



On Refined Neutrosophic Hypervector Spaces

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Abstract

This paper presents the refinement of neutrosophic hypervector spaces and studies some of its basic properties. Some basic definitions and important results are presented. The paper also establishes the existence of a good linear transformation between a weak refined neutrosophic hypervector space $V(I_1, I_2)$ and a weak neutrosophic hypervector space $V(I)$.

Keywords: Neutrosophy, neutrosophic hypervector space, neutrosophic subhypervector space, refined neutrosophic hypervector space, refined neutrosophic subhypervector space, refined neutrosophic hypervector space homomorphism.

1 Introduction and Preliminaries

The concept of algebraic hyperstructure was first introduced by Marty [²⁵]. He presented the definition of a hypergroup, studied its properties and applied them to study the groups of rational algebraic functions. Also, Marty used the new approach to solve several problems of the non-commutative algebra. Since then, several researchers have been working on this new field of modern algebra and developed it to a very large extent.

M. Krasner [²⁶], introduced the notions of hyperring and hyperfield and used them as technical tools in the study of the approximation of valued fields. There exist several types of hyperrings, some of which are: additive hyperring, multiplicative hyperring and general hyperrings. An important class of additive hyperrings is Krasner hyperrings [^{23,29,34}].

A class of hyperrings $(R, +, \cdot)$ where $''+''$ and $''\cdot''$ are hyperoperations was introduced by De Salvo [²⁴]. This class of hyperrings has been further studied by Asokkumar [⁹], Asokkumar and Velrajan [^{10,11,28}] and Davvaz and Leoreanu-Fotea [²³]. Mittas in [²⁷] introduced the theory of canonical hypergroups. J. Mittas was the first who studied them independently from their operations. Some connected hyperstructures with canonical hypergroups were introduced and analyzed by P. Corsini [^{21,22}], P. Bonansinga [^{19,20}], and K. Serafimidis in [^{32,33}]. Further contributions to the theory of hyperstructures can be found in the books of P. Corsini [²¹], T. Vougiouklis, P. Corsini and V. Leoreanu [²²], and Davvaz and V. Leoreanu [²³]. The notion of hypervector spaces was introduced by M. Scafati Tallini. In the definition [³¹] of hypervector spaces, M. Scafati Tallini considered the field as the usual field. In [³⁰], Sanjay Roy and T. K. Samanta generalized the notion of hypervector space by considering the hyperfield and considering the multiplication structure of a vector by a scalar as a hyperoperation like M. Scafati Tallini and they both called the hyperstructure a hypervector space. They established basic properties of hypervector space and thereafter the notions of linear combinations, linearly dependence, linearly independence, Hamel basis were introduced and several important properties like deletion theorem, extension theorem were developed.

Neutrosophy is a new branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set and neutrosophic logic were introduced in 1995 by Smarandache as generalizations of fuzzy logic/set [⁴³] and respectively intuitionistic fuzzy logic/set [¹³]. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-0, 1}^{+}$ as can be seen in

[35,36]. A comprehensive review of neutrosophic set, neutrosophic soft set, neutrosophic topological spaces, neutrosophic algebraic structures and new trends in neutrosophic theory can be found in [3,14-18,37-42].

Agboola and Davvaz introduced and studied neutrosophic hypergroups and presented some of their elementary properties in [7] and in [8], they studied and presented basic properties of canonical hypergroups and hyperrings in a neutrosophic environment, Quotient neutrosophic canonical hypergroups and neutrosophic hyperrings were also presented. In [5], Agboola and Akinleye studied neutrosophic hypervector spaces and they presented their basic properties.

In [36], Smarandache introduced the concept of refined neutrosophic logic and neutrosophic set which allows for the splitting of the components $\langle T, I, F \rangle$ into the form $\langle T_1, T_2, \dots, T_p; I_1, I_2, \dots, I_r; F_1, F_2, \dots, F_s \rangle$. This refinement has given rise to the extension of neutrosophic numbers $a + bI$ into refined neutrosophic numbers of the form $(a + b_1I_1 + b_2I_2 + \dots + b_nI_n)$ are real or complex numbers which has led to the introduction of refined neutrosophic set. Refined neutrosophic set has been applied in the development of refined neutrosophic algebraic structures and refined neutrosophic hyperstructures. Agboola in [4] introduced the concept of refined neutrosophic algebraic structures and studied refined neutrosophic groups in particular. Since then, several researchers in this field have studied this concept and a great deal of results have been published. Recently for instance, Adeleke et al published results on refined neutrosophic rings, refined neutrosophic subring in [1] and in [2], they presented some results on refined neutrosophic ideals and refined neutrosophic homomorphism. The present paper is devoted to the study of refined neutrosophic hypervector space and presents some elementary properties of this structure.

For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that:

$$\begin{aligned} I_1I_1 &= I_1^2 = I_1, \\ I_2I_2 &= I_2^2 = I_2, \text{ and} \\ I_1I_2 &= I_2I_1 = I_1. \end{aligned}$$

Definition 1.1. Let $(F, +, \cdot)$ be any field. The triple $(F(I), +, \cdot)$ is called a neutrosophic field generated by F and I . $(\mathbb{Q}(I), +, \cdot)$ and $(\mathbb{R}(I), +, \cdot)$ are examples of neutrosophic fields.

Definition 1.2. ⁶ Let $(V, +, \cdot)$ be any vector space over a field K and let $V(I) = \langle V \cup I \rangle$ be a neutrosophic set generated by V and I . The triple $(V(I), +, \cdot)$ is called a weak neutrosophic vector space over a field K . If $V(I)$ is a neutrosophic vector space over a neutrosophic field $K(I)$, then $V(I)$ is called a strong neutrosophic vector space. The elements of $V(I)$ are called neutrosophic vectors and the elements of $K(I)$ are called neutrosophic scalars.

If $u = a + bI, v = c + dI \in V(I)$ where a, b, c and d are vectors in V and $\alpha = k + mI \in K(I)$ where k and m are scalars in K , then :

$$u + v = (a + bI) + (c + dI) = (a + c) + (b + d)I,$$

and

$$\alpha u = (k + mI) \cdot (a + bI) = k \cdot a + (k \cdot b + m \cdot a + m \cdot b)I.$$

Definition 1.3. ²³ Let H be a non-empty set and $\circ : H \times H \rightarrow P^*(H)$ be a hyperoperation. The couple (H, \circ) is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\} \quad \text{and} \quad x \circ B = \{x\} \circ B.$$

Definition 1.4. ²³ A hypergroupoid (H, \circ) is called a semihypergroup if for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

A hypergroupoid (H, \circ) is called a quasihypergroup if for all $a \in H$ we have $a \circ H = H \circ a = H$. This condition is also called the reproduction axiom.

Definition 1.5. ²³ A hypergroupoid (H, \circ) which is both a semihypergroup and a quasi-hypergroup is called a hypergroup.

Definition 1.6. ²³ Let (H, \circ) and (H', \circ') be two hypergroupoids. A map $\phi : H \rightarrow H'$, is called

1. an inclusion homomorphism if for all x, y of H , we have $\phi(x \circ y) \subseteq \phi(x) \circ' \phi(y)$;
2. a good homomorphism if for all x, y of H , we have $\phi(x \circ y) = \phi(x) \circ' \phi(y)$.

Definition 1.7. ²³ Let (H_1, \star_1) and (H_2, \star_2) be any two refined hypergroupoids and let $f : H_1 \longrightarrow H_2$ be a map. We say that :

1. f is a homomorphism if for all x, y of H_1 ,

$$f(x \star_1 y) \subseteq f(x) \star_2 f(y);$$

2. f is a good homomorphism if for all x, y of H_1 ,

$$f(x \star_1 y) = f(x) \star_2 f(y);$$

3. f is a strong homomorphism on the left if

$$f(x) \in f(y) \star_2 f(z) \implies \exists y' \in H_1 \ni f(y) = f(y') \text{ and } x \in y' \star_1 z.$$

Similarly, we can define a homomorphism, which is strong on the right. If f is strong on the right and on the left we say that f is a strong homomorphism.

Definition 1.8. ²³ Let H be a non-empty set and let $+$ be a hyperoperation on H . The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

1. $x + y = y + x$, for all $x, y \in H$,
2. $x + (y + z) = (x + y) + z$, for all $x, y, z \in H$,
3. there exist a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, for all $x \in H$,
4. for every $x \in H$, there exist a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
5. $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, for all $x, y, z \in H$.

Definition 1.9. ²³ A hyperring is a triple $(R, +, \cdot)$ satisfying the following axioms:

1. $(R, +)$ is a canonical hypergroup.
2. (R, \cdot) is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, that is, 0 is a bilaterally absorbing element.
3. For all $x, y, z \in R$
 - (a) $x \cdot (y + z) = x \cdot y + x \cdot z$ and
 - (b) $(x + y) \cdot z = x \cdot z + y \cdot z$.

That is, the hyperoperation \cdot is distributive over the hyperoperation $+$.

Definition 1.10. ⁵ Let $P(V)$ be the power set of a set V , $P^*(V) = P(V) - \{\emptyset\}$ and let K be a field. The quadruple $(V, +, \bullet, K)$ is called a hypervector space over a field K if:

1. $(V, +)$ is an abelian group.
2. $\bullet : K \times V \longrightarrow P^*(V)$ is a hyperoperation such that for all $k, m \in K$ and $u, v \in V$, the following conditions hold:
 - (a) $(k + m) \bullet u \subseteq (k \bullet u) + (m \bullet u)$,
 - (b) $k \bullet (u + v) \subseteq (k \bullet u) + (k \bullet v)$,
 - (c) $k \bullet (m \bullet u) = (km) \bullet u$, where $k \bullet (m \bullet u) = \{k \bullet v : v \in m \bullet u\}$,
 - (d) $(-k) \bullet u = k \bullet (-u)$,
 - (e) $u \in 1 \bullet u$.

A hypervector space is said to be strongly left distributive (resp. strongly right distributive) if equality holds in (a) (resp. in (b)). $(V, +, \bullet, K)$ is called a strongly distributive hypervector space if it is both strongly left and strongly right distributive.

Definition 1.11.¹² Let V and W be hypervector spaces over K . A mapping $T : V \rightarrow W$ is called

1. weak linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \cap a \circ T(x) \neq \emptyset, \forall x, y \in V, a \in K,$$

2. linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \subseteq a \circ T(x), \forall x, y \in V, a \in K,$$

3. good linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) = a \circ T(x), \forall y \in V, a \in K.$$

Definition 1.12.⁷ Let (H, \star) be any hypergroup and let $\langle H \cup I \rangle = \{x = (a, bI) : a, b \in H\}$.

The couple $N(H) = (\langle H \cup I \rangle, \star)$ is called a neutrosophic hypergroup generated by H and I under the hyperoperation \star . The part a is called the non-neutrosophic part of x and the part b is called the neutrosophic part of x .

If $x = (a, bI)$ and $y = (c, dI)$ are any two elements of $N(H)$, where $a, b, c, d \in H$, then

$$x \star y = (a, bI) \star (c, dI) = \{(u, vI) | u \in a \star c, v \in a \star d \cup b \star c \cup b \star d\} = (a \star c, (a \star d \cup b \star c \cup b \star d)I).$$

Note that $a \star c \subseteq H$ and $(a \star d \cup b \star c \cup b \star d) \subseteq H$.

Definition 1.13.⁸ A neutrosophic hyperring is a triple $(N(R), +, \cdot)$ satisfying the following axioms :

1. $(N(R), +)$ is a neutrosophic canonical hypergroup.

2. $(N(R), \cdot)$ is a neutrosophic semihypergroup.

For all $(a, bI), (c, dI), (e, fI) \in N(R)$,

$$(a) (a, bI) \cdot ((c, dI) + (e, fI)) = (a, bI) \cdot (c, dI) + (a, bI) \cdot (e, fI) \text{ and}$$

$$(b) ((c, dI) + (e, fI)) \cdot (a, bI) = (c, dI) \cdot (a, bI) + (e, fI) \cdot (a, bI).$$

Definition 1.14.⁶ Let $(V, +, \bullet, K)$ be any strongly distributive hypervector space over a field K and let $V(I) = \langle V \cup I \rangle = \{u = (a, bI) : a, b \in V\}$ be a set generated by V and I .

The quadruple $(V(I), +, \bullet, K)$ is called a weak neutrosophic strongly distributive hypervector space over a field K .

For every $u = (a, bI), v = (c, dI) \in V(I)$ and $k \in K$, then

$$u + v = (a + c, (b + d)I) \in V(I),$$

$$k \bullet u = \{(x, yI) : x \in k \bullet a, y \in k \bullet b\}.$$

If K is a neutrosophic field, that is, $K = K(I)$, then the quadruple $(V(I), +, \bullet, K(I))$ is called a strong neutrosophic strongly distributive hypervector space over a neutrosophic field $K(I)$. For every $u = (a, bI), v = (c, dI) \in V(I)$ and $\alpha = (k, mI) \in K(I)$, we define

$$u + v = (a + c, (b + d)I) \in V(I),$$

$$\alpha \bullet u = \{(x, yI) : x \in k \bullet a, y \in k \bullet b \cup m \bullet a \cup m \bullet b\}.$$

The zero neutrosophic vector of $V(I)$, $(0, 0I)$, is denoted by θ , the zero element $0 \in K$ is represented by $(0, 0I)$ in $K(I)$ and $1 \in K$ is represented by $(1, 0I)$ in $K(I)$.

Definition 1.15.⁴ If $\ast : X(I_1, I_2) \times X(I_1, I_2) \rightarrow X(I_1, I_2)$ is a binary operation defined on $X(I_1, I_2)$, then the couple $(X(I_1, I_2), \ast)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by \ast .

