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FULL LENGTH ARTICLE

Residuation–Galois Calculus for Exact Safety Preimages of Monotone Max–Plus Neural Networks

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ABSTRACT

A proof-only calculus is developed for exact set abstraction and backward safety certification of monotone neural operators. A Galois insertion between concrete sets and interval boxes yields the best correct interval transformer. For every coordinatewise monotone map F, this transformer maps $[\ell, u]$ exactly to $[F(\ell), F(u)]$; hence layerwise interval propagation is hull-exact for networks with nonnegative weights and isotone activations. The analysis is sharpened for max–plus layers $T(x) = W \otimes x \oplus b$. Their upper safety preimages are either empty or principal ideals generated by the residual $W \backslash y$, where $(W \backslash y)_i = \inf_{j: W_{ji} > -\infty}(y_j - W_{ji})$. Reverse residual propagation through a depth-L network computes the greatest input vector satisfying a prescribed upper output bound. Consequently, the largest weighted ℓ_∞ radius around a nominal input is obtained in closed form, without optimization, branching, sampling, or relaxation. Soundness, maximality, compositionality, exactness, homogeneous collapse, and target-bound stability are proved. Four open problems concern signed architectures, two-sided class margins, residuated transformer attention, and completeness beyond boxes.

Keywords: Abstract interpretation ▪ Galois connection ▪ Max–plus algebra ▪ Residuation ▪ Monotone neural network ▪ Exact robustness certificate ▪ Tropical computation

1. INTRODUCTION

Abstract interpretation replaces a concrete semantic domain by an ordered abstraction connected through a Galois relation [1, 2, 3, 4]. Neural-network verification has adopted this principle through interval, zonotope, polyhedral, and linear-relaxation domains [5, 6, 7, 8, 9, 10]. The resulting bounds are generally sound but not exact because correlation is discarded or nonlinear transformers are relaxed. Formal analyses of interval expressivity, fixpoint abstraction, and domain-theoretic robustness sharpen this distinction [11, 12, 13].

This paper isolates a class for which exactness is algebraic. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be isotone under the product order. For every box $[\ell, u]$,

$$\alpha(F([\ell, u])) = [F(\ell), F(u)], \tag{1}$$

where α is the interval hull. Thus interval analysis loses no coordinatewise range information. Nonnegative-weight monotone networks satisfy this hypothesis [14, 15, 16, 17, 18].

A stronger inverse statement becomes available over the max–plus semiring. For

$$T(x) = W \otimes x \oplus b, \qquad (W \otimes x)_j = \max_i (W_{ji} + x_i), \tag{2}$$

residuation gives

$$T(x) \preceq y \iff b \preceq y \text{ and } x \preceq W\backslash y, \quad (W\backslash y)_i = \min_{j:W_{ji}>-\infty}(y_j - W_{ji}). \tag{3}$$

The relation is exact, not a relaxation. It converts output safety into a greatest admissible input and composes backward through depth.

The main results are:

(i) $F^\sharp = \alpha F\gamma$ is the least sound transformer, and $(GF)^\sharp \preceq G^\sharp F^\sharp$;

(ii) isotonicity gives $F^\sharp_\square([\ell,u]) = [F(\ell), F(u)]$;

(iii) principal ideals form a complete safety lattice $\mathfrak{D}_n = \{\varnothing\} \cup \{\downarrow p : p \in \overline{\mathbb{R}}^n\}$;

(iv) reverse residuation yields $\mathscr{N}^{-1}(\downarrow s) = \downarrow p(s)$ or $\varnothing$;

(v) lower preimages admit an exact disjunctive form and may require 2^k minimal generators;

(vi) for $r \in \mathbb{R}^n_{>0}$,

$$\varepsilon^*(x^0, s; r) = \min_i \frac{p_i(s) - x^0_i}{r_i}, \tag{4}$$

with linear sparse-time evaluation and explicit perturbation bounds.

Tropical representations already connect piecewise-linear networks with max–plus geometry [19, 20, 21]. The present contribution is different: it uses residuation to derive exact backward safety preimages and exact radii for a monotone max–plus architecture.

2. ORDERED DOMAINS AND BEST ABSTRACTION

Let $(\mathscr{C}, \sqsubseteq)$ and $(\mathscr{A}, \preceq)$ be complete lattices.

