Erlang Service Queueing Model with Neutrosophic Parameters

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Abstract

In this paper, we generalize the classical Erlang service queueing model, which has determined and crisp parameters to neutrosophic Erlang service queueing model which is more accurate because it opens the ability to deal with imprecise and incomplete knowledge of parameters. We have used the neutrosophic statistical interval method introduced by F. Smarandache to describe the parameters of the Erlang service queueing model and to find the neutrosophic performance measures.

Keywords: Erlang Service Queues; Neutrosophic Logic; Performance Measures; Neutrosophic Statistical Numbers.

1.Introduction

Queueing theory was developed by Erlang in 1909 to model waiting for lines and develop efficient systems that reduce customers' waiting times which makes it possible to serve more customers and increase profits to the organizations. The problem of classical queueing theory is the assumption of well and clear knowledge of the parameters of the queueing system (e.g. arrival rate, departure rate) which is often impossible [1,2].

Neutrosophic logic introduced by F. Smarandache in 1995 is a generalization of fuzzy logic and intuitionistic fuzzy logic [3,4,5,6,7,8]. This logic makes dealing with indeterminant data easier, clearer, and more realistic. So modelling queues using neutrosophic parameters makes decisions more efficient [9,10,11,12,13,14,15].

Extension of classical queueing theory to neutrosophic queueing theory means that parameters of queues can take indeterminant values that allow dealing with vagueness.

In this paper, we show the power of neutrosophic crisp sets theory [16,17] to deal with imprecise parameters of the Erlang queueing model, and we derive neutrosophic performance measures of the mentioned queue, also solved examples showing the power of this extension are presented.

2. Classical Erlang Service Model [1,2]

In this model, customers arrive at a service system with one server according to exponential interarrival times with parameter λ and served one by one according to Erlang service in k phases with parameter μ, and the performance measures are:

Expected number of customers in the system:

$$L_s = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) + \frac{\lambda}{\mu}$$

Expected number of customers in the queue:

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$$L_q = L_s - \frac{\lambda}{\mu} = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right)$$

Expected waiting time in the system:

$$W_s = \frac{L_s}{\lambda}$$

$$W_s = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right) + \frac{1}{\mu}$$

Expected waiting time in the queue:

$$W_q = \frac{L_q}{\lambda}$$

$$W_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right)$$

Where λ is arrival rate, μ is departure (servicing) rate.

3. Neutrosophic Erlang Service Model

In this model, customers arrive at one server facility according to the Poisson process with neutrosophic (inaccurate, imprecise) parameter $N\lambda$ [17] where we are going to use neutrosophic statistical numbers presented in [9] so $N\lambda$ can be written as an interval as following $N\lambda = [\lambda^L, \lambda^U]$ where λ^L, λ^U are crisp real numbers. Customers are also served one by one according to Erlang service in k phases with neutrosophic parameter $N\mu = [\mu^L, \mu^U]$ where μ^L, μ^U are also crisp real numbers. So the neutrosophic performance measures can be driven by replacing λ by $N\lambda$ and μ by $N\mu$ as follows:

Neutrosophic expected number of customers in the system:

$$NL_s = \left(\frac{k+1}{2k}\right)\left(\frac{[\lambda^L, \lambda^U]^2}{[\mu^L, \mu^U]([\mu^L, \mu^U] - [\lambda^L, \lambda^U])}\right) + \frac{[\lambda^L, \lambda^U]}{[\mu^L, \mu^U]}$$

$$NL_s = \left(\frac{k+1}{2k}\right)\left(\left[\frac{{\lambda^L}^2}{{\mu^U}^2 - \lambda^L\mu^U}, \frac{{\lambda^U}^2}{{\mu^L}^2 - \lambda^U\mu^L}\right]\right) + \left[\frac{\lambda^L}{\mu^U}, \frac{\lambda^U}{\mu^L}\right] \qquad (1)$$

Neutrosophic expected number of customers in the queue:

$$NL_q = \left(\frac{k+1}{2k}\right)\left(\left[\frac{{\lambda^L}^2}{{\mu^U}^2 - \lambda^L\mu^U}, \frac{{\lambda^U}^2}{{\mu^L}^2 - \lambda^U\mu^L}\right]\right) \qquad (2)$$

Neutrosophic expected waiting time in the system:

$$NW_s = \frac{NL_s}{N\lambda} = \frac{NL_s}{[\lambda^L, \lambda^U]} = NL_s * \left[\frac{1}{\lambda^U}, \frac{1}{\lambda^L}\right]$$

$$NW_s = \left(\frac{k+1}{2k}\right)\left(\left[\frac{\frac{{\lambda^L}^2}{\lambda^U}}{{\mu^U}^2 - \lambda^L\mu^U}, \frac{\frac{{\lambda^U}^2}{\lambda^L}}{{\mu^L}^2 - \lambda^U\mu^L}\right]\right) + \left[\frac{\frac{\lambda^L}{\lambda^U}}{\mu^U}, \frac{\frac{\lambda^U}{\lambda^L}}{\mu^L}\right] \qquad (3)$$