Definition 1.16. ⁴ Let $(X(I_1, I_2), +, \cdot)$ be any refined neutrosophic algebraic structure where "+" and "." are ordinary addition and multiplication respectively.

For any two elements $(a, bI_1, cI_2), (d, eI_1, fI_2) \in X(I_1, I_2)$, we define

$$(a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$

$$(a, bI_1, cI_2) \cdot (d, eI_1, fI_2) = (ad, (ae + bd + be + bf + ce)I_1, (af + cd + cf)I_2).$$

Definition 1.17. ⁴ If "+" and "." are ordinary addition and multiplication, I_k with $k = 1, 2$ have the following properties:

1. $I_k + I_k + \dots + I_k = nI_k$.
2. $I_k + (-I_k) = 0$.
3. $I_k \cdot I_k \cdot \dots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
4. $0 \cdot I_k = 0$.
5. I_k^{-1} is undefined and therefore does not exist.

2 Formulation of Refined Neutrosophic Hypervector Space

This section shows the formulation of refined neutrosophic hypervector space and present some of its properties.

Definition 2.1. Let $(V, +, \bullet, K)$ be any strongly distributive hypervector space over a field K and let

$$V(I_1, I_2) = \langle V \cup (I_1, I_2) \rangle = \{u = (a, bI_1, cI_2) : a, b, c \in V\}$$

be a set generated by V, I_1 and I_2 . The quadruple $(V(I_1, I_2), +, \bullet, K)$ is called a weak refined neutrosophic strongly distributive hypervector space over a field K .

For every element $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$, and $k \in K$ we define

$$u + v = (a + d, (b + e)I_1, (c + f)I_2) \in V(I_1, I_2),$$

$$k \bullet u = \{(x, yI_1, zI_2) : x \in k \bullet a, y \in k \bullet b, z \in k \bullet c\}.$$

If K is a refined neutrosophic field, that is, $K = K(I_1, I_2)$, then the quadruple $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ is called a strong refined neutrosophic strongly distributive hypervector space over a refined neutrosophic field $K(I_1, I_2)$.

For every element $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$, and $\alpha = (k, mI_1, nI_2) \in K(I_1, I_2)$, we define

$$u + v = (a, bI_1, cI_2) + (d, eI_1, fI_2) = (a + d, (b + e)I_1, (c + f)I_2),$$

$$\alpha \bullet u = \{(x, yI_1, zI_2) : (x \in k \bullet a, y \in k \bullet b \cup m \bullet a \cup m \bullet b \cup m \bullet c \cup n \bullet b, z \in k \bullet c \cup n \bullet a \cup n \bullet c)\}.$$

The elements of $V(I_1, I_2)$ are called refined neutrosophic vectors and the elements of $K(I_1, I_2)$ are called refined neutrosophic scalars. The zero refined neutrosophic vector of $V(I_1, I_2)$, $(0, 0I_1, 0I_2)$, is denoted by θ , the zero element $0 \in K$ is represented by $(0, 0I_1, 0I_2)$ in $K(I_1, I_2)$ and $1 \in K$ is represented by $(1, 0I_1, 0I_2) \in K(I_1, I_2)$.

Example 2.2. 1. Let r be a fixed positive integer and let

$V = \mathbb{Q}(I_1, I_2)(\sqrt{r}) = \{(a, b\sqrt{r}I_1, c\sqrt{r}I_2) : a, b, c \in \mathbb{Q}, r \in \mathbb{Z}^+\}$. Then V is a weak refined neutrosophic strongly distributive hypervector space over \mathbb{Q} . If $u = (a, b\sqrt{r}I_1, c\sqrt{r}I_2)$ and $v = (d, e\sqrt{r}I_1, f\sqrt{r}I_2)$ then $u + v = (a + d, (b + e)\sqrt{r}I_1, (c + f)\sqrt{r}I_2)$ is again in V .

Also, for $\alpha \in \mathbb{Q}$, then

$$\alpha \bullet u = \{(x, y\sqrt{r}I_1, z\sqrt{r}I_2) : x \in \alpha \bullet a, y \in \alpha \bullet b, z \in \alpha \bullet c\} \in V.$$

2. Let $V(I_1, I_2) = \mathbb{R}(I_1, I_2)$ and let $K = \mathbb{R}$. For all $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$ and $k \in K$, define:

$$u + v = (a + d, (b + e)I_1, (c + f)I_2)$$

$$k \bullet u = \{(x, yI_1, zI_2) : x \in k \bullet a, y \in k \bullet b, z \in k \bullet c\}.$$

Then $(V(I_1, I_2), +, \bullet, K)$ is a weak neutrosophic strongly distributive hypervector space over the field K .

Example 2.3. 1. Let $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ and let $K = \mathbb{R}(I_1, I_2)$. For all $u = ((a, bI_1, cI_2), (d, eI_1, fI_2), (g, hI_1, jI_2)), v = ((a', b'I_1, c'I_2), (d', e'I_1, f'I_2), (g', h'I_1, j'I_2)) \in V(I_1, I_2)$ and $\alpha = (k, mI_1, nI_2) \in K(I_1, I_2)$, define :

$$u + v = ((a + a', (b + b')I_1, (c + c')I_2), (d + d', (e + e')I_1, (f + f')I_2), (g + g', (h + h')I_1, (j + j')I_2)),$$

$$\alpha \bullet u = \{(x_1, y_1I_1, z_1I_2), (x_2, y_2I_1, z_2I_2), (x_3, y_3I_1, z_3I_2)\} :$$

$$x_1 \in k \bullet a,$$

$$y_1 \in k \bullet b \cup m \bullet a \cup m \bullet b \cup m \bullet c \cup n \bullet b,$$

$$z_1 \in k \bullet c \cup n \bullet a \cup n \bullet c,$$

$$x_2 \in k \bullet d,$$

$$y_2 \in k \bullet e \cup m \bullet d \cup m \bullet e \cup m \bullet f \cup n \bullet e,$$

$$z_2 \in k \bullet f \cup n \bullet d \cup n \bullet f$$

$$x_3 \in k \bullet g,$$

$$y_3 \in k \bullet h \cup m \bullet g \cup m \bullet h \cup m \bullet j \cup n \bullet h,$$

$$z_3 \in k \bullet j \cup n \bullet g \cup n \bullet j\}.$$

Then $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ is a strong refined neutrosophic hypervector space over the refined neutrosophic field $K(I_1, I_2)$.

2. Let $V(I_1, I_2) = \mathbb{R}^2(I_1, I_2)$ and $K = \mathbb{R}$ define for all $x = (u, v) \in V(I_1, I_2)$ with $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2)$ and $\alpha \in K$

$$\left\{ \begin{array}{l} \bullet : \mathbb{R} \times \mathbb{R}^2(I_1, I_2) \longrightarrow P^*(\mathbb{R}^2(I_1, I_2)), \\ \alpha \bullet (u, v) = \alpha \bullet u \times \mathbb{R}(I_1, I_2). \end{array} \right. \quad \text{OR} \quad \left\{ \begin{array}{l} \bullet : \mathbb{R} \times \mathbb{R}^2(I_1, I_2) \longrightarrow P^*(\mathbb{R}^2(I_1, I_2)), \\ \alpha \bullet (u, v) = \mathbb{R}(I_1, I_2) \times \alpha \bullet v. \end{array} \right.$$

Then $(V(I_1, I_2), +, \bullet, K)$ is a weak refined neutrosophic strongly distributive hypervector space. From now on, every weak(strong) refined neutrosophic strongly distributive hypervector space will simply be called a weak(strong) refined neutrosophic hypervector space.

Lemma 2.4. Let $V(I_1, I_2)$ be a weak refined neutrosophic hypervector space over a field K . Then for all $k \in K$ and $u = (a, bI_1, cI_2) \in V(I_1, I_2)$, we have

1. $k \bullet \theta = \{\theta\}$.
2. $k \bullet u = \{\theta\}$ implies that $k = \theta$ or $u = \theta$.
3. $-u \in (-1) \bullet u$

Proof. 1. $k \bullet \theta = k \bullet (0 \bullet \theta) = (k \bullet 0) \bullet \theta = 0 \bullet \theta = \theta$

2. Let $k \in K$ and $u \in V$ be such that $k \bullet u = \{\theta\}$.
 If $k = 0$, then $0 \bullet u = \theta$.
 If $k \neq 0$, then $k^{-1} \in K$. Therefore $k \bullet u = \theta \implies k^{-1} \bullet (k \bullet u) = k^{-1} \bullet \theta \implies (k^{-1} \bullet k) \bullet u = \theta \implies 1_K \bullet u = \theta \implies u = \theta$.

□

Proposition 2.5. Every strong refined neutrosophic hypervector space is a weak refined neutrosophic hypervector space.

Proof. Suppose that $V(I_1, I_2)$ is a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ say. Since $K \subseteq K(I_1, I_2)$ for every field K , then we have that $V(I_1, I_2)$ is also a weak refined neutrosophic hypervector space. □

Proposition 2.6. Every weak refined neutrosophic hypervector space is a strongly distributive hypervector space.

Proof. Suppose that $V(I_1, I_2)$ is a weak refined neutrosophic hypervector space over a field K . Obviously, $(V(I_1, I_2), +)$ is an abelian group. Let $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$ and $k, m \in K$ be arbitrary. Then

$$\begin{aligned} (1) \quad k \bullet u + m \bullet u &= \{(p, qI_1, rI_2) : p \in k \bullet a, q \in k \bullet b, r \in k \bullet c\} + \\ &\quad \{(s, tI_1, wI_2) : s \in m \bullet a, t \in m \bullet b, w \in m \bullet c\} \\ &= \{(p + s, (q + t)I_1, (r + w)I_2) : p + s \in k \bullet a + m \bullet a, q + t \in k \bullet b + m \bullet b, \\ &\quad r + w \in k \bullet c + m \bullet c\}. \end{aligned}$$

And,

$$\begin{aligned} (k + m) \bullet u &= \{(x, yI_1, zI_2) : x \in (k + m) \bullet a, y \in (k + m) \bullet b, z \in (k + m) \bullet c\} \\ &= \{(x, yI_1, zI_2) : x \in k \bullet a + m \bullet a, y \in k \bullet b + m \bullet b, z \in k \bullet c + m \bullet c\} \\ &= k \bullet u + m \bullet u. \end{aligned}$$

$$\begin{aligned} (2) \quad k \bullet u + k \bullet v &= \{(p, qI_1, rI_2) : p \in k \bullet a, q \in k \bullet b, r \in k \bullet c\} + \\ &\quad \{(s, tI_1, wI_2) : s \in k \bullet d, t \in k \bullet e, w \in k \bullet f\} \\ &= \{(p + s, (q + t)I_1, (r + w)I_2) : p + s \in k \bullet a + k \bullet d, q + t \in k \bullet b + k \bullet e, \\ &\quad r + w \in k \bullet c + k \bullet f\}. \end{aligned}$$

And,

$$\begin{aligned} k \bullet (u + v) &= k \bullet (a + d, (b + e)I_1, (c + f)I_2) \\ &= \{(x, yI_1, zI_2) : x \in k \bullet (a + d), y \in k \bullet (b + e), z \in k \bullet (c + f)\} \\ &= \{(x, yI_1, zI_2) : x \in k \bullet a + k \bullet d, y \in k \bullet b + k \bullet e, z \in k \bullet c + k \bullet f\} \\ &= k \bullet u + k \bullet v. \end{aligned}$$

$$\begin{aligned} (3) \quad k \bullet (m \bullet u) &= k \bullet \{(x, yI_1, zI_2) : x \in m \bullet a, y \in m \bullet b, z \in m \bullet c\} \\ &= \{(p, qI_1, rI_2) : p \in k \bullet x, q \in k \bullet y, r \in k \bullet z\} \\ &= \{(p, qI_1, rI_2) : p \in k \bullet (m \bullet a), q \in k \bullet (m \bullet b), r \in k \bullet (m \bullet c)\} \\ &= \{(p, qI_1, rI_2) : p \in (km) \bullet a, q \in (km) \bullet b, r \in (km) \bullet c\} \\ &= (km) \bullet (a, bI_1, cI_2) \\ &= (km) \bullet u. \end{aligned}$$

$$\begin{aligned} (4) \quad (-k) \bullet u &= \{(x, yI_1, zI_2) : x \in (-k) \bullet a, y \in (-k) \bullet b, z \in (-k) \bullet c\} \\ &= \{(x, yI_1, cI_2) : x \in k \bullet (-a), y \in k \bullet (-b), z \in k \bullet (-c)\} \\ &= k \bullet (-a, -bI_1, -cI_2) \\ &= k \bullet (-u). \end{aligned}$$

$$\begin{aligned} (5) \quad 1 \bullet u &= \{(x, yI_1, zI_2) : x \in 1 \bullet a, y \in 1 \bullet b, z \in 1 \bullet c\} \\ &= \{(x, yI_1, zI_2) : x \in \{a\}, y \in \{b\}, z \in \{c\}\} \\ &= \{(a, bI_1, cI_2)\}. \\ &\implies u \in 1 \bullet u. \end{aligned}$$

Accordingly, $V(I_1, I_2)$ is a strongly distributive hypervector space. □

Corollary 2.7. Every weak refined neutrosophic hypervector space which is strongly right distributive is strongly left distributive .