Definition 1 (Galois insertion)**.** *Maps $\alpha : \mathscr{C} \to \mathscr{A}$ and $\gamma : \mathscr{A} \to \mathscr{C}$ form a Galois insertion, written $\alpha \dashv \gamma$, if*

$$\alpha(c) \preceq a \iff c \sqsubseteq \gamma(a), \qquad \alpha\gamma = \mathrm{id}_{\mathscr{A}}. \tag{5}$$

The closure on $\mathscr{C}$ is $\rho = \gamma\alpha$.

For monotone $F : \mathscr{C} \to \mathscr{C}$, define

$$F^\sharp = \alpha F \gamma : \mathscr{A} \to \mathscr{A}. \tag{6}$$

An abstract transformer $G : \mathscr{A} \to \mathscr{A}$ is sound if $F\gamma \sqsubseteq \gamma G$ pointwise.

Theorem 1 (Best correct transformer)**.** *$F^\sharp$ is sound. If G is sound, then $F^\sharp \preceq G$. Hence $F^\sharp$ is the unique least sound transformer.*

Proof. Since $z \sqsubseteq \rho(z)$, $F\gamma(a) \sqsubseteq \gamma\alpha F\gamma(a) = \gamma F^\sharp(a)$. If $F\gamma(a) \sqsubseteq \gamma G(a)$, adjunction gives $\alpha F\gamma(a) \preceq G(a)$. □

Proposition 1 (Compositional overapproximation)**.** *For monotone $F, G : \mathscr{C} \to \mathscr{C}$,*

$$(GF)^\sharp \preceq G^\sharp F^\sharp. \tag{7}$$

Equality holds if $F\gamma(a)$ is ρ-closed for every $a \in \mathscr{A}$.

Proof. From $F\gamma \sqsubseteq \rho F\gamma$ and monotonicity of G and α, $\alpha GF\gamma \preceq \alpha G\gamma\alpha F\gamma$. If $\rho F\gamma = F\gamma$, the inequality is equality. □

3. EXACT BOX SEMANTICS OF MONOTONE NETWORKS

Let $\mathbb{I}^n$ be the complete lattice of boxes in $\overline{\mathbb{R}}^n$, augmented by $\perp$. On the concrete lattice $\mathscr{P}(\mathbb{R}^n)$, define

$$\alpha_\square(S) = [\inf S, \sup S], \qquad \gamma_\square([\ell,u]) = \{x : \ell \preceq x \preceq u\}. \tag{8}$$

Here $\alpha_\square(\varnothing) = \perp$, unbounded coordinates may take infinite endpoints, and coordinatewise infima and suprema are used. Then $\alpha_\square \dashv \gamma_\square$. For a point map F, write $\widehat{F}(S) = \{F(x) : x \in S\}$ and $F^\sharp_\square = \alpha_\square \widehat{F} \gamma_\square$.

Theorem 2 (Endpoint hull exactness)**.** *If $F : \mathbb{R}^n \to \mathbb{R}^m$ is isotone, then for every $\ell \preceq u$,*

$$F^\sharp_\square([\ell,u]) = [F(\ell), F(u)]. \tag{9}$$

Proof. For $x \in [\ell,u]$, isotonicity gives $F(\ell) \preceq F(x) \preceq F(u)$. Both endpoints belong to the image, so each coordinate infimum and supremum is attained at ℓ and u, respectively. □

Consider a depth-L network

$$h_0 = x, \qquad h_l = \sigma_l(W_l h_{l-1} + b_l), \qquad \mathscr{N}(x) = h_L, \tag{10}$$

with $W_l \geq 0$ entrywise and isotone coordinate activations σ_l.

Corollary 1 (Layerwise exactness)**.** *For (10), naive endpoint propagation is the best interval abstraction:*

$$\mathscr{N}^\sharp_\square([\ell,u]) = [\mathscr{N}(\ell), \mathscr{N}(u)]. \tag{11}$$

Moreover, composing the layerwise box transformers produces exactly the right-hand side.

Proof. Every layer is isotone. Apply Theorem 2 inductively to the lower and upper endpoints. □

Remark 1. Equation (11) is hull-exact, not set-exact: the image $\mathscr{N}([\ell,u])$ need not equal the whole output box. The theorem states that no tighter axis-aligned box exists.

4. RESIDUAL MAX–PLUS LAYERS

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ be the max–plus semiring and let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be its complete ordered extension. Max–plus linear maps are residuated on the product lattice $\overline{\mathbb{R}}^n$ [22, 23, 24]. For $W \in \mathbb{R}^{m\times n}_{\max}$ define

$$(W \otimes x)_j = \bigvee_{i:W_{ji}>-\infty} (W_{ji} + x_i), \tag{12}$$

with $\sup\varnothing = -\infty$. Its residual $W\backslash(\cdot) : \overline{\mathbb{R}}^m \to \overline{\mathbb{R}}^n$ is

$$(W\backslash y)_i = \bigwedge_{j:W_{ji}>-\infty} (y_j - W_{ji}), \tag{13}$$

where $\inf\varnothing = +\infty$. Thus absent entries impose no constraint.

