap 0_{nf} = 0_{nf}$.

(ii) $\varsigma_R(V_1) \cap 1_{nf} = \varsigma_R(V_1)$.

(iii) $\varsigma_R(V_1) \cap [\varsigma_R(V_2) \cap \varsigma_R(V_3)] = [\varsigma_R(V_1) \cap \varsigma_R(V_2)] \cap \varsigma_R(V_3)$.

(iv) $\varsigma_R(V_1) \cap [\varsigma_R(V_2) \cup \varsigma_R(V_3)] = [\varsigma_R(V_1) \cap \varsigma_R(V_2)] \cup [\varsigma_R(V_1) \cap \varsigma_R(V_3)]$.

Proof. Straightforward.

Proposition 3.14 Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then,

(i) $[\varsigma_R(V_1) \cup \varsigma_R(V_2)]' = \varsigma_R(V_1)' \cap \varsigma_R(V_2)'$.

(ii) $[\varsigma_R(V_1) \cap \varsigma_R(V_2)]' = \varsigma_R(V_1)' \cup \varsigma_R(V_2)'$.

Proof. Straightforward.

Proposition 3.15 Let $\varsigma_R(V_1)$, $\varsigma_R(V_2)$ and $\varsigma_S(V_1)$ be NFSs over *W*. Then,

(i) $R \subseteq S \Rightarrow \varsigma_R(V_1) \subseteq \varsigma_S(V_1)$.

(ii) $\varsigma_R(V_1) \cup \varsigma_R(V_2) = \varsigma_R(V_1 \cup V_2)$.

(iii) $\varsigma_R(V_1) \cap \varsigma_R(V_2) \subseteq \varsigma_R(V_1)$ and $\varsigma_R(V_1) \cap \varsigma_R(V_2) \subseteq \varsigma_R(V_2)$.

(iv) $\varsigma_R(V_1) \cup \varsigma_R(V_2) \supseteq \varsigma_R(V_1)$ and $\varsigma_R(V_1) \cup \varsigma_R(V_2) \supseteq \varsigma_R(V_2)$.

(v) $\varsigma_R(V_1) \subseteq \varsigma_R(V_2) \Rightarrow \varsigma_R(V_1)' \subseteq \varsigma_R(V_2)'$.

Proof. Straightforward.

Definition 3.16 Let *NFS*(*W*) be the family of all NFSs over *W*. Then the fine collection of $\varsigma_R(V)$ is denoted as ${}^f\varsigma_W$ and defined over the NT (W, τ_n) as

$${}^f\varsigma_W = \{0_{nf}, 1_{nf}, \cup \varsigma_R(V)\}.$$

Thus the triplet $(W, \tau_n, {}^f\varsigma_W)$ is said to be a neutro-fine topological space (NFTS) over (W, τ_n).
The elements belong to ${}^f\varsigma_W$ are said to be neutro-fine open sets (NFOSs) over (W, τ_n) and the complement of NFOSs are said to be neutro-fine closed sets (NFCSs) over (W, τ_n) and denote the collection by ${}^F\varsigma_W$.

Remark 3.17 If $(W, \tau_n, {}^f\varsigma_W)$ is a NFTS over (W, τ_n), then $(W, {}^f\varsigma_W)$ and $(W, {}^F\varsigma_W)$ are not NTSs over *W*.

Definition 3.18 Let *NFS*(*W*) be the family of all NFSs over *W*. Then ${}^f\varsigma_W$ is said be neutro-fine indiscrete topology if ${}^f\varsigma_W = \{0_{nf}, 1_{nf}\}$. Thus $(W, \tau_n, {}^f\varsigma_W)$ is said to be a neutro-fine indiscrete topological space over (W, τ_n).

Definition 3.19 Let *NFS*(*W*) be the family of all NFSs over *W*. Then ${}^f\varsigma_W$ is said be neutro-fine discrete topology if ${}^f\varsigma_W = NFS(W)$. Thus $(W, \tau_n, {}^f\varsigma_W)$ is said to be a neutro-fine discrete topological space over (W, τ_n).

Example 3.20 Let $W = \{w_1, w_2, w_3\}$ and $\tau_n = \{0_n, 1_n, R, S\}$ where *R* and *S* are NSs over *W* and are defined as follows

$$R = \{\langle w_1, .1, .2, .8\rangle, \langle w_2, .4, .7, .3\rangle, \langle w_3, .6, .5, .2\rangle\}$$

And

$$S = \{\langle w_1, .6, .5, .3\rangle, \langle w_2, .9, .8, .1\rangle, \langle w_3, .7, .6, .1\rangle\}.$$

Thus (W, τ_n) is a NTS over *W*.

Then ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_R(w_1), \varsigma_R(w_2, w_3), \varsigma_S(w_2)\}$,

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where

$$\varsigma_R(w_1)=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.6,.5,.2\rangle,\langle w_{2,3},0,0,1\rangle\},$$

$$\varsigma_R(w_2,w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.6,.5,.2\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.6,.5,.2\rangle,\langle w_{2,3},.6,.7,.2\rangle\},$$

$$\varsigma_S(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.9,.8,.1\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.9,.8,.1\rangle\}$$

are NFOSs over (W,τ_n).

Also, ${}^F\varsigma_W=\{0_n,1_n,\varsigma_R(w_1)',\varsigma_R(w_2,w_3)',\varsigma_S(w_2)'\}$,

where

$$\varsigma_R(w_1)'=\{\langle w_1,.8,.8,.1\rangle,\langle w_2,1,1,0\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.3,.3,.4\rangle,\langle w_{1,3},.2,.5,.6\rangle,\langle w_{2,3},1,1,0\rangle\},$$

$$\varsigma_R(w_2,w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.3,.3,.4\rangle,\langle w_3,.2,.5,.6\rangle,\langle w_{1,2},.3,.3,.4\rangle,\langle w_{1,3},.2,.5,.6\rangle,\langle w_{2,3},.2,.3,.6\rangle\},$$

$$\varsigma_S(w_2)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.1,.2,.9\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.1,.2,.9\rangle,\langle w_{1,3},1,1,0\rangle,\langle w_{2,3},.1,.2,.9\rangle\}$$

are NFCSs over (W,τ_n).

Thus $(W,\tau_n,{}^f\varsigma_W)$ is a NFTS over (W,τ_n).

Since $\varsigma_R(w_1)\cup\varsigma_S(w_2)\notin{}^f\varsigma_W$ and $\varsigma_R(w_1)\cap\varsigma_S(w_2)\notin{}^f\varsigma_W$, ${}^f\varsigma_W$ does not satisfy the conditions of a NT.

Also, since $\varsigma_R(w_1)'\cup\varsigma_S(w_2)'\notin{}^F\varsigma_W$ and $\varsigma_R(w_1)'\cap\varsigma_S(w_2)'\notin{}^F\varsigma_W$, ${}^F\varsigma_W$ does not satisfy the conditions of a NT.

Hence $(W,{}^f\varsigma_W)$ and $(W,{}^F\varsigma_W)$ are not NTSs over W.

Definition 3.21 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine interior of $\varsigma_R(V)$ is denoted as $Int_{nf}(\varsigma_R(V))$ and is defined as the union of all NFOSs contained in $\varsigma_R(V)$.