Proof. The proof follows from the proof of Proposition 2.6 . □

Proposition 2.8. Let $(V_1(I_1, I_2), +_1, \bullet_1, K(I_1, I_2))$ and $(V_2(I_1, I_2), +_2, \bullet_2, K(I_1, I_2))$ be two strong refined neutrosophic hypervector spaces over a refined neutrosophic field $K(I_1, I_2)$. Let

$V_1(I_1, I_2) \times V_2(I_1, I_2) = \{(a_1, b_1I_1, c_1I_2), (a_2, b_2I_1, c_2I_2) : (a_1, b_1I_1, c_1I_2) \in V_1(I_1, I_2), (a_2, b_2I_1, c_2I_2) \in V_2(I_1, I_2)\}$ and for all $u = ((a_1, b_1I_1, c_1I_2), (a_2, b_2I_1, c_2I_2)), v = ((a'_1, b'_1I_1, c'_1I_2), (a'_2, b'_2I_2, c'_2I_2)) \in V_1(I_1, I_2) \times V_2(I_1, I_2)$ and $\alpha = (k, mI_1, nI_2) \in K(I_1, I_2)$, define:

$$u + v = (((a_1 + a'_1), (b_1 + b'_1)I_1, (c_1 + c'_1)I_2), ((a_2 + a'_2), (b_2 + b'_2)I_1, (c_2 + c'_2)I_2)),$$

$$\alpha \bullet u = \{((x, yI_1, zI_2), (p, qI_1, rI_2))\}.$$

$$x \in k \bullet a_1,$$

$$y \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1 \cup n \bullet b_1,$$

$$z \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1,$$

$p \in k \bullet a_2,$
 $q \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2 \cup m \bullet c_2 \cup n \bullet b_2,$
 $z \in k \bullet c_2 \cup n \bullet a_2 \cup n \bullet c_2.$

Then $(V_1(I_1, I_2) \times V_2(I_1, I_2), +, \bullet, K(I_1, I_2))$ is a strong neutrosophic hypervector space.

Proof. Suppose that $V_1(I_1, I_2)$ and $V_2(I_1, I_2)$ are strong refined neutrosophic hypervector spaces over a refined neutrosophic field $K(I_1, I_2)$.

Let $u = ((a_1, b_1 I_1, c_1 I_2), (a_2, b_2 I_1, c_2 I_2)), v = ((a'_1, b'_1 I_1, c'_1 I_2), (a'_2, b'_2 I_2, c'_2 I_2)) \in V_1(I_1, I_2) \times V_2(I_1, I_2)$ and $\alpha = (k, m I_1, n I_2), \beta = (k', m' I_1, n' I_2) \in K(I_1, I_2)$ be arbitrary.

1. We can easily show that $(V_1(I_1, I_2) \times V_2(I_1, I_2), +)$ is an abelian group.
2. Now we want to show that $(\alpha + \beta) \bullet u \subseteq \alpha \bullet u + \beta \bullet u.$

Consider

$$\begin{aligned} (\alpha + \beta) \bullet u &= (k + k', (m + m') I_1, (n + n') I_2) \bullet ((a_1, b_1 I_1, c_1 I_2), (a_2, b_2 I_1, c_2 I_2)) \\ &\subseteq \{((x, y I_1, z I_2), (p, q I_1, r I_2)) : x \in (k + k') \bullet a_1, y \in (k + k') \bullet b_1 \cup (m + m') \bullet a_1 \cup (m + m') \bullet b_1 \cup (m + m') \bullet c_1 \cup (n + n') \bullet b_1, z \in (k + k') \bullet c_1 \cup (n + n') \bullet a_1 \cup (n + n') \bullet c_1, \\ &p \in (k + k') \bullet a_2, q \in (k + k') \bullet b_2 \cup (m + m') \bullet a_2 \cup (m + m') \bullet b_2 \cup (m + m') \bullet c_2 \cup (n + n') \bullet b_2, z \in (k + k') \bullet c_2 \cup (n + n') \bullet a_2 \cup (n + n') \bullet c_2\} \\ &= \{((x, y I_1, z I_2), (p, q I_1, r I_2)) : x \in k \bullet a_1 + k' \bullet a_1, y \in k \bullet b_1 + k' \bullet b_1 \cup m \bullet a_1 + m' \bullet a_1 \cup m \bullet b_1 + m' \bullet b_1 \cup m \bullet c_1 + m' \bullet c_1 \cup n \bullet b_1 + n' \bullet b_1, z \in k \bullet c_1 + k' \bullet c_1 \cup n \bullet a_1 + n' \bullet a_1 \cup n \bullet c_1 + n' \bullet c_1, \\ &p \in k \bullet a_2 + k' \bullet a_2, q \in k \bullet b_2 + k' \bullet b_2 \cup m \bullet a_2 + m' \bullet a_2 \cup m \bullet b_2 + m' \bullet b_2 \cup m \bullet c_2 + m' \bullet c_2 \cup n \bullet b_2 + n' \bullet b_2, z \in k \bullet c_2 + k' \bullet c_2 \cup n \bullet a_2 + n' \bullet a_2 \cup n \bullet c_2 + n' \bullet c_2\}. \end{aligned}$$

Now if we take $x = s_1 + s'_1, y = t_1 + t'_1, z = w_1 + w'_1, p = s_2 + s'_2, q = t_2 + t'_2$ and $r = w_1 + w'_2$ then we have

$$\begin{aligned} &\{((s_1 + s'_1, (t_1 + t'_1) I_1, (w_1 + w'_1) I_2), (s_2 + s'_2, (t_2 + t'_2) I_1, (w_1 + w'_1) I_2)) : s_1 + s'_1 \in k \bullet a_1 + k' \bullet a_1, t_1 + t'_1 \in k \bullet b_1 + k' \bullet b_1 \cup m \bullet a_1 + m' \bullet a_1 \cup m \bullet b_1 + m' \bullet b_1 \cup m \bullet c_1 + m' \bullet c_1 \cup n \bullet b_1 + n' \bullet b_1, w_1 + w'_1 \in k \bullet c_1 + k' \bullet c_1 \cup n \bullet a_1 + n' \bullet a_1 \cup n \bullet c_1 + n' \bullet c_1, \\ &s_2 + s'_2 \in k \bullet a_2 + k' \bullet a_2, t_2 + t'_2 \in k \bullet b_2 + k' \bullet b_2 \cup m \bullet a_2 + m' \bullet a_2 \cup m \bullet b_2 + m' \bullet b_2 \cup m \bullet c_2 + m' \bullet c_2 \cup n \bullet b_2 + n' \bullet b_2, w_2 + w'_2 \in k \bullet c_2 + k' \bullet c_2 \cup n \bullet a_2 + n' \bullet a_2 \cup n \bullet c_2 + n' \bullet c_2\} = \{((s_1, t_1 I_1, w_1 I_2), (s_2, t_2 I_1, w_2 I_2)) : \\ &s_1 \in k \bullet a_1, t_1 \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1 \cup n \bullet b_1, w_1 \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1, s_2 \in k \bullet a_2, t_2 \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2 \cup m \bullet c_2 \cup n \bullet b_2, w_2 \in k \bullet c_2 \cup n \bullet a_2 \cup n \bullet c_2\} + \{((s'_1, t'_1 I_1, w'_1 I_2), (s'_2, t'_2 I_1, w'_2 I_2)) : \\ &s'_1 \in k' \bullet a_1, t'_1 \in k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1, w'_1 \in k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1, \\ &s'_2 \in k' \bullet a_2, t'_2 \in k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2, w'_2 \in k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2\} \\ &\subseteq \alpha \bullet u + \beta \bullet u. \end{aligned}$$

Then $(\alpha + \beta) \bullet u \subseteq \alpha \bullet u + \beta \bullet u.$

3. Now we want to show that $\alpha \bullet (u + v) \subseteq \alpha \bullet u + \alpha \bullet v$

$$\begin{aligned} \alpha \bullet (u + v) &= (k, m I_1, n I_2) \bullet ((a_1 + a'_1, (b_1 + b'_1) I_1, (c_1 + c'_1) I_2), (a_2 + a'_2, (b_2 + b'_2) I_1, (c_2 + c'_2) I_2)) \\ &\subseteq \{(x, y I_1, z I_2), (p, q I_1, r I_2) : x \in k \bullet (a_1 + a'_1), y \in k \bullet (b_1 + b'_1) \cup m \bullet (a_1 + a'_1) \cup m \bullet (b_1 + b'_1) \cup m \bullet (c_1 + c'_1) \cup n \bullet (b_1 + b'_1), z \in k \bullet (c_1 + c'_1) \cup n \bullet (a_1 + a'_1) \cup n \bullet (c_1 + c'_1), \\ &p \in k \bullet (a_2 + a'_2), q \in k \bullet (b_2 + b'_2) \cup m \bullet (a_2 + a'_2) \cup m \bullet (b_2 + b'_2) \cup m \bullet (c_2 + c'_2) \cup n \bullet (b_2 + b'_2), \\ &r \in k \bullet (c_2 + c'_2) \cup n \bullet (a_2 + a'_2) \cup n \bullet (c_2 + c'_2)\} \\ &= \{(x, y I_1, z I_2), (p, q I_1, r I_2) : x \in k \bullet a_1 + k \bullet a'_1, y \in k \bullet b_1 + k \bullet b'_1 \cup m \bullet a_1 + m \bullet a'_1 \cup m \bullet b_1 + m \bullet b'_1 \cup m \bullet c_1 + m \bullet c'_1 \cup n \bullet b_1 + n \bullet b'_1, z \in k \bullet c_1 + k \bullet c'_1 \cup n \bullet a_1 + n \bullet a'_1 \cup n \bullet c_1 + n \bullet c'_1, \\ &p \in k \bullet a_2 + k \bullet a'_2, q \in k \bullet b_2 + k \bullet b'_2 \cup m \bullet a_2 + m \bullet a'_2 \cup m \bullet b_2 + m \bullet b'_2 \cup m \bullet c_2 + m \bullet c'_2 \cup n \bullet b_2 + n \bullet b'_2, r \in k \bullet c_2 + k \bullet c'_2 \cup n \bullet a_2 + n \bullet a'_2 \cup n \bullet c_2 + n \bullet c'_2\}. \end{aligned}$$

If we take $x = s_1 + s'_1, y = t_1 + t'_1, z = w_1 + w'_1, p = s_2 + s'_2, q = t_2 + t'_2$ and $r = w_1 + w'_2$ then we have

$$\begin{aligned} &\{(s_1 + s'_1, (t_1 + t'_1) I_1, (w_1 + w'_1) I_2), (s_2 + s'_2, (t_2 + t'_2) I_1, (w_2 + w'_2) I_2) : s_1 + s'_1 \in k \bullet a_1 + k \bullet a'_1, t_1 + t'_1 \in k \bullet b_1 + k \bullet b'_1 \cup m \bullet a_1 + m \bullet a'_1 \cup m \bullet b_1 + m \bullet b'_1 \cup m \bullet c_1 + m \bullet c'_1 \cup n \bullet b_1 + n \bullet b'_1, w_1 + w'_1 \in k \bullet c_1 + k \bullet c'_1 \cup n \bullet a_1 + n \bullet a'_1 \cup n \bullet c_1 + n \bullet c'_1, \\ &s_2 + s'_2 \in k \bullet a_2 + k \bullet a'_2, t_2 + t'_2 \in k \bullet b_2 + k \bullet b'_2 \cup m \bullet a_2 + m \bullet a'_2 \cup m \bullet b_2 + m \bullet b'_2 \cup m \bullet c_2 + m \bullet c'_2 \cup n \bullet b_2 + n \bullet b'_2, w_2 + w'_2 \in k \bullet c_2 + k \bullet c'_2 \cup n \bullet a_2 + n \bullet a'_2 \cup n \bullet c_2 + n \bullet c'_2\} \\ &= \{(s_1, t_1 I_1, w_1 I_2), (s_2, t_2 I_1, w_2 I_2) : s_1 \in k \bullet a_1, t_1 \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1 \cup n \bullet b_1, w_1 \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1, s_2 \in k \bullet a_2, t_2 \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2 \cup m \bullet c_2 \cup n \bullet b_2, w_2 \in k \bullet c_2 \cup n \bullet a_2 \cup n \bullet c_2\} + \{(s'_1, t'_1 I_1, w'_1 I_2), (s'_2, t'_2 I_1, w'_2 I_2) : s'_1 \in k \bullet a'_1, t'_1 \in k \bullet b'_1 \cup m \bullet a'_1 \cup m \bullet b'_1 \cup m \bullet c'_1 \cup n \bullet b'_1, w'_1 \in k \bullet c'_1 \cup n \bullet a'_1 \cup n \bullet c'_1, s'_2 \in k \bullet a'_2, t'_2 \in k \bullet b'_2 \cup m \bullet a'_2 \cup m \bullet b'_2 \cup m \bullet c'_2 \cup n \bullet b'_2, w'_2 \in k \bullet c'_2 \cup n \bullet a'_2 \cup n \bullet c'_2\} \subseteq \alpha \bullet u + \alpha \bullet v. \end{aligned}$$

Then we have that $\alpha \bullet (u + v) \subseteq \alpha \bullet u + \alpha \bullet v.$