Lemma 1 (Matrix residuation). *For all $x \in \overline{\mathbb{R}}^n$ and $y \in \overline{\mathbb{R}}^m$,*

$$W \otimes x \preceq y \iff x \preceq W \backslash y. \tag{14}$$

Thus $W \otimes (\cdot)$ is left adjoint to $W \backslash (\cdot)$.

Proof. $W \otimes x \preceq y$ iff $W_{ji} + x_i \le y_j$ for every finite entry W_{ji}, iff $x_i \le \inf_{j:W_{ji}>-\infty}(y_j - W_{ji})$ for every i. □

Let $T_{W,b}(x) = W \otimes x \oplus b$ with $b \in \mathbb{R}^m_{\max}$.

Theorem 3 (Exact affine safety preimage). *For $y \in \mathbb{R}^m_{\max}$,*

$$\{x : T_{W,b}(x) \preceq y\} = \begin{cases} \downarrow(W\backslash y), & b \preceq y, \\ \varnothing, & b \not\preceq y. \end{cases} \tag{15}$$

Hence $W\backslash y$ is the greatest safe input whenever the preimage is nonempty.

Proof. $W \otimes x \oplus b \preceq y$ iff both $W \otimes x \preceq y$ and $b \preceq y$. Apply Lemma 1. □

5. DEEP RESIDUAL SAFETY CALCULUS

Let

$$\mathcal{N} = T_L \circ \cdots \circ T_1, \qquad T_l(x) = W_l \otimes x \oplus b_l. \tag{16}$$

For a target upper bound $s \in \mathbb{R}^{d_L}_{\max}$ define the backward sequence

$$y_L = s, \qquad y_{l-1} = W_l \backslash y_l, \quad l = L, L-1, \dots, 1. \tag{17}$$

Call the sequence feasible when $b_l \preceq y_l$ for every l.

Theorem 4 (Exact deep safety preimage). *For (16)–(17),*

$$\{x : \mathcal{N}(x) \preceq s\} = \begin{cases} \downarrow y_0, & b_l \preceq y_l \ \forall l, \\ \varnothing, & \textit{otherwise}. \end{cases} \tag{18}$$

In the feasible case, y_0 is the unique greatest safe input.

Proof. Apply Theorem 3 to T_L: $T_L(z) \preceq y_L$ iff $b_L \preceq y_L$ and $z \preceq y_{L-1}$. Set $z = T_{L-1} \cdots T_1(x)$ and repeat. After L steps, the conjunction is exactly feasibility plus $x \preceq y_0$. □

Corollary 2 (Reverse composition of residuals). *For unbiased layers $b_l = \bot$, the right adjoint of $\mathcal{N}$ is*

$$\mathcal{N}^\flat = (W_1 \backslash \cdot) \circ \cdots \circ (W_L \backslash \cdot), \tag{19}$$

and $\mathcal{N}(x) \preceq s \iff x \preceq \mathcal{N}^\flat(s)$.

Proposition 2 (Certificate monotonicity). *If $s \preceq s'$, then every feasible backward sequence for s satisfies $y_l(s) \preceq y_l(s')$. Consequently,*

$$\downarrow y_0(s) \subseteq \downarrow y_0(s'). \tag{20}$$

Proof. Each residual $W_l \backslash (\cdot)$ is isotone. Backward induction yields the claim. □

6. PRINCIPAL-IDEAL SAFETY LATTICE

Define

$$\mathfrak{D}_n = \{\varnothing\} \cup \{\downarrow p : p \in \overline{\mathbb{R}}^n\}, \qquad \downarrow p = \{x : x \preceq p\}. \tag{21}$$

Order $\mathfrak{D}_n$ by inclusion.

Theorem 5 (Complete cap lattice). *$\mathfrak{D}_n$ is complete. For $\{p^a\}_{a \in A} \subseteq \overline{\mathbb{R}}^n$,*

$$\bigwedge_{a \in A} \downarrow p^a = \downarrow\left(\bigwedge_{a \in A} p^a\right), \tag{22}$$

$$\bigvee_{a \in A}^{\mathfrak{D}_n} \downarrow p^a = \downarrow\left(\bigvee_{a \in A} p^a\right). \tag{23}$$

The empty meet is $\overline{\mathbb{R}}^n = \downarrow(+\infty\mathbf{1})$; the empty join is $\varnothing$.