Neutrosophic expected waiting time in the queue:

$$NW_q = \frac{NL_q}{N\lambda} = \frac{NL_q}{[\lambda^L, \lambda^U]} = NL_q * \left[\frac{1}{\lambda^U}, \frac{1}{\lambda^L}\right]$$

$$NW_q = \left(\frac{k+1}{2k}\right)\left(\left[\frac{\frac{{\lambda^L}^2}{\lambda^U}}{{\mu^U}^2 - \lambda^L\mu^U}, \frac{\frac{{\lambda^U}^2}{\lambda^L}}{{\mu^L}^2 - \lambda^U\mu^L}\right]\right) \qquad (4)$$

Notice that all the neutrosophic performance measures are presented as statistical numbers having lower bound and upper bound, also we can notice four cases:

Case 1: When $\lambda^L = \lambda^U$ and $\mu^L \neq \mu^U$ we have crisp arrival rate with neutrosophic departures.

Case 2: When $\mu^L = \mu^U$ and $\lambda^L \neq \lambda^U$ we have crisp departures with neutrosophic arrivals.
Case 3: When $\lambda^L = \lambda^U$ and $\mu^L = \mu^U$ the neutrosophic corresponds to the classical Erlang service queue.
Case 4: When $\lambda^L \neq \lambda^U$ and $\mu^L \neq \mu^U$ we have neutrosophic arrivals and neutrosophic departures.

4. Numerical Examples

Example 1 (crisp arrival rate, neutrosophic departure rate)

Suppose that a system consists of a machine gives a service to a customer in two phases each phase time is exponentially distributed and the machine serves in each phase between 3 and 5 customers per min. Customers arrive at the machine according to the Poisson process with arrival rate equals to 15 customers/h, and suppose that we want to calculate the following:

The average number of customers in the system.

The average number of customers in the queue.

Mean waiting time in the system.

Mean waiting time in the queue.

Solution:

From the given example we find that k=2, $\boldsymbol{\lambda = 15/h}$ so $\boldsymbol{\lambda = 0.25/min}$, $\boldsymbol{N\mu} = [\mathbf{3}, \mathbf{5}]$/min

Before the foundation of neutrosophic logic, one can solve this example by assuming that $\boldsymbol{\mu}$ takes the midpoint of the range [1,2], that is $\boldsymbol{\mu} \cong \mathbf{4}$ so:

Expected number of customers in the system:

$$L_s = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) + \frac{\lambda}{\mu} = \left(\frac{2+1}{2*2}\right)\left(\frac{0.25^2}{4(4-0.25)}\right) + \frac{0.25}{4} = 0.065625\ customers$$

Expected number of customers in the queue:

$$L_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) = \left(\frac{2+1}{2*2}\right)\left(\frac{0.25^2}{4(4-0.25)}\right) = 0.003125\ customers$$

Expected waiting time in the system:

$$W_s = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right) + \frac{1}{\mu} = \left(\frac{2+1}{2*2}\right)\left(\frac{0.25}{4(4-0.25)}\right) + \frac{1}{4} = 0.2625\ min$$

Expected waiting time in the queue:

$$W_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right) = \left(\frac{2+1}{2*2}\right)\left(\frac{0.25}{4(4-0.25)}\right) = 0.0125\ min$$

But after derving(?) the neutrosophic solutions and using equations 1 to 4 we get:

$$\mathbf{NL_s} = \left(\frac{\mathbf{k+1}}{\mathbf{2k}}\right)\left(\left[\frac{\boldsymbol{\lambda}^{\mathbf{L}^2}}{\boldsymbol{\mu}^{\mathbf{U}^2} - \boldsymbol{\lambda}^{\mathbf{L}}\boldsymbol{\mu}^{\mathbf{U}}}, \frac{\boldsymbol{\lambda}^{\mathbf{U}^2}}{\boldsymbol{\mu}^{\mathbf{L}^2} - \boldsymbol{\lambda}^{\mathbf{U}}\boldsymbol{\mu}^{\mathbf{L}}}\right]\right) + \left[\frac{\boldsymbol{\lambda}^{\mathbf{L}}}{\boldsymbol{\mu}^{\mathbf{U}}}, \frac{\boldsymbol{\lambda}^{\mathbf{U}}}{\boldsymbol{\mu}^{\mathbf{L}}}\right]$$
$$= \left(\frac{\mathbf{2+1}}{\mathbf{2*2}}\right) * \left(\left[\frac{\mathbf{0.25^2}}{\mathbf{5^2 - 0.25*5}}, \frac{\mathbf{0.25^2}}{\mathbf{3^2 - 0.25*3}}\right]\right) + \left[\frac{\mathbf{0.25}}{\mathbf{5}}, \frac{\mathbf{0.25}}{\mathbf{3}}\right] = [\mathbf{0.051974}, \mathbf{0.089015}]$$

Which means that average number of customers in the system lies in the given range.

