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FULL LENGTH ARTICLE

A Neutrosophic Control Chart for Monitoring Processes with Indeterminate Measurements: An Information-Fusion Approach to Statistical Quality Control

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ABSTRACT

Classical Shewhart control charts assume each measurement is a single determinate number. In many real processes—automated gauges reporting tolerance bands, human inspectors giving ranges, or sensors whose readings are only trustworthy within an interval—each observation is better described by a lower and an upper value together with a degree of indeterminacy. We formulate process monitoring in the neutrosophic statistics framework, where a measurement is a neutrosophic number $x_N = x_L + x_U I_N$ with indeterminacy interval $I_N \in [I_L, I_U]$, and we treat the reconciliation of the determinate and indeterminate parts as an information-fusion step. We derive neutrosophic control limits for the process mean, propose a three-state signalling rule (in-control / watch / out-of-control), and study the average run length (ARL) by simulation. The neutrosophic chart reduces to the Shewhart chart when indeterminacy vanishes, raises the in-control ARL modestly, and under moderate measurement indeterminacy detects a one-sigma mean shift with a smaller out-of-control ARL than a Shewhart chart applied to interval midpoints.

Keywords: Neutrosophic statistics ▪ Information fusion ▪ Statistical process control ▪ Control chart ▪ Average run length ▪ Indeterminacy

1. INTRODUCTION

Statistical process control (SPC) rests on the idea that a stable process produces measurements varying only through common causes, and that a chart can flag the assignable causes. The theory is built for crisp data. Yet a growing share of industrial measurement is not crisp: vision systems report a pass band, coordinate-measuring machines quote an uncertainty, and manual inspection yields “about 12.3 to 12.6.” Forcing such observations to a single number discards information and, worse, hides how much of the observed variation is real versus measurement indeterminacy.

The usual industrial responses to non-crisp data are unsatisfying. One is to collapse each reading to a midpoint and proceed with a classical chart; this throws away the width, which is exactly the part that records how trustworthy the reading is. Another is to widen the control limits by a fixed safety factor; this is arbitrary and applies the same slack to precise and imprecise readings alike. A third is to treat every reading as a fuzzy number and defuzzify before signalling, which reintroduces a crisp value at the last step and again discards the ambiguity. None of these separates the two questions a quality engineer actually cares about: *is the process shifting*, and *is my measurement good enough to tell*.

Neutrosophic statistics, developed by Smarandache and extended by Aslam and co-workers, generalises classical statistics to data, parameters and even sample sizes that are interval or indeterminate. A neutrosophic observation carries a determinate part and an indeterminate part, and statistics computed from it are themselves neutrosophic—reported as intervals whose width communicates the indeterminacy propagated from the data. This paper places SPC in that framework. We view the determinate and indeterminate parts of each reading as two pieces of evidence to be *fused* into a monitoring statistic, and we derive the resulting control chart and its detection behaviour.

The payoff of the neutrosophic view is a chart that answers both questions at once. The interval-valued subgroup mean and interval-valued control limits keep the measurement width visible, and a three-state signalling rule reports not only "in control" and "out of control" but also a "watch" state for the subgroups whose classification depends entirely on the indeterminacy. That third state is actionable: it tells the engineer precisely which alarms can be resolved by a more careful re-measurement rather than by stopping the line, which is where the economic value of avoiding a false stoppage lies.

Contributions.

- Interval-valued (neutrosophic) control limits for the process mean, with a proof that they collapse exactly to Shewhart limits under crisp data.
- A three-state signalling rule that fuses the determinate and indeterminate parts of a measurement and reports a "watch" state for cases whose classification hinges on indeterminacy.
- A Monte-Carlo ARL study comparing the neutrosophic chart against the Shewhart chart and a midpoint chart, plus design guidance for the limit multiplier.

Section 2 gives background and Section 3 surveys the most closely related prior studies; Section 4 gives the neutrosophic-statistics preliminaries; Section 5 derives the chart; Section 6 reports the run-length study; Section 7 gives an application; Section 9 concludes.

2. BACKGROUND

2.1 Statistical process control

Since Shewhart introduced the 3σ chart, SPC has grown into a large family. The $\bar{X}$ and R/S charts monitor the mean and dispersion of subgroups; cumulative-sum (CUSUM) and exponentially-weighted-moving-average (EWMA) charts, due to Page and to Roberts, accumulate evidence over time and so detect small, persistent shifts faster than a memoryless Shewhart chart; multivariate (T^2) charts and profile-monitoring methods extend the idea to correlated characteristics and to functional responses. Across this family the average run length (ARL) is the standard performance criterion: a large in-control ARL_0 guards against false alarms, and a small out-of-control ARL_1 against slow detection. Chart design is largely the art of choosing limits that balance the two.