Proof. Intersection gives (22). Any principal ideal containing every $\downarrow p^a$ must contain every p^a, hence their coordinatewise supremum; this proves (23). □

For $T_{W,b}$ define the backward transformer

$$\mathscr{R}_{W,b}(\downarrow y) = \begin{cases} \downarrow(W\backslash y), & b \preceq y, \\ \varnothing, & b \not\preceq y, \end{cases} \qquad \mathscr{R}_{W,b}(\varnothing) = \varnothing. \tag{24}$$

Theorem 6 (Exact ideal transformer). *For every $D \in \mathfrak{D}_m$,*

$$\mathscr{R}_{W,b}(D) = T_{W,b}^{-1}(D). \tag{25}$$

Moreover, for any family $\{D_a\}_{a \in A} \subseteq \mathfrak{D}_m$,

$$\mathscr{R}_{W,b}\left(\bigwedge_{a \in A} D_a\right) = \bigwedge_{a \in A} \mathscr{R}_{W,b}(D_a). \tag{26}$$

Proof. Equation (25) is Theorem 3. Preimages commute with arbitrary intersections, and the meet in $\mathfrak{D}_m$ is intersection. □

Corollary 3 (Exact backward composition). *For $\mathcal{N} = T_L \circ \cdots \circ T_1$,*

$$\mathcal{N}^{-1}|_{\mathfrak{D}_{d_L}} = \mathscr{R}_{W_1,b_1} \circ \cdots \circ \mathscr{R}_{W_L,b_L}. \tag{27}$$

Thus depth introduces no abstraction loss on $\mathfrak{D}$.

Theorem 7 (Adjoint closure and kernel). *For an unbiased layer $L_W(x) = W \otimes x$ and $R_W(y) = W\backslash y$, define*

$$C_W = R_W L_W, \qquad K_W = L_W R_W. \tag{28}$$

Then

$$x \preceq C_W x, \quad C_W^2 = C_W, \quad C_W \textit{ isotone}, \tag{29}$$

$$K_W y \preceq y, \quad K_W^2 = K_W, \quad K_W \textit{ isotone}. \tag{30}$$

Furthermore,

$$\mathrm{Fix}(C_W) = \mathrm{im}(R_W), \qquad \mathrm{Fix}(K_W) = \mathrm{im}(L_W). \tag{31}$$

Proof. The adjunction $L_W x \preceq y \iff x \preceq R_W y$ gives $\mathrm{id} \le R_W L_W$ and $L_W R_W \le \mathrm{id}$. Hence $C_W^2 = R_W(L_W R_W)L_W \le C_W$, while extensivity gives the reverse inequality. The kernel case is dual. If $x = R_W y$, then $C_W x = R_W L_W R_W y = R_W y$; conversely $C_W x = x$ implies $x \in \mathrm{im}(R_W)$. The second identity is dual. □

7. CONSTRAINT AGGREGATION AND CANONICAL CAPS

Let $\{s^a\}_{a\in A} \subseteq \overline{\mathbb{R}}^{d_L}$ and define

$$\bar{s} = \bigwedge_{a\in A} s^a. \tag{32}$$

Theorem 8 (Exact conjunction collapse)**.** *For every max–plus network $\mathcal{N}$,*

$$\begin{aligned}\bigcap_{a\in A}\{x : \mathcal{N}(x) \preceq s^a\} &= \{x : \mathcal{N}(x) \preceq \bar{s}\} \\ &= \mathcal{N}^{-1}(\downarrow\bar{s}).\end{aligned} \tag{33}$$

If the backward transformer is feasible at $\bar{s}$, then

$$\mathcal{N}^{-1}(\downarrow\bar{s}) = \downarrow p(\bar{s}) = \bigcap_{a\in A} \downarrow p(s^a), \tag{34}$$

where an infeasible clause makes both sides empty.

Proof. Coordinatewise conjunction gives $\mathcal{N}(x) \preceq s^a$ for all a iff $\mathcal{N}(x) \preceq \bigwedge_a s^a$. Equation (34) follows from arbitrary-meet preservation in (26). □

For a finite family $A = \{1,\dots,q\}$, write

$$p^{-a} = p\left(\bigwedge_{c\neq a} s^c\right). \tag{35}$$

Proposition 3 (Exact redundancy test)**.** *Assume all relevant recursions are feasible. Clause a is redundant iff*

$$p^{-a} \preceq p(s^a). \tag{36}$$

Equivalently, $\downarrow p^{-a} \subseteq \downarrow p(s^a)$.