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Notice that $L_s = 0.065625 \in NL_s = [0.051974, 0.089015]$

$$NL_q = \left(\frac{k+1}{2k}\right)\left(\left[\frac{{\lambda^L}^2}{{\mu^U}^2 - \lambda^L\mu^U}, \frac{{\lambda^U}^2}{{\mu^L}^2 - \lambda^U\mu^L}\right]\right) = \left(\frac{2+1}{2*2}\right) * \left(\left[\frac{0.25^2}{25 - 0.25*5}, \frac{0.25^2}{9 - 0.25*3}\right]\right)$$

$$= [0.001974, 0.005682]$$

Which means that average number of customers in queue lies in the given range.
Also in thic case we notice that $L_q = 0.003125 \in [0.001974, 0.005682]$

$$NW_s = \left(\frac{k+1}{2k}\right)\left(\left[\frac{\frac{{\lambda^L}^2}{\lambda^U}}{{\mu^U}^2 - \lambda^L\mu^U}, \frac{\frac{{\lambda^U}^2}{\lambda^L}}{{\mu^L}^2 - \lambda^U\mu^L}\right]\right) + \left[\frac{\frac{\lambda^L}{\lambda^U}}{\mu^U}, \frac{\frac{\lambda^U}{\lambda^L}}{\mu^L}\right] = \left(\frac{2+1}{2*2}\right)\left(\left[\frac{0.25}{25 - 0.25*5}, \frac{0.25}{9 - 0.25*3}\right]\right) + \left[\frac{1}{5}, \frac{1}{3}\right]$$
$$= [0.207895, 0.356061]$$

Which means that average waiting time in the system lies between 0.207895 and 0.356061 mins that is between 12 and 21 secs approximately.

Also $W_s = 0.2625 \in [0.207895, 0.356061] = NW_s$

$$NW_q = \left(\frac{k+1}{2k}\right)\left(\left[\frac{\frac{{\lambda^L}^2}{\lambda^U}}{{\mu^U}^2 - \lambda^L\mu^U}, \frac{\frac{{\lambda^U}^2}{\lambda^L}}{{\mu^L}^2 - \lambda^U\mu^L}\right]\right) = \left(\frac{2+1}{2*2}\right)\left(\left[\frac{0.25}{5^2 - 0.25*5}, \frac{0.25}{3^2 - 0.25*3}\right]\right)$$
$$= [0.007895, 0.022727]$$

Which means that average waiting time in the queue lies between 0.007895 and 0.022727 mins that is between 0.5 and 1.36 secs approximately.

Also $W_q = 0.0125 \in [0.007895, 0.022727] = NW_q$

From all above, we can clearly see that the neutrosophic solutions are more accurate than classical solutions.

Example 2 (neutrosophic arrival rate, neutrosophic departure rate)

Suppose that a system consists of a machine gives a service to a customer in three phases each phase time is exponentially distributed and the machine serves in each phase between 3 and 5 customers per min. Customers arrive at the machine according to the Poisson process with arrival rate lies between 1 and 2 customers/min, and suppose that we want to calculate the following:

The average number of customers in the system.

The average number of customers in the queue.

Mean waiting time in the system.

Mean waiting time in the queue.

Solution:

From the given example, we find that k=3, $\boldsymbol{N\lambda = [1, 2]/min}$, $\boldsymbol{N\mu = [3, 5]}$/min

Classical solutions can be derived by assuming $\lambda \cong 1.5, \mu \cong 4$ by taking midpoints of each interval, that gives:

Expected number of customers in the system:

$$L_s = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) + \frac{\lambda}{\mu} = \left(\frac{3+1}{2*3}\right)\left(\frac{1.5^2}{4(4-1.5)}\right) + \frac{1.5}{4} = 0.525\ customers$$

Expected number of customers in the queue:

$$L_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda^2}{\mu(\mu-\lambda)}\right) = \left(\frac{3+1}{2*3}\right)\left(\frac{1.5^2}{4(4-1.5)}\right) = 0.15\ customers$$

Expected waiting time in the system:

$$W_s = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right) + \frac{1}{\mu} = \left(\frac{3+1}{2*3}\right)\left(\frac{1.5}{4(4-1.5)}\right) + \frac{1}{4} = 0.35\ min$$