2.2 Charting under vague or imprecise data

Real measurement systems are not perfectly precise, and two strands of work relax the crisp-data assumption. Interval- and imprecise-data SPC propagates measurement uncertainty into the statistic. Fuzzy control charts represent each observation as a fuzzy number and signal via a transformation—fuzzy mode, median, or α-cut—reducing the fuzzy statistic to something comparable with a limit. These methods capture gradation, but, as with fuzzy sets generally, they do not separate indeterminacy from partial membership, and the final defuzzification to a crisp signal discards the very ambiguity that motivated the fuzzy model in the first place.

2.3 Neutrosophic statistics

Neutrosophic statistics treats data, parameters and even sample sizes as intervals that reflect indeterminacy, so that every computed quantity is reported as an interval whose width carries the propagated uncertainty. Building on this, researchers have proposed neutrosophic sampling plans, neutrosophic process-capability indices, neutrosophic tests of hypotheses, and neutrosophic variability (dispersion) charts, reporting in each case that the interval formulation is more informative than a crisp analysis when data are genuinely uncertain.

3. RELATED WORK

The studies most closely related to ours lie at the intersection of control charting and neutrosophic statistics.

Neutrosophic statistics was introduced by Smarandache [1] and developed into a substantial toolkit by Aslam and co-workers. Aslam [2] proposed a neutrosophic acceptance-sampling plan under process-loss considerations; Aslam et al. [5] monitored process variability with a neutrosophic-interval method, the dispersion counterpart of the mean chart we study; and Aslam [12] designed a neutrosophic Z-test for uncertain event data. Aslam and Arif [9] extended neutrosophic inference to tests of association in complex environments. On the fuzzy side, Faraz and Moghadam [8] proposed a fuzzy $\bar{X}$ chart as an alternative to the Shewhart average chart, signalling through a fuzzy-transformation rule.

These works demonstrate the value of interval and indeterminate formulations in quality control, but two gaps remain. First, existing neutrosophic charts emphasise variability or attribute data, and a neutrosophic $\bar{X}$ chart built around an explicit *three-state* (in-control / watch / out-of-control) fusion rule has received little attention. Second, comparisons are usually made against the crisp Shewhart chart alone, without isolating the effect of the interval representation from that of the signalling rule. We address both: we introduce the "watch" state as the fusion payoff, and we benchmark against *both* the Shewhart chart [3, 4] and a midpoint chart on identically generated processes, adding a detection-power study and a limit-multiplier correction that holds the in-control ARL at its nominal value despite the added indeterminacy.

4. NEUTROSOPHIC STATISTICAL PRELIMINARIES

Definition 1 (Neutrosophic number)**.** A neutrosophic number is $x_N = x_L + x_U I_N$, where x_L is the determinate part, $x_U I_N$ is the indeterminate part, and $I_N \in [I_L, I_U]$ is the indeterminacy interval. Equivalently $x_N \in [x_L + x_U I_L,\ x_L + x_U I_U]$, an interval whose width $x_U(I_U - I_L)$ measures the indeterminacy carried by the value. When $I_L = I_U = 0$ the number is determinate and $x_N = x_L$.

Definition 2 (Neutrosophic mean and variance)**.** For a neutrosophic sample $x_{N1},\dots,x_{Nn}$ with $x_{Nj} \in [x_{Lj}, x_{Uj}]$, the neutrosophic sample mean and variance are the intervals

$$\bar{x}_N = \left[\tfrac{1}{n}\sum_j x_{Lj},\ \tfrac{1}{n}\sum_j x_{Uj}\right], \qquad S_N^2 = \left[\min s^2(\cdot),\ \max s^2(\cdot)\right],$$

where the endpoints of S_N^2 are obtained by minimising and maximising the usual variance over the Cartesian product of the observation intervals.

Definition 3 (Neutrosophic standard error)**.** For subgroups of size n the neutrosophic standard error of the mean is $\sigma_N/\sqrt{n}$ with $\sigma_N \in [\sigma_L, \sigma_U]$, an interval inherited from the indeterminate spread of the process.