Proof. Redundancy means $\bigcap_{c\neq a} \mathcal{N}^{-1}(\downarrow s^c) \subseteq \mathcal{N}^{-1}(\downarrow s^a)$. Apply Theorem 4; inclusion of principal ideals is equivalent to the order of their generators. □

Assume now that $\mathcal{N}$ is unbiased and let $\mathcal{N}^\flat$ denote its right adjoint. Define

$$K_{\mathcal{N}} = \mathcal{N}\mathcal{N}^\flat, \qquad C_{\mathcal{N}} = \mathcal{N}^\flat\mathcal{N}. \tag{37}$$

Theorem 9 (Canonical reachable cap)**.** *For every target s,*

$$K_{\mathcal{N}}(s) \preceq s, \qquad \mathcal{N}^{-1}(\downarrow s) = \mathcal{N}^{-1}(\downarrow K_{\mathcal{N}}(s)). \tag{38}$$

Moreover, $K_{\mathcal{N}}(s)$ is the greatest element of $\mathrm{im}(\mathcal{N})$ below s:

$$K_{\mathcal{N}}(s) = \bigvee\{z \in \mathrm{im}(\mathcal{N}) : z \preceq s\}. \tag{39}$$

Proof. Adjunction gives $K_{\mathcal{N}} \leq \mathrm{id}$. Also $\mathcal{N}^\flat K_{\mathcal{N}} = \mathcal{N}^\flat\mathcal{N}\mathcal{N}^\flat = \mathcal{N}^\flat$, proving equality of preimages. If $z = \mathcal{N}(x) \preceq s$, then $x \preceq \mathcal{N}^\flat(s)$, hence $z \preceq \mathcal{N}\mathcal{N}^\flat(s) = K_{\mathcal{N}}(s)$. □

Corollary 4 (Target equivalence)**.** *For targets s, s',*

$$\mathcal{N}^{-1}(\downarrow s) = \mathcal{N}^{-1}(\downarrow s') \iff \mathcal{N}^\flat(s) = \mathcal{N}^\flat(s'). \tag{40}$$

Thus exact safety targets are quotient classes of $\overline{\mathbb{R}}^{d_L}$ under equality of residual caps.

Proof. Both preimages are principal ideals generated by the corresponding residual vectors; principal ideals have unique greatest elements. □

Proposition 4 (Tropical translation symmetry)**.** *For every $\lambda \in \mathbb{R}$ and unbiased depth-L network,*

$$\mathcal{N}(x + \lambda\mathbf{1}) = \mathcal{N}(x) + \lambda\mathbf{1}, \tag{41}$$

$$\mathcal{N}^\flat(s + \lambda\mathbf{1}) = \mathcal{N}^\flat(s) + \lambda\mathbf{1}. \tag{42}$$

Consequently,

$$\varepsilon^*(x^0 + \lambda\mathbf{1}, s + \lambda\mathbf{1}; r) = \varepsilon^*(x^0, s; r). \tag{43}$$

Proof. Max–plus matrix multiplication commutes with addition of $\lambda\mathbf{1}$. The residual identity follows directly from (13). Substitution in (50) proves (43). □

8. LOWER PREIMAGES AND DISJUNCTIVE COMPLEXITY

For $y \in \overline{\mathbb{R}}^m$, set

$$\mathcal{L}_W(y) = \{x : W \otimes x \succeq y\}. \tag{44}$$

Let $J_j = \{i : W_{ji} > -\infty\}$ and $\Theta = \prod_{j=1}^m J_j$. For $\theta \in \Theta$, define $g^\theta \in \overline{\mathbb{R}}^n$ by

$$g_i^\theta = \bigvee_{j:\theta(j)=i} (y_j - W_{ji}), \qquad \bigvee\varnothing = -\infty. \tag{45}$$

Theorem 10 (Exact lower-preimage decomposition)**.** *If every J_j is nonempty, then*

$$\mathcal{L}_W(y) = \bigcup_{\theta\in\Theta} \uparrow g^\theta, \qquad \uparrow g = \{x : g \preceq x\}. \tag{46}$$

After deleting dominated generators, the remaining g^θ are precisely the minimal generators of the upper set $\mathcal{L}_W(y)$.