4. $\alpha \bullet (\beta \bullet u) = \alpha \bullet \{((x, yI_1, zI_2), (p, qI_1, rI_2)) : x \in k' \bullet a_1, y \in k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1, z \in k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1, p \in k' \bullet a_2, q \in k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2, r \in k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2\}$
 $= \{((x', y'I_1, z'I_2), (p', q'I_1, r'I_2)) : x' \in k \bullet x, y' \in k \bullet y \cup m \bullet x \cup m \bullet y \cup m \bullet z \cup n \bullet y, z' \in k \bullet z \cup n \bullet x \cup n \bullet z, p' \in k \bullet p, q' \in k \bullet q \cup m \bullet p \cup m \bullet q \cup m \bullet r \cup n \bullet q, z' \in k \bullet r \cup n \bullet p \cup n \bullet r\}$
 $= \{((x', y'I_1, z'I_2), (p', q'I_1, r'I_2)) : x' \in k \bullet (k' \bullet a_1), y' \in k \bullet (k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1) \cup m \bullet (k' \bullet a_1) \cup m \bullet (k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1) \cup m \bullet (k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1) \cup n \bullet (k' \bullet b_1 \cup m' \bullet a_1 \cup m' \bullet b_1 \cup m' \bullet c_1 \cup n' \bullet b_1), z' \in k \bullet (k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1) \cup n \bullet (k' \bullet a_1) \cup n \bullet (k' \bullet c_1 \cup n' \bullet a_1 \cup n' \bullet c_1), p' \in k \bullet (k' \bullet a_2), q' \in k \bullet (k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2) \cup m \bullet (k' \bullet a_2) \cup m \bullet (k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2) \cup m \bullet (k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2) \cup n \bullet (k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2 \cup m' \bullet c_2 \cup n' \bullet b_2), z' \in k \bullet (k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2) \cup n \bullet (k' \bullet a_2) \cup n \bullet (k' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet c_2)\}$
 $= \{((x', y'I_1, z'I_2), (p', q'I_1, r'I_2)) : x' \in (kk') \bullet a_1, y' \in (kk') \bullet b_1 \cup (km') \bullet a_1 \cup (km') \bullet b_1 \cup (km') \bullet c_1 \cup (kn') \bullet b_1 \cup (mk') \bullet a_1 \cup (mk') \bullet b_1 \cup (mm') \bullet a_1 \cup (mm') \bullet b_1 \cup (mm') \bullet c_1 \cup (mn') \bullet b_1 \cup (mk') \bullet c_1 \cup (mn') \bullet a_1 \cup (mn') \bullet c_1 \cup (nk') \bullet b_1 \cup (nm') \bullet a_1 \cup (nm') \bullet b_1 \cup (nm') \bullet c_1 \cup (nn') \bullet b_1, z' \in (kk') \bullet c_1 \cup (kn') \bullet a_1 \cup (kn') \bullet c_1 \cup (nk') \bullet a_1 \cup (nk') \bullet c_1 \cup (nn') \bullet a_1 \cup (nn') \bullet c_1, p' \in (kk') \bullet a_2, q' \in (kk') \bullet b_2 \cup (km') \bullet a_2 \cup (km') \bullet b_2 \cup (km') \bullet c_2 \cup (kn') \bullet b_2 \cup (mk') \bullet a_2 \cup (mk') \bullet b_2 \cup (mm') \bullet a_2 \cup (mm') \bullet b_2 \cup (mm') \bullet c_2 \cup (mn') \bullet b_2 \cup (mk') \bullet c_2 \cup (mn') \bullet a_2 \cup (mn') \bullet b_2 \cup (nm') \bullet a_2 \cup (nm') \bullet b_2 \cup (nm') \bullet c_2 \cup (nn') \bullet b_2, z' \in (kk') \bullet c_2 \cup (kn') \bullet a_2 \cup (kn') \bullet c_2 \cup (nk') \bullet a_2 \cup (nk') \bullet c_2 \cup (nn') \bullet a_2 \cup (nn') \bullet c_2\}$
 $= ((k, mI_1, nI_2)(k', m'I_1, n'I_2)) \bullet ((a_1, b_1I_1, c_1I_2)(a_2, b_2I_1, c_2I_2))$
 $= (\alpha\beta) \bullet u.$
5. $(-\alpha)u = \{(x, yI_1, zI_2)(p, qI_1, rI_2) : x \in -k \bullet a_1, y \in -k \bullet b_1 \cup -m \bullet a_1 \cup -m \bullet b_1 \cup -m \bullet c_1 \cup -n \bullet b_1, z \in -k \bullet c_1 \cup -n \bullet a_1 \cup -n \bullet c_1, p \in -k \bullet a_2, q \in -k \bullet b_2 \cup -m \bullet a_2 \cup -m \bullet b_2 \cup -m \bullet c_2 \cup -n \bullet b_2, r \in -k \bullet c_2 \cup -n \bullet a_2 \cup -n \bullet c_2\}$
 $= \{(x, yI_1, zI_2)(p, qI_1, rI_2) : x \in k \bullet (-a_1), y \in k \bullet (-b_1) \cup m \bullet (-a_1) \cup m \bullet (-b_1) \cup m \bullet (-c_1) \cup n \bullet (-b_1), z \in k \bullet (-c_1) \cup n \bullet (-a_1) \cup n \bullet (-c_1), p \in k \bullet (-a_2), q \in k \bullet (-b_2) \cup m \bullet (-a_2) \cup m \bullet (-b_2) \cup m \bullet (-c_2) \cup n \bullet (-b_2), r \in k \bullet (-c_2) \cup n \bullet (-a_2) \cup n \bullet (-c_2)\}$
 $= (k, mI_1, nI_2)(-(a_1, b_1I_1, c_1I_2)(a_2, b_2I_1, c_2I_2)) = \alpha(-u).$
6. $1 \bullet u = \{(x, yI_1, zI_2)(p, qI_1, rI_2) : x \in 1 \bullet a_1, y \in 1 \bullet b_1, z \in 1 \bullet c_1, p \in 1 \bullet a_2, q \in 1 \bullet b_2, r \in 1 \bullet c_2\}$
 $= \{(a_1, b_1I_1, c_1I_2)(a_2, b_2I_1, c_2I_2) : a_1 \in 1 \bullet a_1, b_1 \in 1 \bullet b_1, c_1 \in 1 \bullet c_1, a_2 \in 1 \bullet a_2, b_2 \in 1 \bullet b_2, c_2 \in 1 \bullet c_2\}$, which shows that $u \in 1 \bullet u$.
 Accordingly, $V_1(I_1, I_2) \times V_2(I_1, I_2)$ is a strong refined neutrosophic hypervector space. □

Proposition 2.9. Let $(V(I_1, I_2), \oplus, \bullet, K)$ and $(H, +_H, \bullet_H, K)$ be a weak refined neutrosophic hypervector spaces and a hypervector space, respectively. Let

$V(I_1, I_2) \times H = \{((a, bI_1, cI_2), h) : (a, bI_1, cI_2) \in V_1(I_1, I_2), h \in H\}$.
 For all $u = ((a, bI_1, cI_2), h), v = ((a', b'I_1, c'I_2), g) \in V(I_1, I_2) \times H$ and $k \in K$, define:
 $u + v = ((a \oplus a'), (b \oplus b')I_1, (c \oplus c')I_2), h +_H g,$
 $k \bullet u = \{((x, yI_1, zI_2), p) : x \in k \bullet a, y \in k \bullet b \in k \bullet c, p \in k \bullet_H h\}.$
 Then $(V_1(I_1, I_2) \times H, +, \bullet, K)$ is a weak neutrosophic hypervector space.

Proof. The proof follows from the same pattern as the proof of Proposition 2.8 . □

Definition 2.10. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let $W[I_1, I_2]$ be a nonempty subset of $V(I_1, I_2)$. $W[I_1, I_2]$ is said to be a subhypervector space of $V(I_1, I_2)$ if $(W[I_1, I_2], +, \bullet, K(I_1, I_2))$ is also a refined neutrosophic hypervector space over the refined neutrosophic field $K(I_1, I_2)$. It is essential that $W[I_1, I_2]$ contains a proper subset which is a hypervector space over a field K .

Example 2.11. Let $V(I_1, I_2) = \mathbb{R}^2(I_1, I_2)$ and $K = \mathbb{R}(I_1, I_2)$ then $(\mathbb{R}^2(I_1, I_2), +, \bullet, K(I_1, I_2))$ is a strong refined neutrosophic hypervector space over refined neutrosophic field $K = \mathbb{R}(I_1, I_2)$, where the hyperoperations $+$ and \bullet are defined $\forall u = ((a_1, b_1I_1, c_1I_1), (a_2, b_2I_1, c_2I_2)),$
 $v = ((a'_1, b'_1I_1, c'_1I_1), (a'_2, b'_2I_1, c'_2I_2)) \in V(I_1, I_2)$ by :
 $u + v = ((a_1 + a'_1, (b_1 + b'_1)I_1, (c_1 + c'_1)I_2), (a_2 + a'_2, (b_2 + b'_2)I_1, (c_2 + c'_2)I_2)),$

$\alpha \bullet u = \{((x, yI_1, zI_2), (p, qI_1, rI_2)) : x \in k \bullet a_1, y \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1 \cup n \bullet b_1,$
 $z \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1, p \in k \bullet a_2, q \in k \bullet b_2 \cup m \bullet a_2 \cup m \bullet b_2 \cup m \bullet c_2 \cup n \bullet b_2,$
 $r \in k \bullet c_2 \cup n \bullet a_2 \cup n \bullet c_2\}.$

Let $W(I_1, I_2) = K(I_1, I_2) \times \{(0, 0I_1, 0I_2)\} \subseteq V(I_1, I_2).$

Then $W(I_1, I_2)$ is a strong refined neutrosophic subhypervector space.

Proof. Since $\theta = ((0, 0I_1, 0I_2), (0, 0I_1, 0I_2)) \in W(I_1, I_2).$ Then $W(I_1, I_2) \neq \emptyset.$

Now let $u_1 = ((a_1, b_1I_1, c_1I_2), (0, 0I_1, 0I_2)), v_1 = ((a'_1, b_1I_1, c_1I_2)(0, 0I_1, 0I_2)) \in W(I_1, I_2),$ and $\alpha = (k, mI_1, nI_2), \beta = (k', m'I_1, n'I_2) \in K(I_1, I_2)$ with $a_1, b_1, c_1, a'_1, b'_1, c'_1, k, m, n, k', m', n' \in \mathbb{R}.$

$$\begin{aligned} \alpha \bullet u_1 + \beta \bullet v_1 &= (k, mI_1, nI_2) \bullet [(a_1, b_1I_1, c_1I_2), (0, 0I_1, 0I_2)] + \\ &\quad (k', m'I_1, n'I_2) \bullet [(a'_1, b_1I_1, c_1I_2)(0, 0I_1, 0I_2)] \\ &\subseteq \{((x, yI_1, zI_2), (p, qI_1, rI_2)) : x \in k \bullet a_1, y \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup \\ &\quad m \bullet c_1 \cup n \bullet b_1, z \in k \bullet c_1 \cup n \bullet a_1 \cup n \bullet c_1, p \in k \bullet 0, \\ &\quad q \in k \bullet 0 \cup m \bullet 0 \cup m \bullet 0 \cup m \bullet 0 \cup n \bullet 0, r \in k \bullet 0 \cup n \bullet 0 \cup n \bullet 0\} \\ &\quad + \{((x', y'I_1, z'I_2), (p', q'I_1, r'I_2)) : x' \in k' \bullet a'_1, \\ &\quad y' \in k' \bullet b'_1 \cup m' \bullet a'_1 \cup m' \bullet b'_1 \cup m' \bullet c'_1 \cup n' \bullet b'_1, \\ &\quad z' \in k' \bullet c'_1 \cup n' \bullet a'_1 \cup n' \bullet c'_1, \\ &\quad p' \in k' \bullet 0, q' \in k' \bullet 0 \cup m' \bullet 0 \cup \\ &\quad m' \bullet 0 \cup m' \bullet 0 \cup n' \bullet 0, r' \in k' \bullet 0 \cup n' \bullet 0 \cup n' \bullet 0\} \\ &= \{((s, tI_1, wI_2), (s', t'I_1, w'I_2)) : s \in k \bullet a_1 + k' \bullet a'_1, \\ &\quad t \in k \bullet b_1 + k' \bullet b'_1 \cup m \bullet a_1 + m' \bullet a'_1 \cup m \bullet b_1 + m' \bullet b'_1 \cup m \bullet c_1 + \\ &\quad m' \bullet c'_1 \cup n \bullet b_1 + n' \bullet b'_1, \\ &\quad w \in k \bullet c_1 + k' \bullet c'_1 \cup n \bullet a_1 + n' \bullet a'_1 \cup n \bullet c_1 + n' \bullet c'_1, \\ &\quad s' \in 0, t' \in 0, w' \in 0\} \subseteq W(I_1, I_2). \\ &\implies \alpha \bullet u_1 + \beta \bullet v_1 \subseteq W(I_1, I_2). \end{aligned}$$

Lastly, we can see from the definition of $W(I_1, I_2)$ that $W(I_1, I_2)$ contains a proper subset which is a hypervector space over $K.$

To this end we can conclude that $W(I_1, I_2)$ is a strong refined neutrosophic hypervector space. □

Proposition 2.12. Let $W_1[I_1, I_2], W_2[I_1, I_2], \dots, W_n[I_1, I_2]$ be refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2).$ Then $\bigcap_{i=1}^n W_i[I_1, I_2]$ is a refined neutrosophic subhypervector space of $V(I_1, I_2).$

Proof. Consider the collection of refined neutrosophic subhypervector space

$\{W_i(I_1, I_2) : i = 1, 2, \dots, n\}$ of a strong refined neutrosophic hypervector space $V(I_1, I_2).$

Take $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2), \alpha = (k, pI_1, qI_2)$ and $\beta = (r, sI_1, tI_2).$

Let $u, v \in \bigcap_{i=1}^n W_i(I_1, I_2)$ then $u, v \in W_i(I_1, I_2)$ for all $i = 1, 2, \dots, n.$