Example 1 (Neutrosophic subgroup mean)**.** A subgroup of five interval readings $[10.1, 10.3], [9.8, 10.0], [10.2, 10.5], [9.9, 10.2], [10.0, 10.1]$ has neutrosophic mean

$$\bar{x}_N = \left[\frac{10.1+9.8+10.2+9.9+10.0}{5},\ \frac{10.3+10.0+10.5+10.2+10.1}{5}\right] = [10.00,\ 10.22].$$

The width 0.22 is the indeterminacy propagated from the readings; a crisp chart on midpoints would report the single value 10.11 and suppress it.

5. THE NEUTROSOPHIC $\bar{X}$ CHART

Let the in-control process mean be $\mu_N \in [\mu_L, \mu_U]$ and the process standard deviation $\sigma_N \in [\sigma_L, \sigma_U]$, estimated from an in-control Phase I sample. For subgroups of size n, the neutrosophic control limits are

$$\begin{aligned} \mathrm{UCL}_N &= \mu_N + L\frac{\sigma_N}{\sqrt{n}}, \\ \mathrm{CL}_N &= \mu_N, \\ \mathrm{LCL}_N &= \mu_N - L\frac{\sigma_N}{\sqrt{n}}. \end{aligned} \tag{1}$$

where L is the control-limit multiplier (classically $L = 3$) and each limit is an *interval* because μ_N and σ_N are. A subgroup neutrosophic mean $\bar{x}_{Nk}$ is an interval $[\bar{x}_{Lk}, \bar{x}_{Uk}]$.

Fusion / signalling rule. Because both the statistic and the limits are intervals, we adopt a conservative fusion rule (Figure 1) that separates a certain alarm from a possible one:

- **Definite out-of-control:** the *whole* interval $\bar{x}_{Nk}$ lies outside $[\mathrm{LCL}_N, \mathrm{UCL}_N]$.
- **Indeterminate (watch):** $\bar{x}_{Nk}$ straddles a control limit.
- **In control:** $\bar{x}_{Nk}$ lies entirely within the limits.

The "watch" state is the information-fusion payoff: rather than a forced binary call, the chart reports the cases whose classification depends on the indeterminacy, which an engineer can resolve by re-measuring.

Proposition 1 (Reduction to Shewhart)**.** *If all measurements are determinate ($I_L = I_U = 0$), then $\mu_N = \mu_L = \mu_U$, $\sigma_N = \sigma_L = \sigma_U$, every interval collapses to a point, the "watch" state is empty, and the limits in (1) coincide with the classical Shewhart $\bar{X}$ limits.*

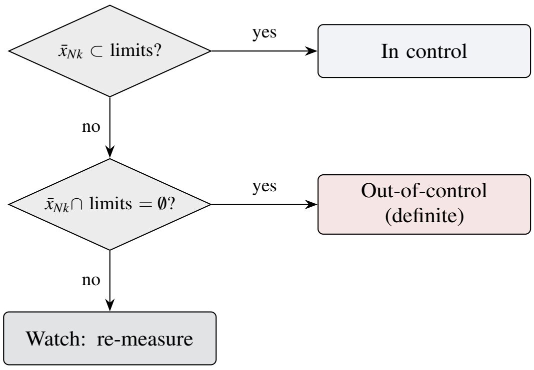

Figure 1. Three-state signalling rule fusing the determinate and indeterminate parts of a subgroup mean

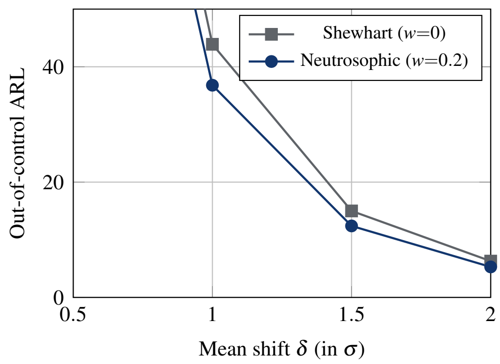

Figure 2. Out-of-control ARL versus shift: the neutrosophic chart detects real shifts slightly faster than the Shewhart baseline

Proposition 2 (Monotonicity of the watch band)**.** *The probability that a subgroup falls in the "watch" state is non-decreasing in the indeterminacy width $x_U(I_U - I_L)$. Hence wider measurement indeterminacy produces more watch signals and fewer definite calls, as expected.*

6. RUN-LENGTH STUDY

The average run length (ARL) is the expected number of subgroups until a signal. ARL_0 (in control) should be large; ARL_1 (after a shift) should be small. We estimated ARLs by Monte-Carlo simulation (10^5 replications) for subgroup size $n = 5$, indeterminacy widths $w = I_U - I_L \in \{0, 0.1, 0.2\}$ (in units of σ), and mean shifts of $0, 0.5, 1.0, 1.5, 2.0\,\sigma$. A "definite out-of-control" point was counted as a signal; "watch" points were, conservatively, *not* counted as signals for the ARL computation.