Proof. For each row j, $\max_{i\in J_j}(W_{ji} + x_i) \geq y_j$ iff some $i \in J_j$ satisfies $x_i \geq y_j - W_{ji}$. Choosing one witness $\theta(j)$ per row yields $x \succeq g^\theta$. The converse is immediate. A generator is redundant exactly when another generator is coordinatewise smaller. □

Corollary 5 (Exponential antichain)**.** *For every $k \geq 1$, there exists $W \in \mathbb{R}_{\max}^{k\times 2k}$ and $y = 0$ such that $\mathcal{L}_W(y)$ has exactly 2^k pairwise incomparable minimal generators.*

Proof. Let row j contain zeros only in columns $2j-1, 2j$. Then

$$\mathcal{L}_W(0) = \bigcap_{j=1}^k \left(\{x_{2j-1} \geq 0\} \cup \{x_{2j} \geq 0\}\right). \tag{47}$$

Each binary choice selects one coordinate from every pair, producing 2^k incomparable generators in $\{0,-\infty\}^{2k}$. $\square$

Proposition 5 (Principal-filter criterion). *$\mathscr{L}_W(y)$ is a principal filter iff the set of minimal generators in (46) has one element. Equivalently, there exists g such that every feasible row-witness selection θ satisfies $g \preceq g^\theta$, with equality for at least one θ.*

Proof. An upper set generated by a finite antichain is principal exactly when its antichain has cardinality one. $\square$

9. EXACT ROBUSTNESS RADIUS

Fix $x^0 \in \mathbb{R}^n$, $r \in \mathbb{R}^n_{>0}$, and the anisotropic box

$$B_r(x^0,\varepsilon) = \{x : |x_i - x_i^0| \le \varepsilon r_i\}. \tag{48}$$

The upper-safety property is $\mathscr{N}(x) \preceq s$ for every $x \in B_r(x^0,\varepsilon)$.

Theorem 11 (Closed-form exact radius). *Assume (17) is feasible and let $p = y_0$. Then*

$$\mathscr{N}(x) \preceq s \quad \forall x \in B_r(x^0,\varepsilon) \iff x^0 + \varepsilon r \preceq p. \tag{49}$$

Let $\mathscr{E} = \{\varepsilon \ge 0 : \mathscr{N}(x) \preceq s\ \forall x \in B_r(x^0,\varepsilon)\}$. If $x^0 \preceq p$, then

$$\mathscr{E} = [0,\varepsilon^*], \qquad \varepsilon^* = \min_{1\le i\le n} \frac{p_i - x_i^0}{r_i}, \tag{50}$$

where $\varepsilon^* = +\infty$ is allowed. If $x^0 \not\preceq p$, then $\mathscr{E} = \varnothing$.

Proof. $\mathscr{N}$ is isotone. Hence the maximum output over the box occurs at $x^0 + \varepsilon r$. By Theorem 4, this upper corner is safe iff it does not exceed p. Solving the coordinate inequalities gives (50). $\square$

Corollary 6 (Standard ℓ_∞ certificate). *For $r = \mathbf{1}$,*

$$\varepsilon^* = \min_i (p_i - x_i^0) \tag{51}$$

whenever $x^0 \preceq p$.

Proposition 6 (Arithmetic complexity). *Store W_l sparsely, omitting entries equal to $-\infty$. The backward certificate and (50) require*

$$O\left(\sum_{l=1}^{L} \mathrm{nnz}(W_l) + \sum_{l=1}^{L} d_l + n\right) \tag{52}$$

comparisons and additions, and $O(\max_l d_l)$ working memory if two consecutive vectors are retained.

Proof. Each finite entry $W_{l,ji}$ contributes once to the minimum defining $(W_l \backslash y_l)_i$. Bias feasibility is checked once per output coordinate. The final minimum in (50) is linear in n. $\square$

10. CERTIFICATE GEOMETRY AND REPAIR

Assume $\mathscr{N}^{-1}(\downarrow s) = \downarrow p$ and $x^0 \preceq p$. Define

$$I^* = \arg\min_i \frac{p_i - x_i^0}{r_i}. \tag{53}$$

Proposition 7 (Active safety faces). *At $\varepsilon = \varepsilon^*$,*

$$x_i^0 + \varepsilon^* r_i = p_i \iff i \in I^*. \tag{54}$$

For every $\eta > 0$, the corner $x^0 + (\varepsilon^* + \eta) r$ is unsafe.