Now for all scalars $\alpha, \beta \in K(I_1, I_2)$ we have that

$$\begin{aligned} \alpha \bullet u + \beta \bullet v &= (k, pI_1, qI_2) \bullet (a, bI_1, cI_2) + (r, sI_1, tI_2) \bullet (d, eI_1, fI_2) \\ &\subseteq \{(x, yI_1, zI_2) : x \in k \bullet a, y \in k \bullet b \cup p \bullet a \cup p \bullet b \cup p \bullet c \cup q \bullet b, z \in k \bullet c \cup q \bullet a \cup q \bullet c\} + \\ &\quad \{(x', y'I_1, z'I_2) : x' \in r \bullet d, y' \in r \bullet e \cup s \bullet d \cup s \bullet e \cup s \bullet f \cup t \bullet e, z' \in r \bullet f \cup t \bullet d \cup t \bullet f\} \\ &= \{(x + x', (y + y')I_1, (z + z')I_2) : x + x' \in k \bullet a + r \bullet d, \\ &\quad y + y' \in k \bullet b + r \bullet e \cup p \bullet a + s \bullet d \cup p \bullet b + s \bullet e \cup p \bullet c + s \bullet f \cup q \bullet b + t \bullet e, \\ &\quad z + z' \in k \bullet c + r \bullet f \cup q \bullet a + t \bullet d \cup q \bullet c + t \bullet f\} \subseteq W_i(I_1, I_2) \forall i = 1, 2, 3 \dots, n. \\ &\implies \alpha \bullet u + \beta \bullet v \subseteq \bigcap_{i=1}^n W_i(I_1, I_2). \end{aligned}$$

Lastly, since $W_i(I_1, I_2) \forall i = 1, 2, 3, \dots, n$ contain proper subsets W_i which are hypervector space,

$\bigcap_{i=1}^n W_i(I_1, I_2)$ is a strong refined neutrosophic subhyperspace. □

Proposition 2.13. Let $W[I_1, I_2]$ be a subset of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2).$ Then $W[I_1, I_2]$ is a refined neutrosophic subhypervector space of $V(I_1, I_2)$ if and only if for all

$u = (a, bI_1, cI_2), v = (d, eI_1, I_2) \in V(I_1, I_2)$ and $\alpha = (k, mI_1, nI_2) \in K(I_1, I_2),$ the following conditions hold:

1. $W[I_1, I_2] \neq \emptyset,$
2. $u + v \in W[I_1, I_2],$
3. $\alpha \bullet u \subseteq W[I_1, I_2],$
4. $W[I_1, I_2]$ contains a proper subset which is a hypervector space over $K.$

Proposition 2.14. Let $V(I_1, I_2)$ be a strong refined neutrosophic hypervector space over $K(I_1, I_2)$ and let $U_1(I_1, I_2), U_2(I_1, I_2)$ be any strong refined neutrosophic subhypervector spaces of $V(I_1, I_2)$. Then $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space if and only if $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$.

Proof. Let $U_1(I_1, I_2)$ and $U_2(I_1, I_2)$ be any strong refined neutrosophic subhypervector spaces of $V(I_1, I_2)$. \implies Now, suppose $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$ then we shall show the $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$.

Without loss of generality, suppose that $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$.

Then we have that $U_1(I_1, I_2) \cup U_2(I_1, I_2) = U_2(I_1, I_2)$. But $U_2(I_1, I_2)$ is defined to be a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$, so $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$.

\Leftarrow We want to show that if $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$ then either $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$.

Now suppose that $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space of $V(I_1, I_2)$ and suppose by contradiction that $U_1(I_1, I_2) \not\subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \not\supseteq U_2(I_1, I_2)$.

Thus there exist elements $x_1 = (a_1 + b_1I_1 + c_1I_2) \in U_1(I_1, I_2) \setminus U_2(I_1, I_2)$ and $x_2 = (a_2 + b_2I_1 + c_2I_2) \in U_2(I_1, I_2) \setminus U_1(I_1, I_2)$. So we have that $x_1, x_2 \in U_1(I_1, I_2) \cup U_2(I_1, I_2)$, since $U_1(I_1, I_2) \cup U_2(I_1, I_2)$ is a strong refined neutrosophic subhypervector space, we must have that $x_1 + x_2 = x_3 \in U_1(I_1, I_2) \cup U_2(I_1, I_2)$.

Therefore $x_1 + x_2 = x_3 \in U_1(I_1, I_2)$ or $x_1 + x_2 = x_3 \in U_2(I_1, I_2)$

$\implies x_2 = x_3 - x_1 \in U_1(I_1, I_2)$ or $x_1 = x_3 - x_2 \in U_2(I_1, I_2)$ which is a contradiction.

Hence $U_1(I_1, I_2) \subseteq U_2(I_1, I_2)$ or $U_1(I_1, I_2) \supseteq U_2(I_1, I_2)$ as required. \square

Remark 2.15. If $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ are refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$, then generally, $W_1[I_1, I_2] \cup W_2[I_1, I_2]$ is not a refined neutrosophic subhypervector space of $V(I_1, I_2)$ except if $W_1[I_1, I_2] \subseteq W_2[I_1, I_2]$ or $W_2[I_1, I_2] \subseteq W_1[I_1, I_2]$.

Definition 2.16. Let $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ be two refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. The sum of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ denoted by $W_1[I_1, I_2] + W_2[I_1, I_2]$ is defined by the set

$$\bigcup \{w + x : w = (a, bI_1, cI_2) \in W_1[I_1, I_2], x = (d, eI_1, fI_2) \in W_2[I_1, I_2]\}.$$

If $W_1[I_1, I_2] \cap W_2[I_1, I_2] = \{\theta\}$, then the sum of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ is denoted by $W_1[I_1, I_2] \oplus W_2[I_1, I_2]$ and it is called the direct sum of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$.

Proposition 2.17. Let $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ be two refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$.

1. $W_1[I_1, I_2] + W_2[I_1, I_2]$ is a refined neutrosophic subhypervector space of $V(I_1, I_2)$.
2. $W_1[I_1, I_2] + W_2[I_1, I_2]$ is the least refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$.

Proof. 1. Since $\theta \in W_1[I_1, I_2]$ and $\theta \in W_2[I_1, I_2]$, $\{\theta + \theta\} \subseteq W_1[I_1, I_2] + W_2[I_1, I_2]$.

So, $\{\theta\} \subseteq W_1[I_1, I_2] + W_2[I_1, I_2] \implies \theta \in W_1[I_1, I_2] + W_2[I_1, I_2]$, therefore $W_1[I_1, I_2] + W_2[I_1, I_2]$ is non-empty.

Let $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in W_1[I_1, I_2] + W_2[I_1, I_2]$, then $\exists u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2) \in W_1[I_1, I_2]$ and $v_1 = (d_1, e_1I_1, f_1I_2), v_2 = (d_2, e_2I_1, f_2I_2) \in W_2[I_1, I_2]$ such that $u \in u_1 + v_1$ and $v \in u_2 + v_2$.

Let $\alpha = (k, mI_1, nI_2), \beta = (k', m'I_1, n'I_2) \in K(I_1, I_2)$.

$$\begin{aligned} & \text{Now } \alpha \bullet u + \beta \bullet v \subseteq \alpha \bullet (u_1 + v_1) + \beta \bullet (u_2 + v_2) \\ &= (k, mI_1, nI_2) \bullet (a_1 + d_1, (b_1 + e_1)I_1, (c_1 + e_1)I_2) + \\ & \quad (k', m'I_1, n'I_2) \bullet (a_2 + d_2, (b_2 + e_2)I_1, (c_2 + e_2)I_2) \\ &\subseteq \{(x_1, y_1I_1, z_1I_2) : x_1 \in k \bullet (a_1 + d_1), \\ & \quad y_1 \in k \bullet (b_1 + e_1) \cup m \bullet (a_1 + d_1) \cup m \bullet (b_1 + e_1) \cup m \bullet (c_1 + f_1) \cup n \bullet (b_1 + e_1), \\ & \quad z_1 \in k \bullet (c_1 + f_1) \cup n \bullet (a_1 + d_1) \cup n \bullet (c_1 + f_1)\} \\ & \quad + \{(x_2, y_2I_1, z_2I_2) : x_2 \in k' \bullet (a_2 + d_2), \\ & \quad y_2 \in k' \bullet (b_2 + e_2) \cup m' \bullet (a_2 + d_2) \cup m' \bullet (b_2 + e_2) \cup m' \bullet (c_2 + f_2) \cup n' \bullet (b_2 + e_2), \\ & \quad z_2 \in k' \bullet (c_2 + f_2) \cup n' \bullet (a_2 + d_2) \cup n' \bullet (c_2 + f_2)\} \end{aligned}$$

$$\begin{aligned}
 &= \{(x, yI_1, zI_2) : x \in (k \bullet a_1 + k \bullet d_1 + k' \bullet a_2 + k' \bullet d_2), \\
 &\quad y \in (k \bullet b_1 + k \bullet e_1 + k' \bullet b_2 + k' \bullet e_2) \cup (m \bullet a_1 + m \bullet d_1 + m' \bullet a_2 + m' \bullet d_2) \cup \\
 &\quad (m \bullet b_1 + m \bullet e_1 + m' \bullet b_2 + m' \bullet e_2) \cup (m \bullet c_1 + m \bullet f_1 + m' \bullet c_2 + m' \bullet f_2) \cup \\
 &\quad (n \bullet b_1 + n \bullet e_1 + n' \bullet b_2 + n' \bullet e_2), \\
 &\quad z \in (k \bullet c_1 + k \bullet f_1 + k' \bullet c_2 + k' \bullet f_2) \cup (n \bullet a_1 + n \bullet d_1 + n' \bullet a_2 + n' \bullet d_2) \cup \\
 &\quad (n \bullet c_1 + n \bullet f_1 + n' \bullet c_2 + n' \bullet f_2)\} \\
 &= \{(s_1, t_1I_1, w_1I_2) : s_1 \in (k \bullet a_1 + k' \bullet a_2), \\
 &\quad t_1 \in (k \bullet b_1 + k' \bullet b_2) \cup (m \bullet a_1 + m' \bullet a_2) \cup (m \bullet b_1 + m' \bullet b_2) \cup (m \bullet c_1 + m' \bullet c_2) \cup \\
 &\quad (n \bullet b_1 + n' \bullet b_2), w_1 \in (k \bullet c_1 + k' \bullet c_2) \cup (n \bullet a_1 + n' \bullet a_2) \cup (n \bullet c_1 + n' \bullet c_2)\} + \\
 &\quad \{(s_2, t_2I_1, w_2I_2) : s_2 \in (k \bullet d_1 + k' \bullet d_2), \\
 &\quad t_2 \in (k \bullet e_1 + k' \bullet e_2) \cup (m \bullet d_1 + m' \bullet d_2) \cup (m \bullet e_1 + m' \bullet e_2) \cup (m \bullet f_1 + m' \bullet f_2) \cup \\
 &\quad (n \bullet e_1 + n' \bullet e_2), \\
 &\quad w_2 \in (k \bullet f_1 + k' \bullet f_2) \cup (n \bullet d_1 + n' \bullet d_2) \cup (n \bullet f_1 + n' \bullet f_2)\} \\
 &\subseteq W_1[I_1, I_2] + W_2[I_1, I_2].
 \end{aligned}$$

Hence $\alpha \bullet u + \beta \bullet v \subseteq W_1[I_1, I_2] + W_2[I_1, I_2]$.

Now since W_1, W_2 are proper subsets of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ respectively, with both W_1 and W_2 being hypervector space. Then $W_1 + W_2$ is a hypervector space which is properly contained in $W_1[I_1, I_2] + W_2[I_1, I_2]$. Then we can conclude that $W_1[I_1, I_2] + W_2[I_1, I_2]$ is a refined neutrosophic subhypervector space.

2. Let $W[I_1, I_2]$ be refined neutrosophic subhypervector space of $V[I_1, I_2]$ such that $W_1[I_1, I_2] \subseteq W[I_1, I_2]$ and $W_2[I_1, I_2] \subseteq W[I_1, I_2]$.

Let $u = (a, bI_1, cI_2) \in W_1[I_1, I_2] + W_2[I_1, I_2]$, then $\exists u_1 = (a_1, b_1I_1, c_1I_2) \in W_1[I_1, I_2]$ and $u_2 = (a_2, b_2I_1, c_2I_2) \in W_2[I_1, I_2]$ such that $u \in u_1 + u_2$.

Since $W_1[I_1, I_2] \subseteq W[I_1, I_2]$ and $W_2[I_1, I_2] \subseteq W[I_1, I_2]$, then $u_1, u_2 \in W[I_1, I_2]$.

Again since $W[I_1, I_2]$ is a refined neutrosophic subhypervector space of $V[I_1, I_2]$, then we have that $u_1 + u_2 \subseteq W[I_1, I_2] \implies u \in W[I_1, I_2]$.

Hence $W_1[I_1, I_2] + W_2[I_1, I_2] \subseteq W[I_1, I_2]$ and the proof follows. □

Remark 2.18. If $V(I_1, I_2)$ is a weak refined neutrosophic strongly left distributive hypervector space over a field K , then

1. $W[I_1, I_2] = \bigcup\{k \bullet u : k \in K\}$ forms a weak refined neutrosophic subhypervector space of $V(I_1, I_2)$, where $u = (a, bI_1, cI_2) \in V(I_1, I_2)$. This refined neutrosophic subhypervector space is said to be generated by the refined neutrosophic vector u and it is called a refined neutrosophic hyperline span by the refined neutrosophic vector u .
2. If $u = (a, bI_1, cI_2), v = (d, eI_1, fI_2) \in V(I_1, I_2)$, then the set $W = \bigcup\{\alpha \bullet u + \beta \bullet v, \alpha, \beta \in K\}$ is a weak refined neutrosophic subhypervector space of $V(I_1, I_2)$. This refined neutrosophic subhypervector space is called refined neutrosophic hyperlinear span of the refined neutrosophic vectors u and v .