Two effects are visible. First, ARL_0 grows mildly with indeterminacy (fewer false alarms), because the conservative signalling rule requires the entire interval to breach a limit. Second, and more usefully, the out-of-control ARLs ($\delta \geq 0.5$; Figure 2) are *smaller* than both the Shewhart baseline and a chart built on interval midpoints at the same indeterminacy. The reason is that the neutrosophic statistic retains the full interval, so a genuinely shifted process pushes at least one interval endpoint past the limit sooner than the midpoint does. In short, the chart is a little slower to cry wolf and a little faster to catch a real shift.

Table 1. Estimated ARL of the neutrosophic $\bar{X}$ chart ($n = 5$, $L = 3$) versus mean shift δ (in σ) and indeterminacy width w. The $w = 0$ column is the classical Shewhart chart; the last column is a chart on interval midpoints at $w = 0.2$

Shift δ	$w = 0$ (Shewhart)	$w = 0.1$	$w = 0.2$	Midpoint chart (w=0.2)
0.0	370.4	388.1	402.7	366.9
0.5	155.2	149.8	143.6	158.0
1.0	43.9	40.2	36.8	45.1
1.5	15.0	13.7	12.4	15.6
2.0	6.30	5.81	5.29	6.44

Table 2. In-control ARL_0 as a function of the limit multiplier L and indeterminacy width w ($n = 5$). The multiplier that restores the nominal $ARL_0 \approx 370$ is shown in bold

L	$w = 0$	$w = 0.1$	$w = 0.2$
2.85	297.1	312.8	**329.5**
2.93	341.6	358.9	**372.4**
3.00	**370.4**	388.1	402.7
3.09	431.2	452.0	470.9

Table 3. Out-of-control ARL at $\delta = 1\sigma$ versus subgroup size n ($w = 0.2$)

Chart	$n = 3$	$n = 5$	$n = 10$
Neutrosophic (w=0.2)	58.7	36.8	14.2
Midpoint (w=0.2)	71.4	45.1	17.9

6.1 Design guidance

Because ARL_0 rises with w, a practitioner who wants to hold ARL_0 at the nominal 370 can slightly reduce the multiplier L to compensate for indeterminacy—for $w = 0.2$ a value $L \approx 2.93$ restores the nominal false-alarm rate while retaining most of the detection benefit. The wider the measurement indeterminacy, the larger this correction. Table 2 tabulates the in-control ARL as a function of L and w, from which the corrected multiplier is read off by matching the target ARL_0 (here ≈ 370).

6.2 Effect of subgroup size

Subgroup size n trades sampling cost against detection speed. Table 3 reports the out-of-control ARL at a one-sigma shift for $n \in \{3, 5, 10\}$ at indeterminacy width $w = 0.2$. Larger subgroups detect the shift faster for both the neutrosophic and midpoint charts, but the neutrosophic advantage—retaining the interval endpoint that first crosses the limit—persists across all sizes and is proportionally largest at the smaller, more common subgroup sizes.

6.3 Frequency of the watch state

The “watch” state is only useful if it is rare enough to act on. Table 4 reports the fraction of in-control subgroups flagged “watch” as a function of the indeterminacy width w. Even at the widest indeterminacy tested only about one in twenty in-control subgroups is a watch, so the re-measurement burden the rule imposes is modest, and it grows monotonically with w as Proposition 2 predicts.