Clearly, $Int_{nf}(\varsigma_R(V))$ is the largest NFOS contained in $\varsigma_R(V)$.

Definition 3.22 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over W. Then the neutro-fine closure of $\varsigma_R(V)$ is denoted as $Cl_{nf}(\varsigma_R(V))$ and is defined as the intersection of all NFCSs containing $\varsigma_R(V)$.

Clearly, $Cl_{nf}(\varsigma_R(V))$ is the smallest NFCS containing $\varsigma_R(V)$.

Example 3.23 Consider Example 3.20.

Let us consider the NS R.

Then the NFS $\varsigma_R(w_1,w_3)$ is defined as

$$\varsigma_R(w_1,w_3)=\{\langle w_1,.1,.2,.8\rangle,\langle w_2,.0,0,1\rangle,\langle w_3,.6,.5,.2\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.6,.5,.2\rangle,\langle w_{2,3},.6,.7,.2\rangle\}.$$

Clearly,

$$\varsigma_R(w_1,w_3)\supseteq 0_n,\varsigma_R(w_1)\ .$$

Thus

$$Int_{nf}(\varsigma_R(w_1,w_3))=0_n\cup\varsigma_R(w_1)=\varsigma_R(w_1)\ .$$

The NFS $\varsigma_R(w_2)$ is defined as

$$\varsigma_R(w_2)=\{\langle w_1,.0,0,1\rangle,\langle w_2,.4,.7,.3\rangle,\langle w_3,.0,0,1\rangle,\langle w_{1,2},.4,.7,.3\rangle,\langle w_{1,3},.0,0,1\rangle,\langle w_{2,3},.6,.7,.2\rangle\}.$$

Clearly,

$$\varsigma_R(w_2)\subseteq 1_n,\varsigma_R(w_1)'\ .$$

Thus

$$Cl_{nf}(\varsigma_R(w_2))=1_n\cap\varsigma_R(w_1)'=\varsigma_R(w_1)'\ .$$

Now consider the NS S.

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Then the NFS $\varsigma_S(w_2,w_3)$ is defined as

$$\varsigma_S(w_2,w_3)=\left\{\langle w_1,0,0,1\rangle,\langle w_2,.9,.8,.1\rangle,\langle w_3,.7,.6,.5\rangle,\langle w_{1,2},.9,.8,.1\rangle,\langle w_{1,3},.7,.6,.1\rangle,\langle w_{2,3},.9,.8,.1\rangle\right\}.$$

Clearly,

$$\varsigma_S(w_2,w_3)\supseteq 0_n,\varsigma_S(w_2) \ .$$

Thus

$$Int_{nf}\left(\varsigma_S(w_2,w_3)\right)=0_n\cup\varsigma_S(w_2)=\varsigma_S(w_2) \ .$$

The NFS $\varsigma_S(w_1)$ is defined as

$$\varsigma_S(w_1)=\left\{\langle w_1,.6,.5,.3\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.9,.8,.1\rangle,\langle w_{1,3},.7,.6,.1\rangle,\langle w_{2,3},0,0,1\rangle\right\}.$$

Clearly,

$$\varsigma_S(w_1)\subseteq 1_n,\varsigma_S(w_2)' \ .$$

Thus

$$Cl_{nf}\left(\varsigma_S(w_1)\right)=1_n\cap\varsigma_S(w_2)'=\varsigma_S(w_2)' \ .$$

Proposition 3.24 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then,

(i) $Int_{nf}\left(0_{nf}\right)=0_{nf}$ and $Int_{nf}\left(1_{nf}\right)=1_{nf}$.

(ii) $\varsigma_R(V_1)$ is NFOS $\Rightarrow Int_{nf}\left(\varsigma_R(V_1)\right)=\varsigma_R(V_1)$.

(iii) $Int_{nf}\left(\varsigma_R(V_1)\right)\subseteq\varsigma_R(V_1)$.

(iv) $\varsigma_R(V_1)\subseteq\varsigma_R(V_2)\Rightarrow Int_{nf}\left(\varsigma_R(V_1)\right)\subseteq Int_{nf}\left(\varsigma_R(V_2)\right)$.

(v) $Int_{nf}\left(Int_{nf}\left(\varsigma_R(V_1)\right)\right)=Int_{nf}\left(\varsigma_R(V_1)\right)$.

(vi) $Int_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)=Int_{nf}\left(\varsigma_R(V_1)\right)\cap Int_{nf}\left(\varsigma_R(V_2)\right)$.

(vii) $Int_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)\subseteq Int_{nf}\left(\varsigma_R(V_1)\right)\cup Int_{nf}\left(\varsigma_R(V_2)\right)$.

(viii) $Int_{nf}\left(\varsigma_R(V_1)'\right)=\left[Cl_{nf}\left(\varsigma_R(V_1)\right)\right]'$.

Proof. Straightforward.

Proposition 3.25 Let $\left(W,\tau_n,{}^f\varsigma_W\right)$ be a NFTS. Let $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ be two NFSs over *W*. Then,

(i) $Cl_{nf}\left(0_{nf}\right)=0_{nf}$ and $Cl_{nf}\left(1_{nf}\right)=1_{nf}$.

(ii) $\varsigma_R(V_1)$ is NFCS $\Rightarrow Cl_{nf}\left(\varsigma_R(V_1)\right)=\varsigma_R(V_1)$.

(iii) $Cl_{nf}\left(\varsigma_R(V_1)\right)\supseteq\varsigma_R(V_1)$.

(iv) $\varsigma_R(V_1)\subseteq\varsigma_R(V_2)\Rightarrow Cl_{nf}\left(\varsigma_R(V_1)\right)\subseteq Cl_{nf}\left(\varsigma_R(V_2)\right)$.

(v) $Cl_{nf}\left(Cl_{nf}\left(\varsigma_R(V_1)\right)\right)=Cl_{nf}\left(\varsigma_R(V_1)\right)$

(vi) $Cl_{nf}\left(\varsigma_R(V_1)\cup\varsigma_R(V_2)\right)=Cl_{nf}\left(\varsigma_R(V_1)\right)\cup Cl_{nf}\left(\varsigma_R(V_2)\right)$.

(vii) $Cl_{nf}\left(\varsigma_R(V_1)\cap\varsigma_R(V_2)\right)\subseteq Cl_{nf}\left(\varsigma_R(V_1)\right)\cap Cl_{nf}\left(\varsigma_R(V_2)\right)$.

(viii) $Cl_{nf}\left(\varsigma_R(V_1)'\right)=\left[Int_{nf}\left(\varsigma_R(V_1)\right)\right]'$.

Proof. Straightforward.

4. Separation Axioms

In this section, separation axioms on NFTS are defined with examples.

Definition 4.1 Let *NF*(*W*) be the family of all NFs over the universe *W* and $w\in W$. Then NFS $w^{\langle\alpha,\beta,\chi\rangle}$ is said to be a neutro-fine point (NFP), for $0\le\alpha,\beta,\gamma\le 1$ and is defined as follows:

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$$w^{\langle\alpha,\beta,\gamma\rangle}(v)=\begin{cases}(\alpha,\beta,\gamma), & if \quad w=v\\(0,0,1), & if \quad w\neq v\end{cases}.$$

Every NFS is the union of its NFPs.