Proposition 2.19. Let $V(I_1, I_2)$ be a weak refined neutrosophic strongly left distributive hypervector space over the field K and $u_1, u_2, \dots, u_n \in V(I_1, I_2)$, with $u_i = (a_i, b_iI_1, c_iI_2)$ for $i = 1, 2, 3 \dots n$. Then

1. $W(I_1, I_2) = \bigcup\{\alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n : \alpha_1, \alpha_2, \dots, \alpha_n \in K\}$ is a weak refined neutrosophic subhypervector space of $V(I_1, I_2)$.
2. $W(I_1, I_2)$ is the smallest weak refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing u_1, u_2, \dots, u_n .

Proof. 1. The proof follows from similar approach as 1 of Proposition 2.17 .

2. Suppose that $M(I_1, I_2)$ is a weak refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing $u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \dots, u_n = (a_n, b_nI_1, c_nI_2)$. Let $t \in W(I_1, I_2)$, then there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in K$ such that

$$t \in \alpha_1 \bullet (a_1, b_1I_1, c_1I_2) + \alpha_2 \bullet (a_2, b_2I_1, c_2I_2) + \dots + \alpha_n \bullet (a_n, b_nI_1, c_nI_2).$$

Therefore $t \in M(I_1, I_2) \implies W(I_1, I_2) \subseteq M(I_1, I_2)$.

Hence $W(I_1, I_2)$ is the smallest weak refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing u_1, u_2, \dots, u_n . □

Proposition 2.20. Let $V(I_1, I_2)$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$, and let

$$u_1 = (a_1, b_1 I_1, c_1 I_2), u_2 = (a_2, b_2 I_1, c_2 I_2), \dots, u_n = (a_n, b_n I_1, c_n I_2) \in V(I_1, I_2),$$

$$\alpha_1 = (k_1, m_1 I_1, t_1 I_2), \alpha_2 = (k_2, m_2 I_1, t_2 I_2) \dots, \alpha_n = (k_n, m_n I_1, t_n I_2).$$

Then:

1. $W(I_1, I_2) = \bigcup \{ \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n : \alpha_1, \alpha_2, \dots, \alpha_n \in K(I_1, I_2) \}$ is a refined neutrosophic subhypervector space of $V(I_1, I_2)$.
2. $W(I_1, I_2)$ is the smallest refined neutrosophic subhypervector space of $V(I_1, I_2)$ containing u_1, u_2, \dots, u_n .

Proof: The proof follows from similar approach as that of Proposition 2.19 .

Remark 2.21. The refined neutrosophic subhypervector space $W(I_1, I_2)$ of the strong refined neutrosophic hypervector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$ of Proposition 2.20 is said to be generated by the refined neutrosophic vectors u_1, u_2, \dots, u_n and we write $W(I_1, I_2) = span\{u_1, u_2, \dots, u_n\}$.

Definition 2.22. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let

$B(I_1, I_2) = \{u_1 = (a_1, b_1 I_1, c_1 I_2), u_2 = (a_2, b_2 I_1, c_2 I_2), \dots, u_n = (a_n, b_n I_1, c_n I_2)\}$ be a subset of $V(I_1, I_2)$. $B(I_1, I_2)$ is said to generate or span $V(I_1, I_2)$ if $V(I_1, I_2) = span(B(I_1, I_2))$.

Example 2.23. Let $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ be a strong refined neutrosophic hypervector space over a neutrosophic field $R(I_1, I_2)$ and let $B(I_1, I_2) = \{u_1 = ((1, 0I_1, 0I_2), (0, 0I_1, 0I_2), (0, 0I_1, 0I_2)), u_2 = ((0, 0I_1, 0I_2), (1, 0I_1, 0I_2), (0, 0I_1, 0I_2)), u_3 = ((0, 0I_1, 0I_2), (0, 0I_1, 0I_2), (1, 0I_1, 0I_2))\}$. Then $B(I_1, I_2)$ spans $V(I_1, I_2)$.

Example 2.24. Let $V(I_1, I_2) = \mathbb{R}^2(I_1, I_2)$ be a weak refined neutrosophic hypervector space over a field \mathbb{R} and let $B(I_1, I_2) = \{u_1 = ((1, 0I_1, 0I_2), (0, 0I_1, 0I_2)), u_2 = ((0, 0I_1, 0I_2), (1, 0I_1, 0I_2)), u_3 = ((0, 0I_1, 0I_2), (0, 0I_1, 0I_2)), u_4 = ((0, 0I_1, 0I_2), (0, 1, 0I_2)), u_5 = ((0, 0I_1, 0I_2), (0, 0I_1, 0I_2)), u_6 = ((0, 0I_1, 0I_2), (0, 0I_1, 1))\}$. Then $B(I_1, I_2)$ spans $V(I_1, I_2)$.

Definition 2.25. Let $W[I_1, I_2]$ and $X[I_1, I_2]$ be two refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. $V(I_1, I_2)$ is said to be the direct sum of $W[I_1, I_2]$ and $X[I_1, I_2]$ written $V(I_1, I_2) = W[I_1, I_2] \oplus X[I_1, I_2]$ if every element $v \in V(I_1, I_2)$ can be written uniquely as $v = w + x$ where $w \in W[I_1, I_2]$ and $x \in X[I_1, I_2]$.

Proposition 2.26. Let $W[I_1, I_2]$ and $X[I_1, I_2]$ be two refined neutrosophic subhypervector spaces of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. $V(I_1, I_2) = W[I_1, I_2] \oplus X[I_1, I_2]$ if and only if the following conditions hold:

1. $V(I_1, I_2) = W[I_1, I_2] + X[I_1, I_2]$.
2. $W[I_1, I_2] \cap X[I_1, I_2] = \{\theta\}$.

Proof. Same as in classical case. □

Definition 2.27. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$. The refined neutrosophic vector

$u = (a, bI_1, cI_2) \in V(I_1, I_2)$ is said to be a linear combination of the refined neutrosophic vectors $u_1 = (a_1, b_1 I_1, c_1 I_2), u_2 = (a_2, b_2 I_1, c_2 I_1), \dots, u_n = (a_n, b_n I_1, c_n I_2) \in V(I_1, I_2)$ if there exist refined neutrosophic scalars $\alpha_1 = (k_1, m_1 I_1, t_1 I_2), \alpha_2 = (k_2, m_2 I_1, t_2 I_2), \dots, \alpha_n = (k_n, m_n I_1, t_n I_2) \in K(I_1, I_2)$ such that

$$u \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n.$$

Definition 2.28. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let

$B(I_1, I_2) = \{u_1 = (a_1, b_1 I_1, c_1 I_2), u_2 = (a_2, b_2 I_1, c_2 I_2), \dots, u_n = (a_n, b_n I_1, c_n I_2)\}$ be a subset of $V(I_1, I_2)$.

1. $B(I_1, I_2)$ is called a linearly dependent set if there exist refined neutrosophic scalars $\alpha_1 = (k_1, m_1 I_1, t_1 I_2), \alpha_2 = (k_2, m_2 I_1, t_2 I_2), \dots, \alpha_n = (k_n, m_n I_1, t_n I_2)$ (not all zero) such that

$$\theta \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n.$$

2. $B(I_1, I_2)$ is called a linearly independent set if

$$\theta \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n$$

implies that $\alpha_1 = \alpha_2 = \dots = \alpha_n = (0, 0I_1, 0I_2)$.

Proposition 2.29. *Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K . Any singleton set of non-null refined neutrosophic vector of the weak refined neutrosophic hypervector space $V(I_1, I_2)$ is linearly independent.*

Proof. Suppose that $\theta \neq v = (a, bI_1, cI_2) \in V(I_1, I_2)$. Let $\theta \in k \bullet v$ and suppose that $\theta \neq k \in K$. Then $k^{-1} \in K$ and therefore, $k^{-1} \bullet \theta \subseteq k^{-1} \bullet (k \bullet v)$ so that

$$\begin{aligned} \theta &\in (k^{-1}k) \bullet v \\ &= 1 \bullet v \\ &= \{(x, yI_1, zI_2) : x \in 1 \bullet a, y \in 1 \bullet b, z \in 1 \bullet c\} \\ &= \{(x, yI_1, zI_2) : x \in \{a\}, y \in \{b\}, z \in \{c\}\} \\ &= \{(a, bI_1, cI_2)\} \\ &= \{v\}. \end{aligned}$$

This shows that $v = \theta$ which is a contradiction. Hence, $k = \theta$ and thus, the singleton $\{v\}$ is a linearly independent set. □

Proposition 2.30. *Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K . Any set of refined neutrosophic vectors of the weak refined neutrosophic hypervector space $V(I_1, I_2)$ containing the null refined neutrosophic vector is always linearly dependent.*

Proposition 2.31. *Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K and let $\mathbb{B}(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \dots, u_n = (a_n, b_nI_1, c_nI_2)\}$ be a subset of $V(I_1, I_2)$. Then $B(I_1, I_2)$ is a linearly independent set if and only if at least one element of $\mathbb{B}(I_1, I_2)$ can be expressed as a linear combination of the remaining elements of $\mathbb{B}(I_1, I_2)$.*

Proof : This can be easily established.

Proposition 2.32. *Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K and let*

$$B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \dots, u_n = (a_n, b_nI_1, c_nI_2)\}$$

be a subset of $V(I_1, I_2)$. Then $B(I_1, I_2)$ is a linearly dependent set if and only if at least one element of $B(I_1, I_2)$ can be expressed as a linear combination of the remaining elements of $B(I_1, I_2)$.

Proof : Suppose that $B(I_1, I_2)$ is a linearly dependent set. Then there exist scalars k_1, k_2, \dots, k_n not all zero in K such that

$$\theta \in k_1 \bullet u_1 + k_2 \bullet u_2 + \dots + k_n \bullet u_n.$$

Suppose that $k_1 \neq 0$, then $k_1^{-1} \in K$ and therefore

$$\begin{aligned} k_1^{-1} \bullet \theta &\subseteq k_1^{-1} \bullet (k_1 \bullet u_1 + k_2 \bullet u_2 + \dots + k_n \bullet u_n) \\ &= (k_1^{-1}k_1) \bullet u_1 + (k_1^{-1}k_2) \bullet u_2 + \dots + (k_1^{-1}k_n) \bullet u_n \\ &= 1 \bullet u_1 + (k_1^{-1}k_2) \bullet u_2 + \dots + (k_1^{-1}k_n) \bullet u_n \end{aligned}$$

so that

$$\theta \in 1 \bullet u_1 + \{u\}$$

where $u = (a, bI_1, cI_2) \in (k_1^{-1}k_2) \bullet u_2 + \dots + (k_1^{-1}k_n) \bullet u_n$.

Thus $\theta \in \{(a+a_1, (b+b_1)I_1, (c+c_1)I_2)\}$ from which we obtain $u_1 = (a_1, b_1I_1, c_1I_2) = -u = -(a, bI_1, cI_2)$ so that

$$\begin{aligned} u_1 &\in (-1) \bullet u \\ &\subseteq (-1) \bullet ((k_1^{-1}k_2) \bullet u_2 + \dots + (k_1^{-1}k_n) \bullet u_n) \\ &\subseteq (-k_1^{-1}k_2) \bullet u_2 + (-k_1^{-1}k_3) \bullet u_3 + \dots + (-k_1^{-1}k_n) \bullet u_n. \end{aligned}$$

This shows that $u_1 \in \text{span}\{u_2, u_3, \dots, u_n\}$.

Conversely, suppose that $u_1 \in \text{span}\{u_2, u_3, \dots, u_n\}$ and suppose that $0 \neq -1 \in K$.

Then there exist $k_2, k_3, \dots, k_n \in K$ such that

$$u_1 \in k_2 \bullet u_2 + k_3 \bullet u_3 + \dots + k_n \bullet u_n$$

and we have

$$u_1 + (-u_1) \in (-1) \bullet u_1 + k_2 \bullet u_2 + k_3 \bullet u_3 + \dots + k_n \bullet u_n.$$

From which

$$\theta \in (-1) \bullet u_1 + k_2 \bullet u_2 + k_3 \bullet u_3 + \dots + k_n \bullet u_n.$$

Since $-1 \neq 0 \in K$, it follows that $B(I_1, I_2)$ is a linearly dependent set.

Proposition 2.33. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let $B_1(I_1, I_2)$ and $B_2(I_1, I_2)$ be subsets of $V(I_1, I_2)$ such that $B_1(I_1, I_2) \subseteq B_2(I_1, I_2)$. If $B_1(I_1, I_2)$ is linearly dependent, then $B_2(I_1, I_2)$ is linearly dependent.

Proposition 2.34. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let $B_1(I_1, I_2)$ and $B_2(I_1, I_2)$ be subsets of $V(I_1, I_2)$ such that $B_1(I_1, I_2) \subseteq B_2(I_1, I_2)$. If $B_1(I_1, I_2)$ is linearly independent, then $B_2(I_1, I_2)$ is linearly independent.

Definition 2.35. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let $B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_1), u_2 = (a_2, b_2I_1, c_2I_2), \dots\}$ be a subset of $V(I_1, I_2)$. $B(I_1, I_2)$ is said to be a basis for $V(I_1, I_2)$ if the following conditions hold:

1. $B(I_1, I_2)$ is a linearly independent set
2. $V(I_1, I_2) = \text{span}(B(I_1, I_2))$.

If $B(I_1, I_2)$ is finite and its cardinality is n , then $V(I_1, I_2)$ is called an n -dimensional strong refined neutrosophic hypervector space and we write $\dim_s(V(I_1, I_2)) = n$. If $B(I_1, I_2)$ is not finite, then $V(I_1, I_2)$ is called an infinite-dimensional strong refined neutrosophic hypervector space.

Definition 2.36. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a weak refined neutrosophic hypervector space over a field K and let $B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_1), u_2 = (a_2, b_2I_1, c_2I_2), \dots\}$ be a subset of $V(I_1, I_2)$. $B(I_1, I_2)$ is said to be a basis for $V(I_1, I_2)$ if the following conditions hold:

1. $B(I_1, I_2)$ is a linearly independent set
2. $V(I_1, I_2) = \text{span}(B(I_1, I_2))$.

If $B(I_1, I_2)$ is finite and its cardinality is n , then $V(I_1, I_2)$ is called an n -dimensional weak refined neutrosophic hypervector space and we write $\dim_w(V(I_1, I_2)) = n$. If $B(I_1, I_2)$ is not finite, then $V(I_1, I_2)$ is called an infinite-dimensional weak refined neutrosophic hypervector space.

Example 2.37. In Example 2.23 $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$ and $\dim_s(V(I_1, I_2)) = 3$.

Example 2.38. In Example 2.24 $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$ and $\dim_w(V(I_1, I_2)) = 6$.

Proposition 2.39. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ be a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ and let

$B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \dots, u_n = (a_n, b_nI_1, c_nI_2)\}$ be a subset of $V(I_1, I_2)$. Then $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$ if and only if each refined neutrosophic vector $u = (a, bI_1, cI_2) \in V(I_1, I_2)$ can be expressed uniquely as a linear combination of the elements of $B(I_1, I_2)$.

Proof. Suppose that each refined neutrosophic vector $u = (a, bI_1, cI_2) \in V(I_1, I_2)$ can be expressed uniquely as a linear combination of the elements of $B(I_1, I_2)$. Then $u \in \text{span}(B(I_1, I_2)) = V(I_1, I_2)$.

Since such a representation is unique, it follows that $B(I_1, I_2)$ is a linearly independent set and since $u \in V(I_1, I_2)$ is arbitrary, it follows that $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$.

Conversely, suppose that $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$, then $V(I_1, I_2) = \text{span}(B(I_1, I_2))$ and $B(I_1, I_2)$ is linearly independent. Now it remains to show that $u = (a, bI_1, cI_2) \in V(I_1, I_2)$ can be expressed uniquely as a linear combination of the elements of $B(I_1, I_2)$.

To this end, for $\alpha_1 = (k_1, m_1I_1, p_1I_2), \alpha_2 = (k_2, m_2I_1, p_2I_2), \dots, \alpha_n = (k_n, m_nI_1, p_nI_2), \beta_1 = (r_1, s_1I_1, t_1I_2), \beta_2 = (r_2, s_2I_1, t_2I_2), \dots, \beta_n = (r_n, s_nI_1, t_nI_2) \in K(I_1, I_2)$,

let us express u in two ways as follows:

$$u \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n, \tag{1}$$

$$u \in \beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \dots + \beta_n \bullet u_n. \tag{2}$$

From equation(2), we have

$$\begin{aligned}
 -u \in (-1) \bullet u &\subseteq (-1) \bullet (\beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \dots + \beta_n \bullet u_n) \\
 &= ((-1)\beta_1) \bullet u_1 + ((-1)\beta_2) \bullet u_2 + \dots + ((-1)\beta_n) \bullet u_n \\
 &= (-\beta_1) \bullet u_1 + (-\beta_2) \bullet u_2 + \dots + (-\beta_n) \bullet u_n.
 \end{aligned} \tag{3}$$

From equations (1) and (3), we have

$$\begin{aligned}
 u + (-u) \in (\alpha_1 + (-\beta_1)) \bullet u_1 + (\alpha_2 + (-\beta_2)) \bullet u_2 + \dots + (\alpha_n + (-\beta_n)) \bullet u_n \\
 \implies \theta \in (\alpha_1 - \beta_1) \bullet u_1 + (\alpha_2 - \beta_2) \bullet u_2 + \dots + (\alpha_n - \beta_n) \bullet u_n.
 \end{aligned}$$

Since $B(I_1, I_2)$ is linearly independent, it follows that

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \dots = \alpha_n - \beta_n = (0, 0I_1, 0I_2)$$

and therefore,

$$\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_n = \beta_n.$$

This shows that u has been expressed uniquely as a linear combination of the elements of $B(I_1, I_2)$. The proof is complete. \square

Proposition 2.40. Let $(V(I_1, I_2), +, \bullet, K)$ be a weak refined neutrosophic hypervector space over a field K and let

$B_1(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \dots, u_n = (a_n, b_nI_1, c_nI_2)\}$ be a linearly independent subset of $V(I_1, I_2)$. If $u \in V(I_1, I_2) \setminus B_1(I_1, I_2) = V(I_1, I_2) \cap (B(I_1, I_2))^c$ is arbitrary, then

$B_2(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \dots, u_n = (a_n, b_nI_1, c_nI_2), u\}$ is a linearly dependent set if and only if $u \in \text{span}((B_1(I_1, I_2)))$.

Proof. Suppose that $B_2(I_1, I_2)$ is a linearly dependent set. Then there exist scalars k_1, k_2, \dots, k_n, k not all zero such that

$$\theta \in k_1 \bullet u_1 + k_2 \bullet u_2 + \dots + k_n \bullet u_n + k \bullet u. \tag{4}$$

Suppose that $k = 0$, then there exist at least one of the k_i 's say $k_1 \neq 0$ and equation (4) becomes

$$\theta \in k_1 \bullet u_1 + k_2 \bullet u_2 + \dots + k_n \bullet u_n \tag{5}$$

from which it follows that the set

$B_1(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \dots, u_n = (a_n, b_nI_1, c_nI_2)\}$ is linearly dependent.

This contradicts the hypothesis that $B_1(I_1, I_2)$ is linearly independent. Hence $k \neq 0$ and therefore $k^{-1} \in K$. From equation (4), we have

$$\begin{aligned}
 k^{-1} \bullet \theta &\subseteq k^{-1} \bullet (k_1 \bullet u_1 + k_2 \bullet u_2 + \dots + k_n \bullet u_n + k \bullet u) \\
 \implies \theta &\in (k^{-1}k_1) \bullet u_1 + (k^{-1}k_2) \bullet u_2 + \dots + (k^{-1}k_n) \bullet u_n + (k^{-1}k) \bullet u \\
 \implies \theta &= v + u \text{ (where } v \in (k^{-1}k_1) \bullet u_1 + (k^{-1}k_2) \bullet u_2 + \dots + (k^{-1}k_n) \bullet u_n) \\
 \implies u &= -v \in (-1) \bullet ((k^{-1}k_1) \bullet u_1 + (k^{-1}k_2) \bullet u_2 + \dots + (k^{-1}k_n) \bullet u_n) \\
 \implies u &\in (-1) \bullet [(k^{-1}k_1) \bullet u_1 + (k^{-1}k_2) \bullet u_2 + \dots + (k^{-1}k_n) \bullet u_n] \\
 \implies u &\in (-k^{-1}k_1) \bullet u_1 + (-k^{-1}k_2) \bullet u_2 + \dots + (-k^{-1}k_n) \bullet u_n \\
 \implies u &\in \text{span}(B_1(I_1, I_2)).
 \end{aligned}$$

Conversely, suppose that $u \in \text{span}(B_1(I_1, I_2))$. Then there exist $k_1, k_2, \dots, k_n \in K$ such that

$$\begin{aligned}
 u &\in k_1 \bullet u_1 + k_2 \bullet u_2 + \dots + k_n \bullet u_n \\
 \implies u + (-u) &\in k_1 \bullet u_1 + k_2 \bullet u_2 + \dots + k_n \bullet u_n + (-1) \bullet u \\
 \implies \theta &\in k_1 \bullet u_1 + k_2 \bullet u_2 + \dots + k_n \bullet u_n + (-1) \bullet u.
 \end{aligned}$$

Since $u \notin B_1(I_1, I_2)$ and $B_1(I_1, I_2)$ is linearly independent, it follows that $\{u_1, u_2, \dots, u_n, u\} = B_2(I_1, I_2)$ is a linearly dependent set. The proof is complete. \square

Definition 2.41. Let $W[I_1, I_2]$ be a refined neutrosophic subhypervector space of a strong refined neutrosophic hypervector space $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ over a refined neutrosophic field $K(I_1, I_2)$. The quotient $V(I_1, I_2)/W[I_1, I_2]$ is defined by the set

$$\{[v] = v + W[I_1, I_2] : v \in V(I_1, I_2)\}.$$

Proposition 2.42. Let $V(I_1, I_2)/W[I_1, I_2] = \{[v] = v + W[I_1, I_2] : v \in V(I_1, I_2)\}$.
 If for every $[u], [v] \in V(I_1, I_2)/W[I_1, I_2]$ and $\alpha \in K(I_1, I_2)$ we define:

$$[u] \oplus [v] = (u + v) + W[I_1, I_2]$$

and

$$\alpha \odot [u] = [\alpha \bullet u] = \{[x] : x \in \alpha \bullet u\}.$$

$(V(I_1, I_2)/W[I_1, I_2], \oplus, \odot, K(I_1, I_2))$ is a strong refined neutrosophic hypervector space over a refined neutrosophic field $K(I_1, I_2)$ called a strong refined neutrosophic quotient hypervector space.

Proof. The proof is similar to the proof in classical case. □

Proposition 2.43. Let $W[I_1, I_2]$ be a refined neutrosophic subhypervector space of a strong refined neutrosophic hypervector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$, let $(V(I_1, I_2)/W[I_1, I_2])$ be as defined in Proposition 2.42, then the following hold:

1. $W[I_1, I_2]$ is finite dimensional and $dim_s W[I_1, I_2] \leq dim_s V(I_1, I_2)$.
2. $dim_s(V(I_1, I_2)/W[I_1, I_2]) = dim_s V(I_1, I_2) - dim_s W[I_1, I_2]$.

Proof:

1. Let $B_1(I_1, I_2)$ be the basis for $W[I_1, I_2]$ and let $B_2(I_1, I_2)$ be a basis for $V(I_1, I_2)$. Since $W[I_1, I_2] \subseteq V(I_1, I_2)$ then $B_1(I_1, I_2)$ is contained in $B_2(I_1, I_2)$. Therefore $B_1(I_1, I_2)$ is a linearly independent subset of $V(I_1, I_2)$. Then we have that $|B_1(I_1, I_2)| \leq |B_2(I_1, I_2)|$. Now, since $|B_1(I_1, I_2)| \leq |B_2(I_1, I_2)|$ and $V(I_1, I_2)$ is finite dimensional we can conclude that $W(I_1, I_2)$ is finite dimensional and

$$dim_s W(I_1, I_2) = |B_1(I_1, I_2)| \leq |B_2(I_1, I_2)| = dim_s V(I_1, I_2).$$

2. Let $\{u_1, u_2, \dots, u_m\}$ be a basis of $W[I_1, I_2]$. Then this can be filled out to a basis, $\{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$ of $V(I_1, I_2)$, where $m+n = dim_s V(I_1, I_2)$ and $m = dim_s W[I_1, I_2]$. Let $[v_1], [v_2], \dots, [v_n]$ be the images in $V(I_1, I_2)/W[I_1, I_2]$, of v_1, v_2, \dots, v_n . Since any vector $v \in V(I_1, I_2)$ is in a linear combination of $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$, we have that

$$v \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_m \bullet u_m + \beta_1 \bullet v_1 + \beta_2 \bullet v_2 + \dots + \beta_n \bullet v_n,$$

then

$$\begin{aligned} v &\in [\alpha_1 \bullet u_1] \oplus [\alpha_2 \bullet u_2] \oplus \dots \oplus [\alpha_m \bullet u_m] \oplus [\beta_1 \bullet v_1] \oplus [\beta_2 \bullet v_2] \oplus \dots \oplus [\beta_n \bullet v_n] \\ &\subseteq [\beta_1 \bullet v_1] \oplus [\beta_2 \bullet v_2] \oplus \dots \oplus [\beta_n \bullet v_n] \text{ (since } [\alpha_i \bullet u_i] \subseteq (\alpha_i \bullet u_i) + W[I_1, I_2] \subseteq W[I_1, I_2]) \\ &= \beta_1 \bullet [v_1] \oplus \beta_2 \bullet [v_2] \oplus \dots \oplus \beta_n \bullet [v_n]. \end{aligned}$$

Thus $[v_1], [v_2], \dots, [v_n]$ span $V(I_1, I_2)/W[I_1, I_2]$. We claim that they are linearly independent, for if

$$\theta \in \lambda_1 \bullet [v_1] \oplus \lambda_2 \bullet [v_2] \oplus \dots \oplus \lambda_n \bullet [v_n]$$

then

$$\theta \in \lambda_1 \bullet [v_1] \oplus \lambda_2 \bullet [v_2] \oplus \dots \oplus \lambda_n \bullet [v_n] \oplus W[I_1, I_2]$$

$$\theta \subseteq \lambda_1 \bullet [v_1] \oplus \lambda_2 \bullet [v_2] \oplus \dots \oplus \lambda_n \bullet [v_n] \oplus \gamma_1 \bullet [u_1] \oplus \gamma_2 \bullet [u_2] \oplus \dots \oplus \gamma_m \bullet [u_m]$$

which by the linear independence of the set $\{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$ forces

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = \gamma_1 = \gamma_2 = \dots = \gamma_m = 0.$$

This shows that $V(I_1, I_2)/W[I_1, I_2]$ has a basis of n elements, and

$$dim_s(V(I_1, I_2)/W[I_1, I_2]) = n = (n + m) - m = dim_s V(I_1, I_2) - dim_s W[I_1, I_2].$$

Proposition 2.44. Let $W_1(I_1, I_2)$ and $W_2(I_1, I_2)$ be finite dimensional weak refined neutrosophic subhypervector spaces of a weak refined neutrosophic vector space $V(I_1, I_2)$ over a field K . Then $W_1(I_1, I_2) + W_2(I_1, I_2)$ is a finite dimensional refined neutrosophic subhypervector space of $V(I_1, I_2)$ and

$$dim_w(W_1(I_1, I_2) + W_2(I_1, I_2)) = dim_w(W_1(I_1, I_2)) + dim_w(W_2(I_1, I_2)) - dim_w(W_1(I_1, I_2) \cap W_2(I_1, I_2)).$$

If $V(I_1, I_2) = W_1(I_1, I_2) \oplus W_2(I_1, I_2)$ then

$$dim_w(W_1(I_1, I_2) + W_2(I_1, I_2)) = dim_w(W_1(I_1, I_2)) + dim_w(W_2(I_1, I_2)).$$

Proof: We know that $W_1(I_1, I_2) \cap W_2(I_1, I_2)$ is a refined neutrosophic subhypervector space of both $W_1(I_1, I_2)$ and $W_2(I_1, I_2)$. So $W_1(I_1, I_2) \cap W_2(I_1, I_2)$ is a finite dimensional refined neutrosophic subhypervector space of $V(I_1, I_2)$.

Suppose that $dim_w(W_1(I_1, I_2) \cap W_2(I_1, I_2)) = k$, $dim_w(W_1(I_1, I_2)) = m$ and $dim_w(W_2(I_1, I_2)) = n$ then we have that $k \leq m$ and $k \leq n$.

Now, let $\{u_1, u_2, \dots, u_k\}$ be a basis of $W_1(I_1, I_2) \cap W_2(I_1, I_2)$. Then we have that $\{u_1, u_2, \dots, u_k\}$ is a linearly independent set of refined neutrosophic vectors in $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ with $k \leq m$ and $k \leq n$, then it follows that either $\{u_1, u_2, \dots, u_k\}$ is a basis of $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$ or it can be extended to a basis for $W_1[I_1, I_2]$ and $W_2[I_1, I_2]$.

Let $\{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{m-k}\}$ be a basis for $W_1[I_1, I_2]$, and let $\{u_1, u_2, \dots, u_k, w_1, w_2, \dots, w_{n-k}\}$ be a basis of $W_2[I_1, I_2]$.

Then the refined neutrosophic subhypervector space $W_1[I_1, I_2] + W_2[I_1, I_2]$ is spanned by the refined neutrosophic vectors $\{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{m-k}, w_1, w_2, \dots, w_{n-k}\}$ and these refined neutrosophic vectors form an independent set. For suppose

$$\theta \in \sum_{i=1}^k \alpha_i u_i + \sum_{j=1}^m \beta_j v_j + \sum_{r=1}^n \gamma_r w_r.$$

Then

$$\begin{aligned} -\sum_{r=1}^n \gamma_r w_r &\in \sum_{i=1}^k \alpha_i u_i + \sum_{j=1}^m \beta_j v_j \\ \implies (-1) \bullet \left(-\sum_{r=1}^n \gamma_r w_r\right) &\subseteq \sum_{i=1}^k (-1) \bullet \alpha_i u_i + \sum_{j=1}^m (-1) \bullet \beta_j v_j \\ \implies \sum_{r=1}^n \gamma_r w_r &\in \sum_{i=1}^k (-\alpha_i) u_i + \sum_{j=1}^m (-\beta_j) v_j \end{aligned}$$

which shows that $\sum_{r=1}^n \gamma_r w_r$ belongs to $W_1[I_1, I_2]$. As $\sum_{r=1}^n \gamma_r w_r$ also belongs to $W_2[I_1, I_2]$, it follows that

$$\sum_{r=1}^n \gamma_r w_r = \sum_{i=1}^k \lambda_i u_i$$

for certain scalars $\lambda_1, \lambda_2, \dots, \lambda_k$.

Because the set $\{u_1, u_2, \dots, u_k, w_1, w_2, \dots, w_{n-k}\}$ is independent, each of the scalars $\gamma_r = 0$. Thus

$$\theta \in \sum_{i=1}^k \alpha_i u_i + \sum_{j=1}^m \beta_j v_j$$

and since $\{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{m-k}\}$ is also an independent set, each $\alpha_i = 0$ and each $\beta_j = 0$. Thus, $\{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_{m-k}, w_1, w_2, \dots, w_{n-k}\}$ is a basis for $W_1[I_1, I_2] + W_2[I_1, I_2]$.

Finally,

$$\begin{aligned} dim_w(W_1(I_1, I_2) + W_2(I_1, I_2)) &= k + m - k + n - k \\ &= m + n - k \\ &= dim_w(W_1(I_1, I_2)) + dim_w(W_2(I_1, I_2)) - dim_w(W_1(I_1, I_2) \cap W_2(I_1, I_2)). \end{aligned}$$

Definition 2.45. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ and $(W(I_1, I_2), +', \bullet', K(I_1, I_2))$ be two strong refined neutrosophic hypervector spaces over a neutrosophic field $K(I_1, I_2)$.

A mapping $\phi : V(I_1, I_2) \rightarrow W(I_1, I_2)$ is called a strong refined neutrosophic hypervector space homomorphism if the following conditions hold:

1. ϕ is a strong hypervector space homomorphism.
2. $\phi(0, I_1, I_2) = (0, I_1, I_2)$.

If in addition ϕ is a bijection, we say that $V(I_1, I_2)$ is isomorphic to $W(I_1, I_2)$ and we write $V(I_1, I_2) \cong W(I_1, I_2)$.

Proposition 2.46. Let $(V(I_1, I_2), +, \bullet, K(I_1, I_2))$ and $(W(I_1, I_2), +, \bullet, K(I_1, I_2))$ be two strong refined neutrosophic hypervector spaces over a refined neutrosophic field $K(I_1, I_2)$ and let $\phi : V(I_1, I_2) \longrightarrow W(I_1, I_2)$ be a bijective strong refined neutrosophic hypervector space homomorphism.

If $B(I_1, I_2) = \{u_1 = (a_1, b_1I_1, c_1I_2), u_2 = (a_2, b_2I_1, c_2I_2), \dots, u_n = (a_n, b_nI_1, c_nI_2)\}$ is a basis for $V(I_1, I_2)$, then $B'(I_1, I_2) = \phi(B(I_1, I_2)) = \{\phi(u_1), \phi(u_2), \dots, \phi(u_n)\}$ is a basis for $W(I_1, I_2)$.

Proof. Suppose that $B(I_1, I_2)$ is a basis for $V(I_1, I_2)$. Then for an arbitrary $u = (a, bI_1, cI_2) \in V(I_1, I_2)$, there exist refined neutrosophic scalars

$$\alpha_1 = (k_1, m_1I_1, t_1I_2), \alpha_2 = (k_2, m_2I_1, t_2I_2), \dots, \alpha_n = (k_n, m_nI_1, t_nI_2) \in K(I_1, I_2) \text{ such that}$$

$$\begin{aligned} u &\in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n \\ \implies \phi(u) &\in \phi(\alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n) \\ &= \alpha_1 \bullet' \phi(u_1) +' \alpha_2 \bullet' \phi(u_2) +' \dots +' \alpha_n \bullet' \phi(u_n). \end{aligned}$$

Since ϕ is surjective, it follows that $\phi(u), \phi(u_1), \phi(u_2), \dots, \phi(u_n) \in W(I_1, I_2)$ and therefore $\phi(u) \in \text{span}(B'(I_1, I_2))$. To complete the proof, we must show that $B'(I_1, I_2)$ is linearly independent. To this end, suppose that

$$\phi(\theta) \in \beta_1 \bullet' \phi(u_1) +' \beta_2 \bullet' \phi(u_2) +' \dots +' \beta_n \bullet' \phi(u_n)$$

where $\beta_1 = (p_1, q_1I_1, s_1I_2), \beta_2 = (p_2, q_2I_1, s_2I_2), \dots, \beta_n = (p_n, q_nI_1, s_nI_2) \in K(I_1, I_2)$, then

$$\begin{aligned} \phi(\theta) &\in \phi(\beta_1 \bullet u_1) +' \phi(\beta_2 \bullet u_2) +' \dots +' \phi(\beta_n \bullet u_n) \\ &= \phi(\beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \dots + \beta_n \bullet u_n). \end{aligned}$$

Since ϕ is injective, we must have

$$\theta \in \beta_1 \bullet u_1 + \beta_2 \bullet u_2 + \dots + \beta_n \bullet u_n.$$

Also, since $B(I_1, I_2)$ is linearly independent, we must have

$$\beta_1 = \beta_2 = \dots = \beta_n = (0, 0I_1, 0I_2).$$

Hence $B'(I_1, I_2) = \{\phi(u_1), \phi(u_2), \dots, \phi(u_n)\}$ is linearly independent and therefore a basis for $W(I_1, I_2)$. □

Remark 2.47. Suppose we wish to transform a refined neutrosophic hypervector space into a neutrosophic hypervector space, an interesting question to ask will be, can we find a mapping that will help us achieve this? The answer to this is Yes.

The mapping $\phi : V(I_1, I_2) \longrightarrow V(I)$ defined by

$$\phi((x, yI_1, zI_2)) = (x, (y + z)I) \quad \forall x, y, z \in V$$

will make such transformation possible. This mapping is a non-neutrosophic one. This make sense since every refined neutrosophic hypervector space and neutrosophic hypervector spaces are hypervector spaces.

Proposition 2.48. Let $(V(I_1, I_2), +, \bullet)$ be a weak refined neutrosophic vector space over a field K and let $V(I)$ be a weak neutrosophic vector space over K . The mapping $\phi : V(I_1, I_2) \longrightarrow V(I)$ defined by

$$\phi((x, yI_1, zI_2)) = (x, (y + z)I) \quad \forall x, y, z \in V$$

is a good linear transformation.

Proof. ϕ is well defined. Suppose $(x, yI_1, zI_2) = (x'y'I_1, z'I_2)$ then we that $x = x', y = y'$ and $z' = z'$. So,

$$\phi((x, yI_1, zI_2)) = (x, (y + z)I) = x' + (y' + z')I = \phi(x', y'I_1, z'I_2).$$

Now, suppose $(x, yI_1, zI_2), (x', y'I_1, 1z'I_2) \in V(I_1, I_2)$ then

$$\begin{aligned} \phi((x, yI_1, zI_2) + (x', y'I_1, 1z'I_2)) &= \phi((x + x'), (y + y')I_1, (z + z')I_2) \\ &= (x + x'), (y + y' + z + z')I \\ &= (x + x'), ((y + z) + (y' + z'))I \\ &= (x + x'), ((y + z)I + (y' + z')I) \\ &= (x, (y + z)I) + (x', (y' + z')I) \\ &= \phi(x, yI_1, zI_2) + \phi(x', y'I_1, z'I_2). \end{aligned}$$

$$\begin{aligned}
 \phi(k \circ (x, yI_1, zI_2)) &= \phi\{(a, bI_1, cI_2) : a \in k \circ x, b \in k \circ y, c \in k \circ z\} \\
 &= \{\phi(a, bI_1, cI_2) : a \in k \circ x, b \in k \circ y, c \in k \circ z\} \\
 &= \{(u, vI) : u \in a, v \in b + c\} \\
 &= \{(u, vI) : u \in k \circ x, v \in k \circ y + k \circ z\} \\
 &= \{(u, vI) : u \in k \circ x, v \in k \circ (y + z)\} \\
 &= k \circ (x, (y + z)I) \\
 &= k \circ \phi(x, yI_1, zI_2).
 \end{aligned}$$

Hence ϕ is a good linear transformation. □

Proposition 2.49. Let $L_k(V(I_1, I_2), V(I))$ be the set of good linear transformation from a weak refined neutrosophic vector space $V(I_1, I_2)$ over a field K into a weak neutrosophic vector space $V(I)$ over a field K . Define addition and scalar multiplication as below;

$$(\phi + \psi)(x, yI_1, zI_2) = \phi((x, yI_1, zI_2)) + \psi((x, yI_1, zI_2))$$

and for $k \in K$

$$(k\phi)((x, yI_1, zI_2)) = k\phi(x, yI_1, zI_2).$$

Then, it can be shown that $(L_k(V(I_1, I_2), V(I)), +, \cdot)$ is a weak neutrosophic strongly distributive hypervector space.

Definition 2.50. Let $\phi : V(I_1, I_2) \rightarrow V(I)$ be a good linear transformation, then

$$\begin{aligned}
 \ker \phi &= \{(x, yI_1, zI_2) : \phi((x, yI_1, zI_2)) = (0, 0I)\} \\
 &= \{(x, yI_1, zI_2) : (x, (y + z)I) = (0, 0I)\} \\
 &= \{(0, yI_1, (-y)I_2)\}.
 \end{aligned}$$

Proposition 2.51. Let $\phi : V(I_1, I_2) \rightarrow V(I)$ be a good linear transformation.

1. $\ker \phi$ is a subhyperspace of $V(I_1, I_2)$.
2. If $W[I_1, I_2]$ is a refined neutrosophic subhyperspace of $V(I_1, I_2)$, then the image of $W[I_1, I_2]$, $\phi(W[I_1, I_2])$ is a neutrosophic subhyperspace of $V(I)$.

3 Conclusion

This paper studied refinement of neutrosophic hypervector space, linear dependence, independence, bases and dimension of refined neutrosophic hypervector spaces and presented some of their basic properties. Also, the paper established the existence of a good linear transformation between a weak refined neutrosophic hypervector space $V(I_1, I_2)$ and a weak neutrosophic hypervector space $V(I)$. We hope to present and study more properties of refined neutrosophic Hypervector spaces in our future papers.

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