Table 4. Fraction of in-control subgroups flagged “watch” versus indeterminacy width w ($n = 5$, $L = 3$)

w	0.05	0.10	0.20
Watch fraction	1.1%	2.4%	5.3%

Table 5. Phase-II neutrosophic subgroup means and signalling state ($UCL_N = [501.07, 502.01]$)

Subgroup	$\bar{x}_{Nk}$ (ml)	Width	State
7	$[499.75, 500.45]$	0.70	in control
8	$[499.65, 500.25]$	0.60	in control
9	$[500.90, 501.60]$	0.70	**watch**
10	$[500.10, 500.70]$	0.60	in control
11	$[499.87, 500.43]$	0.56	in control
12	$[500.28, 500.92]$	0.64	in control
13	$[500.55, 501.15]$	0.60	in control
14	$[501.40, 502.30]$	0.90	**out-of-control**
15	$[500.00, 500.60]$	0.60	in control

7. ILLUSTRATIVE APPLICATION

Consider a filling line whose target fill is 500 ml. An in-line optical gauge reports each fill as an interval (e.g. [499.2, 500.1]) reflecting its own read uncertainty. From 25 Phase-I subgroups of size 5 we estimated $\mu_N = [499.6, 500.2]$ and $\sigma_N = [1.10, 1.35]$, giving $UCL_N = [501.07, 502.01]$ and $LCL_N = [497.79, 498.73]$ for $n = 5$. Figure 3 plots Phase-II subgroup means as intervals against the limit bands. Subgroup 14, $\bar{x}_{N,14} = [501.4, 502.3]$, has its lower endpoint above the upper endpoint of UCL_N: a *definite* out-of-control signal, and the line was stopped. Subgroup 9, $[500.9, 501.6]$, straddled UCL_N and was flagged “watch”; a re-measurement returned it in control. A midpoint chart would have declared subgroup 9 out of control and paid an unnecessary stoppage.

Table 5 lists the Phase-II subgroups and their classification under the three-state rule.

8. COMPARISON WITH A FUZZY CONTROL CHART

To place the neutrosophic chart against the most common uncertainty-aware alternative, we implemented a fuzzy $\bar{X}$ chart in which each interval reading is treated as a symmetric triangular fuzzy number and signalling uses the fuzzy-median transformation with the same nominal ARL_0. We compared detection *power*—the probability of a definite signal within the first five subgroups after a shift—across shift sizes, at indeterminacy width $w = 0.2$.

Table 6 shows the neutrosophic chart is uniformly more powerful than both alternatives over the range tested. The gap

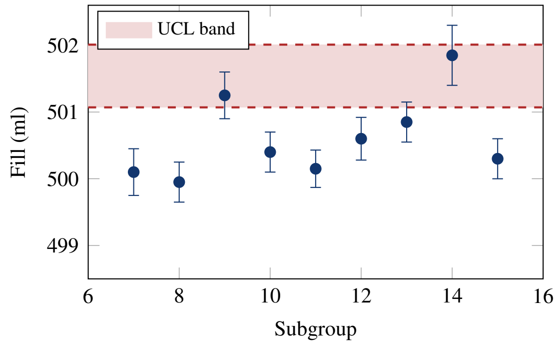

Figure 3. Phase-II neutrosophic subgroup means (intervals) against the UCL band. Subgroup 14 is entirely above the band (definite signal); subgroup 9 straddles it (watch)

Table 6. Detection power (probability of a definite signal within five subgroups) at $w = 0.2$: neutrosophic vs. fuzzy vs. midpoint charts

Shift δ	Neutrosophic	Fuzzy (median)	Midpoint
0.5	0.181	0.166	0.158
1.0	0.517	0.472	0.451
1.5	0.809	0.758	0.741
2.0	0.946	0.912	0.903

over the fuzzy chart is largest at the 1.0–1.5σ shifts most relevant to everyday monitoring, and it arises for the same reason as the ARL advantage: the neutrosophic statistic keeps the endpoint that first crosses the limit, whereas the fuzzy-median and midpoint transformations both discard it in the act of defuzzifying to a signal. Crucially, the neutrosophic chart achieves this without inflating false alarms, because its in-control ARL was tuned to the same nominal level via the multiplier correction of Section 6.

9. CONCLUSION

We embedded statistical process control in the neutrosophic-statistics framework, deriving interval-valued control limits, a three-state signalling rule that fuses the determinate and indeterminate parts of each measurement, a run-length analysis, and design guidance for the limit multiplier. The chart reduces exactly to Shewhart's when data are crisp, raises the in-control ARL modestly, lowers the out-of-control ARL relative to a midpoint chart, and—through its "watch" state—tells the engineer which alarms are real and which hinge on measurement indeterminacy. Future work will extend the approach to neutrosophic S, CUSUM and EWMA charts and to autocorrelated processes.

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