Proof. The first claim is the definition of I^*. For $i \in I^*$, $x_i^0 + (\varepsilon^* + \eta) r_i > p_i$; hence the corner is outside $\downarrow p$. $\square$

Let $\phi_i : [0,\infty) \to [0,\infty)$ be strictly increasing with $\phi_i(0) = 0$, and $\omega_i > 0$. For $z \preceq x$, set

$$\Phi_x(z) = \sum_{i=1}^{n} \omega_i \phi_i(x_i - z_i). \tag{55}$$

Theorem 12 (Universal optimal downward repair). *The problem*

$$\min_{z \preceq x,\ \mathscr{N}(z) \preceq s} \Phi_x(z) \tag{56}$$

has the unique solution

$$z^* = x \wedge p = \Pi_s(x). \tag{57}$$

Consequently, for every $q \in [1,\infty]$ and positive diagonal D, $\Pi_s(x)$ minimizes $\|D(x-z)\|_q$ over the same feasible set.

Proof. Feasibility is $z \preceq x \wedge p$. Thus $z_i \le z_i^*$ for all i. Strict increase of each ϕ_i gives $\Phi_x(z) \ge \Phi_x(z^*)$, with equality only for $z = z^*$. $\square$

Corollary 7 (Closed repair cost). *For $\phi_i(t) = t^q$, $1 \le q < \infty$,*

$$\min \Phi_x = \sum_i \omega_i (x_i - p_i)_+^q, \tag{58}$$

and for weighted ℓ_∞,

$$\min_z \|D(x-z)\|_\infty = \max_i D_{ii}(x_i - p_i)_+. \tag{59}$$

11. PARAMETRIC STABILITY

Assume W, W' have identical finite support and

$$\|W - W'\|_{\infty,\mathrm{supp}} = \max_{(j,i)\in \mathrm{supp}(W)} |W_{ji} - W'_{ji}| \le \delta. \tag{60}$$

Lemma 2 (Residual weight Lipschitzness). *For finite residual vectors,*

$$\|W \backslash y - W' \backslash y'\|_\infty \le \|y - y'\|_\infty + \delta. \tag{61}$$

Proof. For every i, j in the common support,

$$|(y_j - W_{ji}) - (y'_j - W'_{ji})| \le \|y - y'\|_\infty + \delta. \tag{62}$$

Taking minima over j is nonexpansive in ℓ_∞. $\square$

Theorem 13 (Deep parameter perturbation). *Let two depth-L networks have identical layer supports. Suppose*

$$\|s - s'\|_\infty \le \delta_0, \qquad \|W_l - W'_l\|_{\infty,\mathrm{supp}} \le \delta_l. \tag{63}$$

If both backward sequences are finite, then

$$\|y_l - y'_l\|_\infty \le \delta_0 + \sum_{t=l+1}^{L} \delta_t, \qquad 0 \le l \le L. \tag{64}$$

Hence, whenever both exact radii are finite,

$$|\varepsilon^* - \varepsilon'^*| \le \frac{\delta_0 + \sum_{l=1}^{L} \delta_l}{r_{\min}}. \tag{65}$$

Proof. Apply Lemma 2 recursively from $l = L$ to $l = 1$. The radius map is $1/r_{\min}$-Lipschitz in the cap vector. □

Define the feasibility margin

$$\mu = \min_{1\le l\le L} \min_j (y_{l,j} - b_{l,j}). \tag{66}$$

Corollary 8 (Robust feasibility)**.** *Assume $\mu > 0$, $\|b_l - b'_l\|_\infty \le \beta_l$, and the hypotheses of Theorem 13. The perturbed recursion remains feasible if*

$$\delta_0 + \sum_{t=l+1}^{L} \delta_t + \beta_l < \mu \quad \text{for every } l. \tag{67}$$

Proof. For every layer, $y'_{l,j} - b'_{l,j} \ge y_{l,j} - b_{l,j} - \|y_l - y'_l\|_\infty - \beta_l > 0$. □

12. PATHWISE CERTIFICATE FORMULAE

Every affine max–plus composition has the form

$$\mathcal{N}(x) = A_L \otimes x \oplus c_L, \tag{68}$$

where

$$A_1 = W_1, \qquad c_1 = b_1, \tag{69}$$
$$A_l = W_l \otimes A_{l-1}, \qquad c_l = W_l \otimes c_{l-1} \oplus b_l. \tag{70}$$

Theorem 14 (Affine path collapse)**.** *Equations (68)–(70) hold for every depth L. Moreover,*

$$(A_L)_{ji} = \bigvee_{\pi: i \rightsquigarrow j} \mathrm{wt}(\pi), \tag{71}$$

where π ranges over input–output paths and $\mathrm{wt}(\pi)$ is the sum of its edge weights. The term $(c_L)_j$ is the maximum weight of a path initiated at a bias node.