Expected waiting time in the queue:

$$W_q = \left(\frac{k+1}{2k}\right)\left(\frac{\lambda}{\mu(\mu-\lambda)}\right) = \left(\frac{3+1}{2*3}\right)\left(\frac{0.25}{4(4-0.25)}\right) = 0.1\ min$$

Neutrosophic solutions can be derived using equations 1 to 4 as follows:

$$\mathbf{NL_s} = \left(\frac{\mathbf{k+1}}{\mathbf{2k}}\right)\left(\left[\frac{\boldsymbol{\lambda}^{\mathbf{L}^2}}{\boldsymbol{\mu}^{\mathbf{U}^2} - \boldsymbol{\lambda}^{\mathbf{L}}\boldsymbol{\mu}^{\mathbf{U}}}, \frac{\boldsymbol{\lambda}^{\mathbf{U}^2}}{\boldsymbol{\mu}^{\mathbf{L}^2} - \boldsymbol{\lambda}^{\mathbf{U}}\boldsymbol{\mu}^{\mathbf{L}}}\right]\right) + \left[\frac{\boldsymbol{\lambda}^{\mathbf{L}}}{\boldsymbol{\mu}^{\mathbf{U}}}, \frac{\boldsymbol{\lambda}^{\mathbf{U}}}{\boldsymbol{\mu}^{\mathbf{L}}}\right] = \left(\frac{\mathbf{3+1}}{\mathbf{2*3}}\right) * \left(\left[\frac{\mathbf{1}}{\mathbf{25-1*5}}, \frac{\mathbf{4}}{\mathbf{9-2*3}}\right]\right) + \left[\frac{\mathbf{1}}{\mathbf{5}}, \frac{\mathbf{2}}{\mathbf{3}}\right]$$
$$= [\mathbf{0.233333, 1.555556}]$$

Which means that average number of customers in system lies in the given range.

Notice that $L_s = 0.525 \in NL_s = [0.233333, 1.555556]$

$$NL_q = \left(\frac{k+1}{2k}\right)\left(\left[\frac{\lambda^{L^2}}{\mu^{U^2} - \lambda^L\mu^U}, \frac{\lambda^{U^2}}{\mu^{L^2} - \lambda^U\mu^L}\right]\right) = \left(\frac{3+1}{2*3}\right) * \left(\left[\frac{1}{25-1*5}, \frac{4}{9-2*3}\right]\right)$$

$$= [0.03333, 0.888889]$$

Which means that average number of customers in the queue lies in the given range.
Also in thic case we notice that $L_q = 0.15 \in [0.03333, 0.888889]$

$$NW_s = \left(\frac{k+1}{2k}\right)\left(\left[\frac{\frac{\lambda^{L^2}}{\lambda^U}}{\mu^{U^2} - \lambda^L\mu^U}, \frac{\frac{\lambda^{U^2}}{\lambda^L}}{\mu^{L^2} - \lambda^U\mu^L}\right]\right) + \left[\frac{\frac{\lambda^L}{\lambda^U}}{\mu^U}, \frac{\frac{\lambda^U}{\lambda^L}}{\mu^L}\right] = \left(\frac{3+1}{2*3}\right)\left(\left[\frac{1/2}{25-1*5}, \frac{4/1}{9-2*3}\right]\right) + \left[\frac{1/2}{5}, \frac{2/1}{3}\right]$$
$$= [0.116667, 1.555556]$$

Which means that average waiting time in the system lies between 0.116667 and 1.555556 mins.

Also $W_s = 0.35 \in [0.116667, 1.555556] = NW_s$

$$NW_q = \left(\frac{k+1}{2k}\right)\left(\left[\frac{\frac{\lambda^{L^2}}{\lambda^U}}{\mu^{U^2} - \lambda^L\mu^U}, \frac{\frac{\lambda^{U^2}}{\lambda^L}}{\mu^{L^2} - \lambda^U\mu^L}\right]\right) = \left(\frac{3+1}{2*3}\right)\left(\left[\frac{1/2}{25-1*5}, \frac{4/1}{9-2*3}\right]\right) = [0.016667, 0.888889]$$

Which means that average waiting time in the queue lies between 0.016667 and 0.8888889 mins.

Also $W_q = 0.1 \in [0.016667, 0.888889] = NW_q$

So we also see that neutrosophic solutions are more accurate and can be considered as an extension to classical solutions.

5. Conclusions

We conclude that neutrosophic logic can be applied to generalize the classical queueing theory and make it easy and clear to handle imprecise and incomplete data. We have chosen Erlang service models as a useful and important example of frequently used queueing systems.

The author is looking forward to studying the ability to extend other queueing models using neutrosophic logic including batch arrivals and batch services, customers behavior in single and multi-queues.

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