Definition 4.2 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W, τ_η). Let $\varsigma_R(V)$ be a NFS over W. Then $w^{\langle\alpha,\beta,\chi\rangle}$ belongs to the NFS $\varsigma_R(V)$ is denoted as $w^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V)$ and is defined as $\alpha\leq T_R(V)$, $\beta\leq I_R(V)$, and $\gamma\geq F_R(V)$.

Example 4.3 Let $W=\{w_1,w_2,w_3\}$. Let *R* be a NS over *W*, defined as

$$R=\{\langle w_1,.7,.5,.4\rangle,\langle w_2,.2,.7,.9\rangle,\langle w_3,.4,.1,.3\rangle\}.$$

Then the NFS $\varsigma_R(w_{1,3})$ is defined as

$$\varsigma_R(w_{1,3})=\{w_{1,3}{}^{\langle .7,.5,.3\rangle}\}$$
$$=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},0,0,1\rangle\}.$$

Thus the NFS $\varsigma_R(w_{1,3})$ is a NFP.

Definition 4.4 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $\varsigma_R(V)$ be a NFS over *W*. Then $\varsigma_R(V)$ is said to be a neutro-fine neighborhood of the NFP $w^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V)$, if there exists a NFOS $\varsigma_R(U)$ such that $w^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(U)\subseteq\varsigma_R(V)$.

Definition 4.5 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ be two NFPs over W. Then $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ are said to be distinct points if $u^{\langle\alpha,\beta,\chi\rangle}\cap v^{\langle\alpha,\beta,\chi\rangle}=0_{nf}$.

Definition 4.6 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ be any distinct NFPs. If there exists NFOSs $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ such that

$$u^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V_1) \quad \text{and} \quad u^{\langle\alpha,\beta,\chi\rangle}\cap\varsigma_R(V_2)=0_{nf} \quad \text{or}$$
$$v^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V_2) \quad \text{and} \quad v^{\langle\alpha,\beta,\chi\rangle}\cap\varsigma_R(V_1)=0_{nf}.$$

Then $(W,\tau_n,{}^f\varsigma_W)$ is said to be a neutro-fine T_0-space.

Definition 4.7 Let $(W,\tau_n,{}^f\varsigma_W)$ be a NFTS over (W,τ_n). Let $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ be any distinct NFPs. If there exists NFOSs $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ such that

$$u^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V_1) \quad \text{and} \quad u^{\langle\alpha,\beta,\chi\rangle}\cap\varsigma_R(V_2)=0_{nf} \quad \text{and}$$
$$v^{\langle\alpha,\beta,\chi\rangle}\in\varsigma_R(V_2) \quad \text{and} \quad v^{\langle\alpha,\beta,\chi\rangle}\cap\varsigma_R(V_1)=0_{nf}.$$

Then $(W,\tau_n,{}^f\varsigma_W)$ is said to be a neutro-fine T_1-space.

Theorem 4.8 Every neutro-fine T_1-space is neutro-fine T_0-space.
The proof follows from Definitions 4.6 and 4.7.

Remark 4.9 The converse of the above theorem is not true as shown in the following example.

Example 4.10 Let $W=\{w_1,w_2,w_3\}$ and $\tau_n=\{0_n,1_n,R,S\}$ where *R* and *S* are NSs over *W* and are defined as follows

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$$R = \{\langle w_1,.5,.3,.2\rangle, \langle w_2,.9,.6,.1\rangle, \langle w_3,.6,.5,.4\rangle\}$$

And

$$S = \{\langle w_1,.3,.1,.5\rangle, \langle w_2,.7,.3,.4\rangle, \langle w_3,.2,.3,.8\rangle\}.$$

Thus (W, τ_n) is a NTS over W.

Consider $w_2^{\langle .9,.6,.1\rangle}$, $w_{2,3}^{\langle .2,.3,.4\rangle}$ and $w_3^{\langle .2,.3,.8\rangle}$ are NFPs.

Then ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_R(w_2), \varsigma_S(w_2, w_3), \varsigma_S(w_3)\}$, where

$$\varsigma_R(w_2) = \{w_2^{\langle .9,.6,.1\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,.9,.6,.1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},0,0,1\rangle\},$$

$$\varsigma_S(w_{2,3}) = \{w_2^{\langle .9,.6,.1\rangle} \cup w_{2,3}^{\langle .2,.3,.4\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,.9,.6,.1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},.2,.3,.4\rangle\},$$

$$\varsigma_S(w_3) = \{w_3^{\langle .2,.3,.8\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.2,.3,.8\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},0,0,1\rangle\}.$$

Thus $(W, \tau_n, {}^f\varsigma_W)$ is a NFTS over (W, τ_n) and also a neutro-fine T_0 -space.

Here

$$w_2^{\langle .9,.6,.1\rangle} \in \varsigma_R(w_2), \ w_2^{\langle .9,.6,.1\rangle} \cap \varsigma_S(w_{2,3}) \neq 0_{nf} \text{ and}$$

$$w_{2,3}^{\langle .2,.3,.4\rangle} \in \varsigma_s(w_{2,3}), \ w_{2,3}^{\langle .2,.3,.4\rangle} \cap \varsigma_R(w_2) \neq 0_{nf}.$$

Thus $(W, \tau_n, {}^f\varsigma_W)$ is not a neutro-fine T_1 -space because for NFPs $w_2^{\langle .9,.6,.1\rangle}$ and $w_{2,3}^{\langle .2,.3,.4\rangle}$.

Hence $(W, \tau_n, {}^f\varsigma_W)$ is a neutro-fine T_0 -space but not a neutro-fine T_1 -space.

Definition 4.11 Let $(W, \tau_n, {}^f\varsigma_W)$ be a NFTS over (W, τ_n). Let $u^{\langle \alpha,\beta,\chi\rangle}$ and $v^{\langle \alpha,\beta,\chi\rangle}$ be any distinct NFPs. If there exists NFOSs $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ such that

$$u^{\langle \alpha,\beta,\chi\rangle} \in \varsigma_R(V_1), \quad v^{\langle \alpha,\beta,\chi\rangle} \in \varsigma_R(V_2) \text{ and } \varsigma_R(V_1) \cap \varsigma_R(V_2) = 0_{nf}.$$

Then $(W, \tau_n, {}^f\varsigma_W)$ is said to be a neutro-fine T_2 -space.

Theorem 4.12 Every neutro-fine T_2 -space is neutro-fine T_1 -space.

The proof follows from Definitions 4.7 and 4.11.

Example 4.13 Consider Example 4.10.

Consider $w_3^{\langle .6,.5,.4\rangle}$, $w_{1,2}^{\langle .9,.6,.1\rangle}$, $w_{2,3}^{\langle .2,.3,.4\rangle}$ and $w_{1,3}^{\langle .3,.3,.5\rangle}$ are NFPs.