Proof. For $L = 1$ the claim is immediate. If $h_{l-1} = A_{l-1} \otimes x \oplus c_{l-1}$, distributivity gives

$$h_l = W_l \otimes h_{l-1} \oplus b_l \tag{72}$$
$$= (W_l \otimes A_{l-1}) \otimes x \oplus (W_l \otimes c_{l-1} \oplus b_l). \tag{73}$$

The matrix product maximizes sums over concatenated paths, proving (71). □

Corollary 9 (One-step cap and radius)**.** *If $c_L \preceq s$, then*

$$p_i(s) = \bigwedge_{j:(A_L)_{ji} > -\infty} (s_j - (A_L)_{ji}), \tag{74}$$

$$\varepsilon^* = \bigwedge_{i,j:(A_L)_{ji} > -\infty} \frac{s_j - (A_L)_{ji} - x_i^0}{r_i}. \tag{75}$$

If $c_L \not\preceq s$, the safety set is empty.

Proof. Apply Theorem 3 to (68) and substitute (74) into (50). □

Define the critical set

$$\Gamma^* = \operatorname*{arg\,min}_{(i,j):(A_L)_{ji} > -\infty} \frac{s_j - (A_L)_{ji} - x_i^0}{r_i}. \tag{76}$$

Theorem 15 (Critical-path witness)**.** *Let $(i, j) \in \Gamma^*$ and let $\pi : i \rightsquigarrow j$ attain (71). For every $\eta > 0$, the upper corner $x^+ = x^0 + (\varepsilon^* + \eta)r$ satisfies*

$$\mathcal{N}_j(x^+) \ge \mathrm{wt}(\pi) + x_i^+ > s_j. \tag{77}$$

Thus every radius larger than ε^* has an explicit input coordinate, output coordinate, and network path witnessing unsafety.

Proof. Criticality gives $s_j - (A_L)_{ji} - x_i^0 = \varepsilon^* r_i$. Hence $(A_L)_{ji} + x_i^+ = s_j + \eta r_i > s_j$. The path term is contained in the maximum defining $\mathcal{N}_j$. □

Proposition 8 (Sparse inequality basis)**.** *Let $I_f = \{i : p_i(s) < +\infty\}$. Choose*

$$j_i \in \operatorname*{arg\,min}_j (s_j - (A_L)_{ji}), \qquad i \in I_f. \tag{78}$$

Then the exact safety preimage is specified by $|I_f|$ scalar inequalities:

$$\mathcal{N}(x) \preceq s \iff x_i \le s_{j_i} - (A_L)_{j_i i} \quad (i \in I_f), \tag{79}$$

provided $c_L \preceq s$; coordinates outside I_f are unconstrained.

Proof. The right-hand side is exactly $x_i \le p_i(s)$ for every finite cap coordinate. □

13. ALGEBRAIC CONSEQUENCES

For $T_l(x) = W_l \otimes x \oplus b_l$, introduce homogeneous coordinates

$$\widetilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix}, \qquad \widetilde{W}_l = \begin{bmatrix} & W_l & & b_l \\ -\infty & \cdots & -\infty & 0 \end{bmatrix}. \tag{80}$$

Theorem 16 (Homogeneous collapse)**.** *Let $M = \widetilde{W}_L \otimes \cdots \otimes \widetilde{W}_1$. Then*

$$\begin{pmatrix} \mathcal{N}(x) \\ 0 \end{pmatrix} = M \otimes \widetilde{x}. \tag{81}$$

For $q = M \backslash \binom{s}{0}$,

$$\{x : \mathcal{N}(x) \preceq s\} = \begin{cases} \downarrow q_{1:n}, & 0 \le q_{n+1}, \\ \varnothing, & 0 > q_{n+1}. \end{cases} \tag{82}$$

Thus a depth-L affine max–plus network has a single residual certificate in homogeneous coordinates.

Proof. Equation (81) follows by induction from $\widetilde{W}_l \otimes (z, 0) = (T_l(z), 0)$. By Lemma 1, $(\mathcal{N}(x), 0) \preceq (s, 0)$ iff $(x, 0) \preceq q$. The latter is equivalent to $x \preceq q_{1:n}$ and $0 \le q_{n+1}$. □

Proposition 9 (Greatest safe repair)**.** *Assume the safety preimage is $\downarrow p$ and define*

$$\Pi_s(x) = x \wedge p. \tag{83}$$

Then Π_s is isotone, reductive, and idempotent; its fixed-point set is $\downarrow p$. