NeutroRings Revisited

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Abstract

The objective of this paper is to revisit the studies on NeutroRings introduced in [5]. It is shown that there are 511 types/classes of NeutroRings and there are 19171 types/classes of AntiRings. A particular type/class of NeutroRings is studied. Several interesting examples and properties of the type/class of the NeutroRings are presented. It is shown that fundamental theorem of homomorphism and isomorphism theorems of the classical rings are holding in the type/class of NeutroRings studied.
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1 Introduction and Preliminaries

In the classical rings $R(+,.)$, addition and multiplication closure laws are 100% true for all the elements of R. Also, associative and distributive axioms over R are 100% true for all the elements of R. There are no provisions in the classical ring R to have addition and multiplication laws to be either partially true or partially indeterminate or partially false for the elements of R. Also, there are no provisions for associative and distributive axioms over R to be either partially true or partially indeterminate or partially false for the elements of R. Lack of these provisions in the classical rings is a problem because such rings cannot be used to model the reality accurately. This problem was solved by Smarandache in [10] by introducing the concepts of NeutroAlgebraicStructures and AntiAlgebraicStructures. Smarandache further studied these new concepts in [9] and [8] respectively. With these new concepts, a lot of research activities have begun with some papers already published. For instance in [6], Akbar and Smarandache studied Neutro-BE-algebras and Anti-BE-algebras and they showed that any classical algebra S with n operations (laws and axioms) where $n \geq 1$ will have $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras. In [3], Agboola et al. studied NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems, in [4], Agboola studied NeutroGroups and in [5], he studied NeutroRings. Also in [2], Agboola revisited NeutroGroups and in [1], he studied AntiGroups. In the present paper, the concept of NeutroRings introduced in [5] is revisited. It is shown that there are 511 types/classes of NeutroRings and there are 19171 types/classes of AntiRings. A particular type/class of NeutroRings is studied. Several interesting examples and properties of the type/class of the NeutroRings are presented. It is shown that fundamental theorem of homomorphism and isomorphism theorems of the classical rings are holding in the type/class of NeutroRings studied.

Definition 1.1. [8]

(i) A classical operation is an operation well defined for all the set's elements.

(ii) A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set.

(iii) An AntiOperation is an operation that is outer defined for all set's elements.

(iv) A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e. true for all set's elements).

(v) A NeutroLaw/NeutroAxiom (or Neutrosophic Law/Neutrosophic Axiom) defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where $T, I, F \in [0,1]$, with $(T, I, F) \neq (1,0,0)$ that represents the classical axiom, and $(T, I, F) \neq (0,0,1)$ that represents the AntiAxiom.

(vi) An AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.

(vii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements).

(viii) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

Theorem 1.2. *[6] Let $\mathbb{U}$ be a nonempty finite or infinite universe of discourse and let S be a finite or infinite subset of $\mathbb{U}$. If n classical operations (laws and axioms) are defined on S where $n \geq 1$, then there will be $(2^n - 1)$ NeutroAlgebras and $(3^n - 2^n)$ AntiAlgebras.*

2 NeutroRings Revisited

Definition 2.1. [Classical ring][7]
Let R be a nonempty set and let $+, . : R \times R \to R$ be binary operations of the usual addition and multiplication respectively defined on R. The triple $(R, +, .)$ is called a classical ring if the following conditions $(R1 - R9)$ hold:

(R1) $x + y \in R \; \forall x, y \in R$ [closure law of addition].

(R2) $x + (y + z) = (x + y) + z \; \forall x, y, z \in R$ [axiom of associativity].

(R3) There exists $0 \in R$ such that $x + e = e + x = x \; \forall x \in R$ [axiom of existence of neutral element].

(R4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e \; \forall x \in G$ [axiom of existence of inverse element]

(R5) $x + y = y + x \; \forall x, y \in R$ [axiom of commutativity].

(R6) $x.y \in R \; \forall x, y \in R$ [closure law of multiplication].

(R7) $x.(y.z) = (x.y).z \; \forall x, y, z \in R$ [axiom of associativity].

(R8) $x.(y + z) = (x.y) + (x.z) \; \forall x, y, z \in R$ [axiom of left distributivity].

(R9) $(y + z).x = (y.x) + (z.x) \; \forall x, y, z \in R$ [axiom of right distributivity].

If in addition we have,

(R10) $x.y = y.x \; \forall x, y \in R$ [axiom of commutativity],

then $(R, +, .)$ is called a commutative ring.

Definition 2.2. [NeutroSophication of the laws and axioms of the classical ring]

(NR1) $x + y \in R$ for at least one duplet $(x, y) \in R$ and $u + v \notin R$ for at least one duplet $(u, v) \in R$ [NeutroClosure law of addition].

(NR2) $x + (y + z) = (x + y) + z$ for at least one triplet $(x, y, z) \in R$ and $u + (v + w) \neq (u + v) + w$ for at least one triplet $(u, v, w) \in R$ [NeutroAxiom of associativity (NeutroAssociativity)].

(NR3) There exists an element $e \in R$ such that $x + e = x + e = x$ for at least one $x \in R$ and $y + e = e + y \neq y$ for at least one $y \in R$ [NeutroAxiom of existence of neutral element (NeutroNeutralElement)].

(NR4) There exists $-x \in R$ such that $x + (-x) = (-x) + x = e$ for at least one $x \in R$ and there exists $-y \in R$ such that $y + (-y) = (-y) + y \neq e$ for at least one $y \in R$ [NeutroAxiom of existence of inverse element (NeutroInverseElement)].

(NR5) $x + y = y + x$ for at least one duplet $(x, y) \in R$ and $u + v \neq v + u$ for at least one duplet $(u, v) \in R$ [NeutroAxiom of commutativity (NeutroCommutativity)].

(NR6) $x.y \in R$ for at least one duplet $(x, y) \in R$ and $u.v \notin R$ for at least one duplet $(u, v) \in R$ [NeutroClosure law of multiplication].

(NR7) $x.(y.z) = (x.y).z$ for at least one triplet $(x, y, z) \in R$ and $u.(v.w) \neq (u.v).w$ for at least one triplet $(u, v, w) \in R$ [NeutroAxiom of associativity (NeutroAssociativity)].

(NR8) $x.(y + z) = (x.y) + (x.z)$ for at least one triplet $(x, y, z) \in R$ and $u.(v + w) \neq (u.v) + (u.w)$ for at least one triplet $(u, v, w) \in R$ [NeutroAxiom of left distributivity (NeutroLeftDistributivity)].

(NR9) $(y + z).x = (y.x) + (z.x)$ for at least one triplet $(x, y, z) \in R$ and $(v + w).u \neq (v.u) + (w.u)$ for at least one triplet $(u, v, w) \in R$ [NeutroAxiom of right distributivity (NeutroRightDistributivity)].

(NR10) $x.y = y.x$ for at least one duplet $(x, y) \in R$ and $u.v \neq v.u$ for at least one duplet $(u, v) \in R$ [NeutroAxiom of commutativity (NeutroCommutativity)].

Definition 2.3. [AntiSophication of the law and axioms of the classical ring]

(AR1) $x + y \notin R \, \forall x, y \in R$ [AntiClosure law of addition].

(AR2) $x + (y + z) \neq (x + y) + z \, \forall x, y, z \in R$ [AntiAxiom of associativity (AntiAssociativity)].

(AR3) There doest not exist an element $e \in R$ such that $x + e = x + e = x \, \forall x \in R$ [AntiAxiom of existence of neutral element (AntiNeutralElement)].

(AR4) There does not exist $-x \in R$ such that $x + (-x) = (-x) + x = e \, \forall x \in R$ [AntiAxiom of existence of inverse element (AntiInverseElement)].

(AR5) $x + y \neq y + x \, \forall x, y \in R$ [AntiAxiom of commutativity (AntiCommutativity)].

(AR6) $x.y \notin R \, \forall x, y \in R$ [AntiClosure law of multiplication].

(AR7) $x.(y.z) \neq (x.y).z \, \forall x, y, z \in R$ [AntiAxiom of associativity (AntiAssociativity)].

(AR8) $x.(y + z) \neq (x.y) + (x.z) \, \forall x, y, z \in R$ [AntiAxiom of left distributivity (AntiLeftDistributivity)].

(AR9) $(y + z).x \neq (y.x) + (z.x) \, \forall x, y, z \in R$ [AntiAxiom of right distributivity (AntiRightDistributivity)].

(AR10) $x.y \neq y.x \, \forall x, y \in R$ [AntiAxiom of commutativity (AntiCommutativity)].

Definition 2.4. [NeutroRing]
A NeutroRing NR is an alternative to the classical ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ with no AntiLaw or AntiAxiom.

Definition 2.5. [AntiRing]
An AntiRing AR is an alternative to the classical ring R that has at least one AntiLaw or at least one of $\{AR1, AR2, AR3, AR4, AR5, AR6, AR7, AR8, AR9\}$.

Definition 2.6. [NeutroCommutativeRing]
A NeutroNoncommutativeRing NR is an alternative to the classical noncommutative ring R that has at least one NeutroLaw or at least one of $\{NR1, NR2, NR3, NR4, NR5, NR6, NR7, NR8, NR9\}$ and $NR10$ with no AntiLaw or AntiAxiom.

Definition 2.7. [AntiCommutativeRing]
An AntiCommutativeRing AR is an alternative to the classical commutative ring R that has at least one AntiLaw or at least one of $\{AR1, AR2, AR3, AR4, AR5, AR6, AR7, AR8, AR9\}$ and $AR10$.

Theorem 2.8. *Let $(R, +, .)$ be a finite or infinite classical ring. Then:*

(i) There are 511 types/classes of NeutroRings.

(ii) There are 19171 types/classes of AntiRings.

Proof. Follows from Theorem 1.2. □

Theorem 2.9. *Let $(R, +, .)$ be a finite or infinite classical commutative ring. Then:*

(i) There are 1023 types/classes of NeutroCommutativeRings.

(ii) There are 58025 types/classes of AntiCommutativeRings.

Proof. Follows from Theorem 1.2. □

Remark 2.10. It is evident from Theorem 2.8 and Theorem 2.9 that there are many types of NeutroRings and NeutroCommutativeRings. The type of NeutroRings studied by Agboola in [5] are those for which $NR1, NR2, \cdots, NR10$ were considered.

3 A Study of a Class of NeutroRings

In this section, we are going to study a class of NeutroRings $(NR, +, .)$ where $R1, R2, R3, R4, R5, R6$ are totally true for all the elements of NR and where $R7, R8, R9, R10$ are partially true, partially indeterminate and partially false for some elements of NR. $x.y$ will be written as xy $\forall x, y \in NR$.

Definition 3.1. Let $(NR, +, .)$ be a NeutroRing.

(i) NR is called a finite NeutroRing of order n if the cardinality of NR is n that is $o(NR) = n$. Otherwise, NR is called an infinite NeutroRing and we write $o(NR) = \infty$.

(ii) NR is called a NeutroRing with unity if there exists a multiplicative unit element $u \in NR$ such that $ux = xu = x$ for at least one $x \in R$.

(iii) If there exists a least positive integer n such that $nx = e$ for at least one $x \in NR$, then NR is called a NeutroRing of characteristic n. If no such n exists, then NR is called a NeutroRing of characteristic zero.

(iv) An element $x \in NR$ is called an idempotent element if $x^2 = x$.

(v) An element $x \in NR$ is called a nilpotent element if for the least positive integer n, we have $x^n = e$.

(vi) An element $e \neq x \in NR$ is called a zero divisor element if there exists an element $e \neq y \in NR$ such that $xy = e$ or $yx = e$.

(vii) An element $x \in NR$ is called a multiplicative inverse element if there exists at least one $y \in NR$ such that $xy = yx = u$ where u is the multiplicative unity element in NR.

Definition 3.2. Let $(NR, +, .)$ be a NeutroCommutativeRing with unity. Then

(i) NR is called a NeutroIntegralDomain if NR has no at least one zero divisor element.

(ii) NR is called a NeutroField if NR has at least one multiplicative inverse element.

Example 3.3. Let $NR = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and let $\oplus$ and $\odot$ be two binary operations defined on NR by

$$x \oplus y = x + y - 1, \;\; x \odot y = x + xy \;\; \forall\, x, y \in NR.$$

It is clear that $(NR, \oplus)$ is an abelian group.

(1) **NeutroAssociativity:** Let $x, y, z \in NR$. Then

$$\begin{aligned} x \odot (y \odot z) &= x + xy + xyz, \\ (x \odot y) \odot z &= x + xy + xz + xyz, \\ \therefore \quad x + xy + xyz &= x + xy + xz + xyz \\ \Rightarrow \quad xz &= 0 \\ \therefore \quad x &= 0 \;\text{ or }\; z = 0. \end{aligned}$$

This shows that only the triplets $(0, y, z), (x, y, 0), (0, y, 0)$ can verify associativity with 60% degree of associativity.

(2) **NeutroLeftDistributivity:** Let $x, y, z \in NR$. Then

$$\begin{aligned} x \odot (y \oplus z) &= x + xy + xz - x, \\ (x \odot y) \oplus (x \odot z) &= x + xy + x + xz - 1, \\ \therefore \quad x + xy + xz - x &= x + xy + x + xz - 1 \\ \Rightarrow \quad 2x &= 1 \\ \therefore \quad x &= 3. \end{aligned}$$

This shows that only the triplet $(3, y, z)$ can verify left distributivity with 20% degree of left distributivity.

(3) **NeutroRightDistributivity:** Let $x, y, z \in NR$. Then

$$\begin{aligned} (y \oplus z) \odot x &= y + z - 1 + yx + zx - x, \\ (y \odot x) \oplus (z \odot x) &= y + yx + z + zx - 1, \\ \therefore \quad y + z - 1 + yx + zx - x &= y + yx + z + zx - 1 \\ \Rightarrow \quad -x &= 0 \\ \therefore \quad x &= 0. \end{aligned}$$

This shows that only the triplet $(0, y, z)$ can verify right distributivity with 20% degree of right distributivity.

(4) **NeutroCommutativity:** Let $x, y \in NR$. Then

$$\begin{aligned} x \odot y &= x + xy, \\ y \odot x &= y + yx, \\ \therefore \quad x + xy &= y + yx \\ \Rightarrow \quad x &= y \\ \therefore \quad x &= y. \end{aligned}$$

This shows that only the duplet (x, x) can verify commutativity with 20% degree of commutativity.

We have just shown according to Definition 2.6 that $(NR, \oplus, \odot)$ is a NeutroCommutativeRing.

Example 3.4. Let $NR = \{a, b, c, d\}$ and let $''+''$ and $''.''$ be binary operations defined on R as shown in the Cayley tables below:

$+$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$.$	a	b	c	d
a	a	b	c	d
b	a	c	b	c
c	c	d	c	d
d	d	a	d	a

It is clear that $(NR, +)$ is an abelian group. From the tables we have:

(1) **NeutroAssociativity:**

$$\begin{aligned} a(bc) &= (ab)c = b, \\ b(bb) &= b \text{ but } (bb)b = d \neq b. \end{aligned}$$

This shows NeutroAssociativity of $''+''$.

(2) **NeutroLeftDistributivity:**

$$\begin{aligned} a(b + c) &= ab + ac = d, \\ b(c + d) &= c \text{ but } bc + bd = d \neq c. \end{aligned}$$

This shows NeutroLeftDistributivity of $''.''$ over $''+''$.

(3) **NeutroRightDistributivity:**

$$\begin{aligned}(b+c)c &= bc+cc=d,\\ (b+c)a &= d \text{ but } ba+ca=c\neq d.\end{aligned}$$

This shows NeutroRightDistributivity of $''.''$ over $''+''$.

(4) **NeutroCommutativity:**

$$\begin{aligned}ac &= ca=a,\\ bc &= b \text{ but } cb=d\neq b.\end{aligned}$$

This shows NeutroCommutativity of $''.''$.

We have just shown according to Definition 2.6 that $(NR,+,.)$ is a NeutroCommutativeRing.

Example 3.5. From Example 3.4, we note that $e=a$ is the additive neutral element. We now have the following:

(i) NR is a NeutroCommutativeRing with unity since $aa=a, ac=ca=c, ad=da=d$.

(ii) $\{a,c\}$ are idempotent elements.

(iii) $\{d\}$ is a nilpotent element.

(iv) $\{b,d\}$ are zero divisor elements.

(v) $\{a,d\}$ are invertible elements.

(vi) NR is not a NeutroIntegralDomain.

(vii) NR is a NeutroField.

(viii) NR is a NeutroCommutativeRing of characteristic 2.

Theorem 3.6. *Let $(NR_i,+,.), i=1,2,\cdots,n$ be a family of NeutroRings. Then*

(i) $NR=\bigcap_{i=1}^n NR_i$ is a NeutroRing.

(ii) $NR=\prod_{i=1}^n NR_i$ is a NeutroRing.

Definition 3.7. Let $(NR,+,.)$ be a NeutroRing. A nonempty subset NS of NR is called a NeutroSubring of NR if $(NS,+,.)$ is also a NeutroRing.

The only trivial NeutroSubring of NR is NR.

Example 3.8. Let $(NR,+,.)$ be the NeutroRing of Example 3.4 and let $NS=\{a,c\}$. The compositions of elements of NS are given in the Cayley tables below.

$+$	a	c
a	a	c
c	c	a

$.$	a	c
a	a	c
c	c	c

Then, NS is a NeutroSubring of NR.

Example 3.9. Let $NR=\mathbb{Z}$ and let $*$ be a binary operation of ordinary addition and let $\circ$ be a binary operation defined on NR by

$$x\circ y=x+xy \ \ \forall\, x,y\in NR.$$

It can be shown that $(NR,*,\circ)$ is a NeutroCommutativeRing. Let $NS=4\mathbb{Z}$. Under the binary operations $*$ and $\circ$ of NR, it can be shown that NS is a NeutroSubring of NR.

Theorem 3.10. *Let $(NR,+,.)$ be a NeutroRing and let $\{NS_i\}, i=1,2,\cdots,n$ be a family of NeutroSubrings of NR. Then* *(i)* $NS = \bigcap_{i=1}^{n} NS_i$ *is a NeutroSubring of* NR.

(ii) $NS = \prod_{i=1}^{n} NS_i$ *is a NeutroSubring of* NR.

(iii) $NS = \bigcup_{i=1}^{n} NS_i$ *is not necessarily a NeutroSubring of* NR.

Definition 3.11. Let $(NR, +, .)$ be a NeutroRing. A nonempty subset NI of NR is called a left NeutroIdeal of NR if the following conditions hold:

(i) NI is a NeutroSubring of NR.

(ii) $x \in NI$ and $r \in NR$ imply that at least one $xr \in NI$ for all $r \in NR$.

Definition 3.12. Let $(NR, +, .)$ be a NeutroRing. A nonempty subset NI of NR is called a right NeutroIdeal of NR if the following conditions hold:

(i) NI is a NeutroSubring of NR.

(ii) $x \in NI$ and $r \in NR$ imply that at least one $rx \in NI$ for all $r \in NR$.

Definition 3.13. Let $(NR, +, .)$ be a NeutroRing. A nonempty subset NI of NR is called a two-sided NeutroIdeal of NR if the following conditions hold:

(i) NI is a NeutroSubring of NR.

(ii) $x \in NI$ and $r \in NR$ imply that at least one $xr, rx \in NI$ for all $r \in NR$.

Example 3.14. Let $NI = NS$ be the NeutroSubring of Example 3.8. It can be shown that NI is a NeutroIdeal of NR. To see this, consider the following:

$$aa, ac, ca, cc \in NI, ab, ad, cb, cd \notin NI.$$

This shows that NI is a left NeutroIdeal of NR. Also,

$$aa, ba, ca, ac, cc \in NI, da, bc, dc \notin NI.$$

This shows that NI is a right NeutroIdeal of NR. Hence, NI is a NeutroIdeal of NR.

Example 3.15. Let $NI = NS = 4\mathbb{Z}$ where NS is the NeutroSubring of Example 3.9. Then NI is a left NeutroIdeal of NR and not a two-sided NeutroIdeal of NR. To see this, let $r \in NR, 4k = x \in NI$ with $k \in \mathbb{Z}$. Then

$$\begin{aligned} x \circ r &= x + xr \\ &= 4k + 4kr \\ &= 4k(1 + r) \in NI. \\ r \circ x &= r + rx \\ &= r + 4kr \\ &= r(1 + 4k) \notin NI. \end{aligned}$$

We have just shown that NI is a left NeutroIdeal of NR and not a right NeutroIdeal of NR. Hence, NI is a left NeutroIdeal of NR and not a two-sided NeutroIdeal of NR.

Theorem 3.16. *Let* $(NR, +, .)$ *be a NeutroRing and let* $\{NI_i\}, i = 1, 2, \cdots, n$ *be a family of NeutroIdeals of* NR. *Then*

(i) $NI = \bigcap_{i=1}^{n} NI_i$ *is a NeutroIdeal of* NR.

(ii) $NI = \sum_{i=1}^{n} NI_i$ *is a NeutroIdeal of* NR.

Definition 3.17. Let $(NR, +, .)$ be a NeutroRing and let NI be a NeutroIdeal of NR. The set NR/NI is defined by

$$NR/NI = \{x + NI : x \in NR\}.$$

For $x + NI, y + NI \in NR/NI$ with $x, y \in NR$, let $\oplus$ and $\odot$ be binary operations on NR/NI defined as follows:

$$\begin{aligned}(x + NI) \oplus (y + NI) &= (x + y) + NI,\\ (x + NI) \odot (y + NI) &= (xy) + NI.\end{aligned}$$

Then it can be shown that the tripple $(NR/NI, \oplus, \odot)$ is a NeutroRing which we call a NeutroQuotientRing.

Example 3.18. Let $(NR, *, \circ)$ be the NeutroRing of Example 3.9 and let NI be the NeutroIdeal of Example 3.15. Then

$$\begin{aligned}NR/NI &= \{x * NI : x \in NR\}\\ &= \{x + NI : x \in NR\}\\ &= \{NI, 1 + NI, 2 + NI, 3 + NI\}.\end{aligned}$$

Let $\oplus$ and $\odot$ be binary operations on NR/NI defined as follows:

$$\begin{aligned}(x + NI) \oplus (y + NI) &= (x * y) + NI\\ &= (x + y) + NI.\\ (x + NI) \odot (y + NI) &= (x \circ y) + NI\\ &= (x + xy) + NI.\end{aligned}$$

Now consider the following Cayley tables:

$\oplus$	NI	$1+NI$	$2+NI$	$3+NI$
NI	NI	$1+NI$	$2+NI$	$3+NI$
$1+NI$	$1+NI$	$2+NI$	$3+NI$	NI
$2+NI$	$2+NI$	$3+NI$	NI	$1+NI$
$3+NI$	$3+NI$	NI	$1+NI$	$2+NI$

$\odot$	NI	$1+NI$	$2+NI$	$3+NI$
NI	NI	NI	NI	NI
$1+NI$	$1+NI$	$2+NI$	$3+NI$	NI
$2+NI$	$2+NI$	NI	$2+NI$	NI
$3+NI$	$3+NI$	$2+NI$	$1+NI$	NI

.

It can easily be shown that $(NR/NI, \oplus, \odot)$ is a NeutroCommutativeRing with unity.

Theorem 3.19. *Let NI be a NeutroIdeal of the NeutroRing NR. Then NR/NI is a NeutroCommutativeRing with unity if and only if NR is a NeutroCommutativeRing with unity.*

Definition 3.20. Let $(NR, +, .)$ and $(NS, +', .')$ be any two NeutroRings. The mapping $\phi : NR \to NS$ is called a NeutroRingHomomorphism if ϕ preserves the binary operations of NR and NS that is if for at least a pair $(x, y) \in NR$, we have:

$$\begin{aligned}\phi(x + y) &= \phi(x) +' \phi(y),\\ \phi(x.y) &= \phi(x).'\phi(y).\end{aligned}$$

The kernel of ϕ denoted by $Ker\phi$ is defined as

$$Ker\phi = \{x : \phi(x) = e_{NR}\}.$$

The image of ϕ denoted by $Im\phi$ is defined as

$$Im\phi = \{y \in NS : y = \phi(x) \text{ for at least one } y \in NS\}.$$

If in addition ϕ is a NeutroBijection, then ϕ is called a NeutroRingIsomorphism and we write $NR \cong NS$. NeutroRingEpimorphism, NeutroRingMonomorphism, NeutroRingEndomorphism and NeutroRingAutomorphism are defined similarly.

Example 3.21. Let NR be the NeutroRing of Example 3.4 and let $\phi : NR \times NR \to NR$ be a projection defined by

$$\phi(x,y) = x.$$

Then

$$\begin{aligned}
\phi(a,a) &= \phi(a,b) = \phi(a,c) = \phi(a,d) = a,\\
\phi(b,a) &= \phi(b,b) = \phi(b,c) = \phi(b,d) = b,\\
\phi(c,a) &= \phi(c,b) = \phi(c,c) = \phi(c,d) = c,\\
\phi(d,a) &= \phi(d,b) = \phi(d,c) = \phi(d,d) = d,\\
\phi((a,b)+(c,d)) &= \phi(a+c,b+d) = \phi(c,a) = c,\\
\phi(a,b)+\phi(c,d) &= a+c = c,\\
\phi((a,b).(c,d)) &= \phi(ac,bd) = \phi(c,c) = c,\\
\phi(a,b)\phi(c,d) &= ac = c.
\end{aligned}$$

We have just shown that ϕ is a NeutroRingHomomorphism. Next,

$$\begin{aligned}
Im\phi &= \{a,b,c,d\} = NR,\\
Ker\phi &= \{(a,a),(a,b),(a,c),(a,d)\}.
\end{aligned}$$

Consider the Cayley tables below:

$+$	(a,a)	(a,b)	(a,c)	(a,d)
(a,a)	(a,a)	(a,b)	(a,c)	(a,d)
(a,b)	(a,b)	(a,c)	(a,d)	(a,a)
(a,c)	(a,c)	(a,d)	(a,a)	(a,b)
(a,d)	(a,d)	(a,a)	(a,b)	(a,c)

$.$	(a,a)	(a,b)	(a,c)	(a,d)
(a,a)	(a,a)	(a,b)	(a,c)	(a,d)
(a,b)	(a,a)	(a,c)	(a,b)	(a,c)
(a,c)	(a,c)	(a,d)	(a,c)	(a,d)
(a,d)	(a,d)	(a,a)	(a,d)	(a,a)

It is evident from the Cayley tables that $(Ker\phi, +, .)$ is a NeutroRing and since $Ker\phi$ is a subset of $NR\times NR$, it follows that $Ker\phi$ is a NeutroSubring of $NR \times NR$. It can equally be shown that $Ker\phi$ is a NeutroIdeal of $NR \times NR$.

Theorem 3.22. *Let NR and NS be two NeutroRings and suppose that $\phi : NR \to NS$ is a NeutroRingHomomorphism. Then:*

(i) $\phi(e_{NR}) = e_{NS}$.

(ii) $Ker\phi$ *is a NeutroIdeal of* NR.

(iii) $Im\phi$ *is a NeutroSubring of* NS.

(iv) ϕ *is NeutroInjective if and only if* $Ker\phi = \{e_{NR}\}$.

Example 3.23. Let NR/NI be the NeutroQuotientRing of Example 3.18 and let $\phi : NR \to NR/NI$ be a mapping defined by $\phi(x) = x + NI$ for at least one $x \in NR$. Then

$$\begin{aligned}
\phi(0) &= 0 + NI = NI,\\
\phi(1) &= 1 + NI,\\
\phi(2) &= 2 + NI,\\
\phi(3) &= 3 + NI,\\
\phi(2+3) &= \phi(5) = 5 + NI = 1 + NI,\\
\phi(2)\oplus\phi(3) &= (2+NI)\oplus(3+NI)\\
&= (2+NI)*(3+NI)\\
&= (2+NI)+(3+NI)\\
&= 5+NI = 1+NI.\\
\phi(2\circ 3) &= \phi(8) = 8+NI = NI,\\
\phi(2)\odot\phi(3) &= (2+NI)\odot(3+NI)\\
&= (2\circ 3)+NI = 8+NI = NI.
\end{aligned}$$

We have just shown that ϕ is a NeutroRingHomomorphism.

$$\begin{aligned} Ker\phi &= \{x \in NR : \phi(x) = e_{NR/NI}\} \\ &= \{x \in NR : x + NI = e_{NR/NI} = NI\} \\ &= NI. \end{aligned}$$

Theorem 3.24. *Let NI be a NeutroIdeal of the NeutroRing NR. Then the mapping $\psi : NR \to NR/NI$ defined by*

$$\psi(x) = x + NI \quad \textit{for at least one } \; x \in NR$$

is a NeutroRingEpimomorphism and the $Ker\psi = NI$.

Theorem 3.25. [Fundamental Theorem of NeutroRingHomomorphism]. *Let $\phi : NR \to NS$ be a NeutroRingHomomorpism and let $K = Ker\phi$. Then the mapping $\psi : NR/K \to Im\phi$ defined by*

$$\psi(x + K) = \phi(x) \quad \textit{for at least one } \; x \in NR$$

is a NeutroRingIsomorphism.

Proof. Let $x + K, y + K \in NR/K$ with at least a pair $(x, y) \in NR$. Then

$$\begin{aligned} \psi((x + K) \oplus (y + K)) &= \psi((x + y) + K) \\ &= \phi(x + y) \\ &= \phi(x) + \phi(y) \\ &= \psi(x + K) \oplus \psi(y + K). \\ \psi((x + K) \odot (y + K)) &= \psi((xy) + K) \\ &= \phi(xy) \\ &= \phi(x)\phi(y) \\ &= \psi(x + K) \odot \psi(y + K). \\ Ker\psi &= \{x + K \in NR/K : \psi(x + K) = e_{\phi(x)}\} \\ &= \{x + K \in NR/K : \phi(x) = e_{\phi(x)}\} \\ &= K. \end{aligned}$$

This shows that ψ is a NeutroBijectiveRingHomomorphism and therefore it is a NeutroRingIsomorphism that is $NR/K \cong Im\phi$ which is the same as what is obtainable in the classical rings. □

Theorem 3.26. *Let NI and NJ be any two NeutroIdeals of the NeutroRing NR. Then*

(i) $(NI + NJ)/NI \cong NI/(NI \cap NJ)$.

(ii) $(NI+NJ)/(NI\cap NJ) \cong (NI+NJ)/NI\times(NI+NJ)/NJ \cong NJ/(NI\cap NJ)\times NI/(NI\cap NJ)$.

(iii) If $NR = NI + NJ$, then $NR/(NI \cap NJ) \cong NR/NI \times NR/NJ$.

Proof. The proof is the same as the classical rings. □

Theorem 3.27. *NeutroRingIsomorphism of NeutroRings is an equivalence relation.*

Proof. The proof is the same as the classical rings. □

4 Conclusion

We have in this paper revisited the concept of NeutroRings introduced in [5]. We have shown that there are 511 types/classes of NeutroRings as well as there are 19171 types/classes of AntiRings. We have studied a particular type/class of NeutroRings. We have presented several interesting examples and properties of the type/class of the NeutroRings studied. We have shown that fundamental theorem of homomorphism and isomorphism theorems of the classical rings are holding in the type/class of NeutroRings we studied. In our future papers, we will study AntiRings and their types/classes. Also, more types/classes of NeutroRings will be studied.

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