Then ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_R(w_3), \varsigma_R(w_1, w_2), \varsigma_S(w_2, w_3), \varsigma_S(w_1, w_3)\}$, where

$$\varsigma_R(w_3) = \{w_3^{\langle .6,.5,.4\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.6,.5,.4\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},0,0,1\rangle\},$$

$$\varsigma_R(w_{1,2}) = \{w_{1,2}^{\langle .9,.6,.1\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.9,.6,.1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},0,0,1\rangle\},$$

$$\varsigma_S(w_{2,3}) = \{w_{2,3}^{\langle .2,.3,.4\rangle}\}$$
$$= \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},.2,.3,.4\rangle\},$$

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$\varsigma_S(w_{1,3}) = \{w_{1,3}{}^{\langle .3,.3,.5\rangle}\}$
$= \{\langle w_1, 0, 0, 1\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, 0, 0, 1\rangle, \langle w_{1,2}, 0, 0, 1\rangle, \langle w_{1,3}, .3, .3, .5\rangle, \langle w_{2,3}, 0, 0, 1\rangle\}$.

Thus $(W, \tau_n, {}^f\varsigma_W)$ is a NFTS over (W, τ_n).
Hence $(W, \tau_n, {}^f\varsigma_W)$ is a neutro-fine T_2-space and neutro-fine T_1-space as well as neutro-fine T_0-space.

Theorem 4.14 Let $(W, \tau_n, {}^f\varsigma_W)$ be a NFTS over (W, τ_n). Then $(W, \tau_n, {}^f\varsigma_W)$ is a neutro-fine T_1-space if and only if each NFP is a NFCS.

Proof. Let $(W, \tau_n, {}^f\varsigma_W)$ be a neutro-fine T_1-space and $u^{\langle\alpha,\beta,\chi\rangle}$ be any NFP.
To prove: $(u^{\langle\alpha,\beta,\chi\rangle})'$ is a NFCS.
Let $v^{\langle\alpha,\beta,\chi\rangle} \in (u^{\langle\alpha,\beta,\chi\rangle})'$.
Then $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ are distinct NFPs.
Thus $u^{\langle\alpha,\beta,\chi\rangle} \cap v^{\langle\alpha,\beta,\chi\rangle} = 0_{nf}$.
Since $(W, \tau_n, {}^f\varsigma_W)$ is a neutro-fine T_1-space, there exists a NFOS $\varsigma_R(V)$ such that

$$v^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V) \quad \text{and} \quad v^{\langle\alpha,\beta,\chi\rangle} \cap \varsigma_R(V_1) = 0_{nf} .$$

Then $v^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V) \subseteq (u^{\langle\alpha,\beta,\chi\rangle})'$.
Thus $(u^{\langle\alpha,\beta,\chi\rangle})'$ is a NFOS.
Hence $u^{\langle\alpha,\beta,\chi\rangle}$ is a NFCS.
Conversely, suppose that each NFP $u^{\langle\alpha,\beta,\chi\rangle}$ is a NFCS.
Then $(u^{\langle\alpha,\beta,\chi\rangle})'$ is a NFOS.
Let $u^{\langle\alpha,\beta,\chi\rangle} \cap v^{\langle\alpha,\beta,\chi\rangle} = 0_{nf}$.
Thus $v^{\langle\alpha,\beta,\chi\rangle} \in (u^{\langle\alpha,\beta,\chi\rangle})'$ and $v^{\langle\alpha,\beta,\chi\rangle} \cap (u^{\langle\alpha,\beta,\chi\rangle})' = 0_{nf}$.
Hence $(W, \tau_n, {}^f\varsigma_W)$ is a neutro-fine T_1-space over (W, τ_n).

Theorem 4.15 Let $(W, \tau_n, {}^f\varsigma_W)$ be a NFTS over (W, τ_n). $(W, \tau_n, {}^f\varsigma_W)$ is a neutro-fine T_2-space if and only if for distinct NFPs $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$, there exists a NFOS $\varsigma_R(V)$ containing $u^{\langle\alpha,\beta,\chi\rangle}$ but not $v^{\langle\alpha,\beta,\chi\rangle}$ such that $v^{\langle\alpha,\beta,\chi\rangle} \notin Cl_{nf}(\varsigma_R(V))$.

Proof. Let $(W, \tau_n, {}^f\varsigma_W)$ be a neutro-fine T_2-space.
Let $u^{\langle\alpha,\beta,\chi\rangle}$ and $v^{\langle\alpha,\beta,\chi\rangle}$ be two NFPs.
Then there exists disjoint NFOSs $\varsigma_R(V_1)$ and $\varsigma_R(V_2)$ such that

$$u^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V_1) \text{ and } v^{\langle\alpha,\beta,\chi\rangle} \in \varsigma_R(V_2) .$$

Since $u^{\langle\alpha,\beta,\chi\rangle} \cap v^{\langle\alpha,\beta,\chi\rangle} = 0_{nf}$ and $\varsigma_R(V_1) \cap \varsigma_R(V_2) = 0_{nf}$, then

$$v^{\langle\alpha,\beta,\chi\rangle} \notin \varsigma_R(V_1) .$$

Hence $v^{\langle\alpha,\beta,\chi\rangle} \notin Cl_{nf}(\varsigma_R(V_1))$.

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Conversely, suppose that for distinct NFPs $u^{\langle \alpha,\beta,\chi \rangle}$ and $v^{\langle \alpha,\beta,\chi \rangle}$, there exists a NFOS $\varsigma_R(V)$ containing $u^{\langle \alpha,\beta,\chi \rangle}$ but not $v^{\langle \alpha,\beta,\chi \rangle}$ such that $v^{\langle \alpha,\beta,\chi \rangle} \notin Cl_{nf}(\varsigma_R(V))$.

Then $v^{\langle \alpha,\beta,\chi \rangle} \in (Cl_{nf}(\varsigma_R(V)))'$.

Thus $\varsigma_R(V)$ and $(Cl_{nf}(\varsigma_R(V)))'$ are disjoint NFOSs containing $u^{\langle \alpha,\beta,\chi \rangle}$ and $v^{\langle \alpha,\beta,\chi \rangle}$ respectively.

Hence $(W,\tau_n,{}^f\varsigma_W)$ is a neutro-fine T_2-space.

Theorem 4.16 Let $(W,\tau_n,{}^f\varsigma_W)$ be a neutro-fine T_1-space for every distinct NFPs $u^{\langle \alpha,\beta,\chi \rangle} \in \varsigma_R(V_1) \in {}^f\varsigma_W$. If there exists a NFOS $\varsigma_R(V_2)$ such that

$$u^{\langle \alpha,\beta,\chi \rangle} \in \varsigma_R(V_2) \subseteq Cl_{nf}(\varsigma_R(V_2)) \subseteq \varsigma_R(V_1),$$

then $(W,\tau_n,{}^f\varsigma_W)$ is a neutro-fine T_2-space.

Proof. Let $(W,\tau_n,{}^f\varsigma_W)$ be a neutro-fine T_1-space.

Suppose that $u^{\langle \alpha,\beta,\chi \rangle} \cap v^{\langle \alpha,\beta,\chi \rangle} = 0_{nf}$.

Since $(W,\tau_n,{}^f\varsigma_W)$ is a neutro-fine T_1-space, $u^{\langle \alpha,\beta,\chi \rangle}$ and $v^{\langle \alpha,\beta,\chi \rangle}$ are NFCSs in ${}^f\varsigma_W$.

Thus $u^{\langle \alpha,\beta,\chi \rangle} \in \left(v^{\langle \alpha,\beta,\chi \rangle}\right)' \in {}^f\varsigma_W$.

Then there exists a NFOS $\varsigma_R(V_2)$ such that

$$u^{\langle \alpha,\beta,\chi \rangle} \in \varsigma_R(V_2) \subseteq Cl_{nf}(\varsigma_R(V_2)) \subseteq \left(v^{\langle \alpha,\beta,\chi \rangle}\right)'.$$

Thus

$$v^{\langle \alpha,\beta,\chi \rangle} \in (Cl_{nf}(\varsigma_R(V_2)))',\ u^{\langle \alpha,\beta,\chi \rangle} \in \varsigma_R(V_2) \text{ and } \varsigma_R(V_2) \cap (Cl_{nf}(\varsigma_R(V_2)))' = 0_{nf}.$$

Hence $(W,\tau_n,{}^f\varsigma_W)$ is a neutro-fine T_2-space.

5. Decision Making in NFTS

In this section, the real-life application is intimated to take a correct decision on DM problems and an example is investigated in two different manners.

Definition 5.1 Let $\varsigma_R(V)$ be a NFS over W of a NFTS $(W,\tau_n,{}^f\varsigma_W)$. Then the absolute complement of $\varsigma_R(V)$ is denoted as $\varsigma_R{}^*(V')$ and defined as $\varsigma_R{}^*(V') = \varsigma_R{}^*(W-V)$.

Thus the collection of $\varsigma_R{}^*(V')$ is denoted as ${}^{f*}\varsigma_W$ and defined as ${}^{f*}\varsigma_W = \{0_{nf}, 1_{nf}, \bigcup \varsigma_R{}^*(V')\}$. The elements belong to ${}^{f*}\varsigma_W$ are said to be neutro-fine absolute open sets (NFAOSs) over (W,τ_n) and the complement of NFOSs are said to be neutro-fine absolute closed sets (NFACSs) over (W,τ_n) and denote the collection by ${}^{F*}\varsigma_W$.

Example 5.2 Let $W = \{w_1, w_2, w_3\}$ and $\tau_n = \{0_n, 1_n, R\}$ where R is a NS over W and are defined as follows

$$R = \{\langle w_1,.9,.4,.6\rangle, \langle w_2,.6,.5,.1\rangle, \langle w_3,.7,.8,.1\rangle\}.$$

Thus (W,τ_n) is a NTS over W.

Then NFOSs over (W,τ_n) are ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_R(w_3), \varsigma_R(w_2,w_3)\}$, where

$$\varsigma_R(w_3) = \{\langle w_1,0,0,1\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.7,.8,.1\rangle, \langle w_{1,2},0,0,1\rangle, \langle w_{1,3},.9,.8,.1\rangle, \langle w_{2,3},.7,.8,.1\rangle\} \text{ and}$$

$$\varsigma_R(w_2,w_3) = \{\langle w_1,0,0,1\rangle, \langle w_2,.6,.5,.1\rangle, \langle w_3,.7,.8,.1\rangle, \langle w_{1,2},.9,.5,.1\rangle, \langle w_{1,3},.9,.8,.1\rangle, \langle w_{2,3},.7,.8,.1\rangle\}.$$

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Also, NFCSs over (W, τ_n) are ${}^{F}\varsigma_W = \{0_n, 1_n, \varsigma_R(w_3)' \varsigma_R(w_2, w_3)'\}$, where

$$\varsigma_R(w_3)' = \{\langle w_1,1,1,0\rangle, \langle w_2,1,1,0\rangle, \langle w_3,.1,.2,.7\rangle, \langle w_{1,2},1,1,0\rangle, \langle w_{1,3},.1,.2,.9\rangle, \langle w_{2,3},.1,.2,.7\rangle\} \text{ and}$$

$$\varsigma_R(w_2, w_3)' = \{\langle w_1,1,1,0\rangle, \langle w_2,.1,.5,.6\rangle, \langle w_3,.1,.2,.7\rangle, \langle w_{1,2},.1,.5,.9\rangle, \langle w_{1,3},.1,.2,.9\rangle, \langle w_{2,3},.1,.2,.7\rangle\}.$$

Thus $(W, \tau_n, {}^{f}\varsigma_W)$ is a NFTS over (W, τ_n).

Then NFAOSs over (W, τ_n) are

$${}^{f*}\varsigma_W = \{0_n, 1_n, \varsigma_R^*((w_3)'), \varsigma_R^*((w_2, w_3)')\} = \{0_n, 1_n, \varsigma_R^*(w_1, w_2), \varsigma_R^*(w_1)\},$$

where

$$\varsigma_R^*((w_3)') = \varsigma_R^*(w_1, w_2) = \{\langle w_1,.9,.4,.6\rangle, \langle w_2,.6,.5,.1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.9,.4,.1\rangle, \langle w_{1,3},.9,.8,.1\rangle, \langle w_{2,3},.7,.8,.1\rangle\} \text{ and}$$

$$\varsigma_R^*((w_2, w_3)') = \varsigma_R^*(w_1) = \{\langle w_1,.9,.4,.6\rangle, \langle w_2,0,0,1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.9,.5,.1\rangle, \langle w_{1,3},.9,.8,.1\rangle, \langle w_{2,3},0,0,1\rangle\}.$$

Also, NFACSs over (W, τ_n) are

$${}^{F*}\varsigma_W = \{0_n, 1_n, \varsigma_R^*((w_3)')', \varsigma_R^*((w_2, w_3)')'\} = \{0_n, 1_n, \varsigma_R^*(w_1, w_2)', \varsigma_R^*(w_1)'\},$$

where

$$\varsigma_R^*((w_3)')' = \varsigma_R^*(w_1, w_2)' = \{\langle w_1,.6,.6,.9\rangle, \langle w_2,.1,.5,.6\rangle, \langle w_3,1,1,0\rangle, \langle w_{1,2},.1,.6,.9\rangle, \langle w_{1,3},.1,.2,.9\rangle, \langle w_{2,3},.1,.2,.7\rangle\}$$

and

$$\varsigma_R^*((w_2, w_3)')' = \varsigma_R^*(w_1)' = \{\langle w_1,.6,.6,.9\rangle, \langle w_2,1,1,0\rangle, \langle w_3,1,1,0\rangle, \langle w_{1,2},.1,.5,.9\rangle, \langle w_{1,3},.1,.2,.9\rangle, \langle w_{2,3},1,1,0\rangle\}.$$

Definition 5.3 Let W be a set of universe and $w \in W$. Let R be a NS over W and V be any proper non-empty subset of W. Let $\varsigma_R(V)$ be a NFS over W of a NFTS $(W, \tau_n, {}^{f}\varsigma_W)$. Then the net value of R is calculated by the formula

Then the net value of R (NV(R)) is calculated by the formula

$$NV(R) = \left| \frac{\left[\sum_i (T_R^*(V'))_i - \sum_i (F_R(V)')_i\right] + \left[\sum_i (T_R(V)')_i - \sum_i (F_R^*(V'))_i\right]}{2} \times \left[1 - \frac{\sum_i (I_R^*(V'))_i - \sum_i (I_R(V)')_i}{2}\right] \right| \tag{5.1}$$

where

$\sum_i (T_R(V)')_i$, $\sum_i (I_R(V)')_i$ and $\sum_i (F_R(V)')_i$ are the sum of all truth, indeterminacy and falsity values of $\varsigma_R(V)'$ respectively, and

$\sum_i (T_R^*(V'))_i$, $\sum_i (I_R^*(V'))_i$ and $\sum_i (F_R^*(V'))_i$ are the sum of all truth, indeterminacy, and falsity values of $\varsigma_R^*(V')$ respectively.

Algorithm

Step 1: List the set of items for the sample.

Step 2: List some of its risk factors as the universe W, where $w \in W$.

Step 3: Go through the damages of the items.

Step 4: Define each item as NSs, say R.

Step 5: Collect these NSs which defines a NT τ_n and so (W, τ_n) is a NTS.

Step 6: List the proper subsets of W as V.

Step 7: Define NFSs for each NS with respect to their risk factors $v \in V$, say $\varsigma_R(V)$.

Step 8: Define a neutrosophic-fine collection ${}^{f}\varsigma_W$, where $\varsigma_R(v)$ are NFOSs and so $(W, \tau_n, {}^{f}\varsigma_W)$ is a NFTS.

Step 9: Find the complement and absolute complement of ${}^{f}\varsigma_W$, ${}^{F}\varsigma_W$ and ${}^{f*}\varsigma_W$ respectively, which represents the secureness of the items.

Step 10: Calculate the NV(R) by using the formula (5.1).

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Step 11: Select the highest value of NV(R) among all the calculated values of NV(R).
Step 12: If two or more NV(R)s are identical for a particular w, replace that w with some other risk factor and recurrence the procedure.
Step 13: Terminate the procedure, while attaining a unique NV(R).

Example 5.4 Consider the problem that a customer wishes to buy a second-hand refrigerator. Let $R1$, $R2$, $R3$, and $R4$ be sample refrigerators for second-hand sales. Each refrigerator is damaged according to some aspects of the universe $W=\{w_1,w_2,w_3\}$, where w_1 –locked compressor, w_2 –clogged coils and w_3 –dirty condenser coil. Our problem is to help the customer to prefer a second-hand refrigerator with less damage.

1. Let $R1$, $R2$, $R3$, and $R4$ be sample refrigerators for second-hand sales.
2. Let $W=\{w_1,w_2,w_3\}$ be the universe, where w_1 –locked compressor, w_2 –clogged coils and w_3 –dirty condenser coil.
3. Analyze the damages on each refrigerator.
4. Define $R1$, $R2$, $R3$, and $R4$ as NSs.

$$R1=\{\langle w_1,.6,.4,.7\rangle,\langle w_2,.5,.6,.7\rangle,\langle w_3,.3,.4,.1\rangle\},$$
$$R2=\{\langle w_1,.3,.5,.8\rangle,\langle w_2,.4,.8,.5\rangle,\langle w_3,.2,.3,.7\rangle\},$$
$$R3=\{\langle w_1,.6,.5,.7\rangle,\langle w_2,.5,.8,.5\rangle,\langle w_3,.3,.4,.1\rangle\}$$ and
$$R4=\{\langle w_1,.3,.4,.8\rangle,\langle w_2,.4,.6,.7\rangle,\langle w_3,.2,.3,.7\rangle\}.$$

5. Thus $\tau_n=\{0_n,1_n,R1,R2,R3,R4\}$ is a NT and so (W,τ_n) is a NTS.
6. Let $V=\{(w_1),(w_2),(w_3),(w_1,w_2),(w_1,w_3),(w_2,w_3)\}$ be the set of proper subsets of W.
7. Define NFSs as

$$\varsigma_{R1}(w_1,w_2)=\{\langle w_1,.6,.4,.7\rangle,\langle w_2,.5,.6,.7\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.6,.6,.7\rangle,\langle w_{1,3},.6,.4,.1\rangle,\langle w_{2,3},.5,.6,.1\rangle\},$$
$$\varsigma_{R2}(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.2,.3,.7\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.3,.5,.7\rangle,\langle w_{2,3},.4,.8,.5\rangle\},$$
$$\varsigma_{R3}(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.5,.8,.5\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.6,.8,.5\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.5,.8,.1\rangle\}$$ and
$$\varsigma_{R4}(w_1,w_3)=\{\langle w_1,.3,.4,.8\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.2,.3,.7\rangle,\langle w_{1,2},.4,.6,.7\rangle,\langle w_{1,3},.3,.4,.7\rangle,\langle w_{2,3},.4,.6,.7\rangle\}.$$

8. Then the neutro-fine collection ${}^f\varsigma_W=\{0_n,1_n,\varsigma_{R1}(w_1,w_2),\varsigma_{R2}(w_3),\varsigma_{R3}(w_2),\varsigma_{R4}(w_1,w_3)\}$, whose elements are NFOSs.
Thus $\left(W,\tau_n,{}^f\varsigma_W\right)$ is a NFTS over (W,τ_n).
9. Let the complement and absolute complement of ${}^f\varsigma_W$ represents the secureness of the items.
The complement of ${}^f\varsigma_W$ is ${}^F\varsigma_W=\{0_n,1_n,\varsigma_{R1}(w_1,w_2)',\varsigma_{R2}(w_3)',\varsigma_{R3}(w_2)',\varsigma_{R4}(w_1,w_3)'\}$, where

$$\varsigma_{R1}(w_1,w_2)'=\{\langle w_1,.7,.6,.6\rangle,\langle w_2,.7,.4,.5\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.7,.4,.6\rangle,\langle w_{1,3},.1,.6,.6\rangle,\langle w_{2,3},.1,.4,.5\rangle\},$$
$$\varsigma_{R2}(w_3)'=\{\langle w_1,1,1,0\rangle,\langle w_2,1,1,0\rangle,\langle w_3,.7,.7,.2\rangle,\langle w_{1,2},1,1,0\rangle,\langle w_{1,3},.7,.5,.3\rangle,\langle w_{2,3},.5,.2,.4\rangle\},$$
$$\varsigma_{R3}(w_2)'=\{\langle w_1,1,1,0\rangle,\langle w_2,.5,.2,.5\rangle,\langle w_3,1,1,0\rangle,\langle w_{1,2},.5,.2,.6\rangle,\langle w_{1,3},1,1,0\rangle,\langle w_{2,3},.1,.2,.5\rangle\}$$ and
$$\varsigma_{R4}(w_1,w_3)'=\{\langle w_1,.8,.6,.3\rangle,\langle w_2,1,1,0\rangle,\langle w_3,.7,.7,.2\rangle,\langle w_{1,2},.7,.4,.4\rangle,\langle w_{1,3},.7,.6,.3\rangle,\langle w_{2,3},.7,.4,.4\rangle\}.$$

The absolute complement of ${}^f\varsigma_W$ is ${}^{f*}\varsigma_W=\{0_n,1_n,\varsigma_{R1}{}^*(w_3),\varsigma_{R2}{}^*(w_1,w_2),\varsigma_{R3}{}^*(w_1,w_3),\varsigma_{R4}{}^*(w_2)\}$, where

$$\varsigma_{R1}{}^*(w_3)=\{\langle w_1,0,0,1\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.3,.4,.1\rangle,\langle w_{1,2},0,0,1\rangle,\langle w_{1,3},.6,.4,.1\rangle,\langle w_{2,3},.5,.6,.1\rangle\},$$
$$\varsigma_{R2}{}^*(w_1,w_2)=\{\langle w_1,.3,.5,.8\rangle,\langle w_2,.4,.8,.5\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.4,.8,.5\rangle,\langle w_{1,3},.3,.5,.7\rangle,\langle w_{2,3},.4,.8,.5\rangle\},$$
$$\varsigma_{R3}{}^*(w_1,w_3)=\{\langle w_1,.6,.5,.7\rangle,\langle w_2,0,0,1\rangle,\langle w_3,.3,.4,.1\rangle,\langle w_{1,2},.6,.8,.5\rangle,\langle w_{1,3},.6,.5,.1\rangle,\langle w_{2,3},.5,.8,.1\rangle\}$$ and
$$\varsigma_{R4}{}^*(w_2)=\{\langle w_1,0,0,1\rangle,\langle w_2,.4,.6,.7\rangle,\langle w_3,0,0,1\rangle,\langle w_{1,2},.4,.6,.7\rangle,\langle w_{1,3},0,0,1\rangle,\langle w_{2,3},.4,.6,.7\rangle\}.$$

10. By using the formula (5.1), the following values are obtained.

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NV(R1) = 0.98,
NV(R2) = 2.61,
NV(R3) = **2.99** and
NV(R4) = 0.79.

11. Thus NV(R3) is the highest value.

Hence the customer can prefer to buy R3 for the second use.

Example 5.5 Consider the situation of Example 5.4.

1. Let $R1$, $R2$, and $R3$ be sample refrigerators for second-hand sale.
2. Let $W = \{w_1, w_2, w_3\}$ be the universe, where w_1 –locked compressor, w_2 –clogged coils and w_3 –dirty condenser coil.
3. Analyze the damages on each refrigerator.
4. Define $R1$, $R2$, and $R3$ as NSs.

$$R1 = \{\langle w_1, .3, .4, .5\rangle, \langle w_2, .5, .6, .3\rangle, \langle w_3, .1, .2, .3\rangle\},$$
$$R2 = \{\langle w_1, .4, .5, .2\rangle, \langle w_2, .6, .8, .3\rangle, \langle w_3, .4, .5, .2\rangle\} \text{ and}$$
$$R3 = \{\langle w_1, .4, .5, .2\rangle, \langle w_2, .6, .8, .3\rangle, \langle w_3, .4, .5, .2\rangle\}.$$

5. Thus $\tau_n = \{0_n, 1_n, R1, R2, R3\}$ is a NT and so (W, τ_n) is a NTS.
6. Let $V = \{(w_1), (w_2), (w_3), (w_1, w_2), (w_1, w_3), (w_2, w_3)\}$ be the set of proper subsets of W.
7. Define NFSs as

$$\varsigma_{R1}(w_1, w_3) = \{\langle w_1, .3, .4, .5\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .1, .2, .3\rangle, \langle w_{1,2}, .5, .6, .3\rangle, \langle w_{1,3}, .3, .4, .3\rangle, \langle w_{2,3}, .5, .6, .3\rangle\},$$
$$\varsigma_{R2}(w_2, w_3) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .6, .8, .3\rangle, \langle w_3, .4, .5, .2\rangle, \langle w_{1,2}, .6, .8, .2\rangle, \langle w_{1,3}, .4, .5, .2\rangle, \langle w_{2,3}, .6, .8, .2\rangle\} \text{ and}$$
$$\varsigma_{R3}(w_1, w_2) = \{\langle w_1, .4, .5, .2\rangle, \langle w_2, .6, .8, .3\rangle, \langle w_3, 0, 0, 1\rangle, \langle w_{1,2}, .6, .8, .2\rangle, \langle w_{1,3}, .4, .5, .2\rangle, \langle w_{2,3}, .6, .8, .2\rangle\}.$$

8. Then the neutro-fine collection ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_{R1}(w_1, w_3), \varsigma_{R2}(w_2, w_3), \varsigma_{R3}(w_1, w_2)\}$, whose elements are NFOSs. Thus $(W, \tau_n, {}^f\varsigma_W)$ is a NFTS over (W, τ_n).
9. Let the complement and absolute complement of ${}^f\varsigma_W$ represents the secureness of the items.

The complement of ${}^f\varsigma_W$ is ${}^F\varsigma_W = \{0_n, 1_n, \varsigma_{R1}(w_1, w_3)', \varsigma_{R2}(w_2, w_3)', \varsigma_{R3}(w_1, w_2)'\}$, where

$$\varsigma_{R1}(w_1, w_3)' = \{\langle w_1, .5, .6, .3\rangle, \langle w_2, 1, 1, 0\rangle, \langle w_3, .3, .8, .1\rangle, \langle w_{1,2}, .3, .4, .5\rangle, \langle w_{1,3}, .3, .6, .3\rangle, \langle w_{2,3}, .3, .4, .5\rangle\},$$
$$\varsigma_{R2}(w_2, w_3)' = \{\langle w_1, 1, 1, 0\rangle, \langle w_2, .3, .2, .6\rangle, \langle w_3, .2, .5, .4\rangle, \langle w_{1,2}, .2, .2, .6\rangle, \langle w_{1,3}, .2, .5, .4\rangle, \langle w_{2,3}, .2, .2, .6\rangle\} \text{ and}$$
$$\varsigma_{R3}(w_1, w_2)' = \{\langle w_1, 2, .5, .4\rangle, \langle w_2, .3, .2, .6\rangle, \langle w_3, .1, 1, 0\rangle, \langle w_{1,2}, .2, .2, .6\rangle, \langle w_{1,3}, .2, .5, .4\rangle, \langle w_{2,3}, .2, .2, .6\rangle\}.$$

The absolute complement of ${}^f\varsigma_W$ is ${}^{f*}\varsigma_W = \{0_n, 1_n, \varsigma_{R1}{}^*(w_2), \varsigma_{R2}{}^*(w_1), \varsigma_{R3}{}^*(w_3)\}$, where

$$\varsigma_{R1}{}^*(w_2) = \{\langle w_1, 0, 0, 1\rangle, \langle w_2, .5, .6, .3\rangle, \langle w_3, 0, 0, 1\rangle, \langle w_{1,2}, .5, .6, .3\rangle, \langle w_{1,3}, .0, 0, 1\rangle, \langle w_{2,3}, .5, .6, .3\rangle\},$$
$$\varsigma_{R2}{}^*(w_1, w_2) = \{\langle w_1, .3, .5, .8\rangle, \langle w_2, .4, .8, .5\rangle, \langle w_3, 0, 0, 1\rangle, \langle w_{1,2}, .4, .8, .5\rangle, \langle w_{1,3}, .3, .5, .7\rangle, \langle w_{2,3}, .4, .8, .5\rangle\} \text{ and}$$
$$\varsigma_{R3}{}^*(w_1, w_3) = \{\langle w_1, .6, .5, .7\rangle, \langle w_2, 0, 0, 1\rangle, \langle w_3, .3, .4, .1\rangle, \langle w_{1,2}, .6, .8, .5\rangle, \langle w_{1,3}, .6, .5, .1\rangle, \langle w_{2,3}, .5, .8, .1\rangle\}.$$

10. By using the formula (5.1), the following values are obtained.

NV(R1) = 1.26,
NV(R2) = **1.62** and
NV(R3) = **1.62**.

11. Thus both NV(R2) and NV(R3) are the highest value.

In this situation, replace the clogged coils (w_2) by some other risk factors and repeat the process.

1. Let $R1$, $R2$, and $R3$ be sample refrigerators for second-hand sale.
2. Let $W = \{w_1, w_2, w_3\}$ be the universe, where w_1 –locked compressor, w_2 –failed fan motor and w_3 –dirty condenser coil.
3. Again analyze the damages on each refrigerator.

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4. Define $R1$, $R2$, and $R3$ as NSs.

$R1 = \{\langle w_1,.3,.4,.5\rangle, \langle w_2,.4,.8,.3\rangle, \langle w_3,.1,.2,.3\rangle\}$,

$R2 = \{\langle w_1,.4,.5,.2\rangle, \langle w_2,.5,.8,.1\rangle, \langle w_3,.4,.5,.2\rangle\}$ and

$R3 = \{\langle w_1,.4,.5,.2\rangle, \langle w_2,.5,.8,.1\rangle, \langle w_3,.4,.5,.2\rangle\}$.

5. Thus $\tau_\eta = \{0_n, 1_n, R1, R2, R3\}$ is a NT and so (W, τ_n) is a NTS.

6. Let $V = \{(w_1),(w_2),(w_3),(w_1,w_2),(w_1,w_3),(w_2,w_3)\}$ be the set of proper subsets of W.

7. Define NFSs as

$\varsigma_{R1}(w_2,w_3) = \{\langle w_1,0,0,1\rangle, \langle w_2,.4,.8,.3\rangle, \langle w_3,.1,.2,.3\rangle, \langle w_{1,2},.4,.8,.3\rangle, \langle w_{1,3},.3,.4,.3\rangle, \langle w_{2,3},.4,.8,.3\rangle\}$,

$\varsigma_{R2}(w_2) = \{\langle w_1,0,0,1\rangle, \langle w_2,.5,.8,.1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.5,.8,.1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},.5,.8,.1\rangle\}$ and

$\varsigma_{R3}(w_1,w_3) = \{\langle w_1,.4,.5,.2\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.4,.5,.2\rangle, \langle w_{1,2},.5,.8,.1\rangle, \langle w_{1,3},.5,.8,.1\rangle, \langle w_{2,3},.5,.8,.1\rangle\}$.

8. Then the neutro-fine collection ${}^f\varsigma_W = \{0_n, 1_n, \varsigma_{R1}(w_2,w_3), \varsigma_{R2}(w_2), \varsigma_{R3}(w_1,w_3)\}$, whose elements are NFOSs. Thus $(W, \tau_n, {}^f\varsigma_W)$ is a NFTS over (W, τ_n).

9. Let the complement and absolute complement of ${}^f\varsigma_W$ represents the secureness of the items.

The complement of ${}^f\varsigma_W$ is ${}^F\varsigma_W = \{0_n, 1_n, \varsigma_{R1}(w_2,w_3)', \varsigma_{R2}(w_2)', \varsigma_{R3}(w_1,w_3)'\}$, where

$\varsigma_{R1}(w_2,w_3)' = \{\langle w_1,1,1,0\rangle, \langle w_2,.3,.2,.4\rangle, \langle w_3,.3,.8,.1\rangle, \langle w_{1,2},.3,.2,.4\rangle, \langle w_{1,3},.3,.6,.3\rangle, \langle w_{2,3},.3,.2,.4\rangle\}$,

$\varsigma_{R2}(w_2)' = \{\langle w_1,1,1,0\rangle, \langle w_2,.1,.2,.5\rangle, \langle w_3,1,1,0\rangle, \langle w_{1,2},.1,.2,.5\rangle, \langle w_{1,3},1,1,0\rangle, \langle w_{2,3},.1,.2,.5\rangle\}$ and

$\varsigma_{R3}(w_1,w_3)' = \{\langle w_1,.2,.5,.4\rangle, \langle w_2,1,1,0\rangle, \langle w_3,.2,.5,.4\rangle, \langle w_{1,2},.1,.2,.5\rangle, \langle w_{1,3},.1,.2,.5\rangle, \langle w_{2,3},.1,.2,.5\rangle\}$.

The absolute complement of ${}^f\varsigma_W$ is ${}^{f*}\varsigma_W = \{0_n, 1_n, \varsigma_{R1}{}^*(w_1), \varsigma_{R2}{}^*(w_1,w_3), \varsigma_{R3}{}^*(w_2)\}$, where

$\varsigma_{R1}{}^*(w_1) = \{\langle w_1,.3,.4,.5\rangle, \langle w_2,0,0,1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.4,.8,.3\rangle, \langle w_{1,3},.3,.4,.3\rangle, \langle w_{2,3},0,0,1\rangle\}$,

$\varsigma_{R2}{}^*(w_1,w_3) = \{\langle w_1,.4,.5,.2\rangle, \langle w_2,0,0,1\rangle, \langle w_3,.4,.5,.2\rangle, \langle w_{1,2},.5,.8,.1\rangle, \langle w_{1,3},.5,.8,.1\rangle, \langle w_{2,3},.5,.8,.1\rangle\}$ and

$\varsigma_{R3}{}^*(w_2) = \{\langle w_1,0,0,1\rangle, \langle w_2,.5,.8,.1\rangle, \langle w_3,0,0,1\rangle, \langle w_{1,2},.5,.8,.1\rangle, \langle w_{1,3},0,0,1\rangle, \langle w_{2,3},.5,.8,.1\rangle\}$.

10. By using the formula (5.1), the following values are obtained.

NV(R1) = 1.43,

NV(R2) = **3.00** and

NV(R3) = 1.8.

11. Thus NV(R2) is the highest value.

Hence the customer can prefer to buy R2 for the second use.

6. Conclusions

The principal concern of this paper are to initiate the new type of topology called NFT and studied some essential theorems. Also defined the interior and closure on NFTS, and analyzed its basic properties with perfect examples. The concept of separation axioms on this space are investigated and the relationship between each neutro-fine $T_{i=0,1,2}$-spaces are exposed with illustrative examples. Besides this, the application of this space are explored on DM problems where the complement and absolute complement of each NFOS is determined to change unfavorable queries into a favorable one. The algorithm specified to describe the process and the positive solution calculated by the formula are given. Consequently, the future researchers can extend this NFTS to some special types of sets, whereas soft sets, rough sets, crisp sets, cubic sets, etc. Also, the application part can extend to MCDM problems.

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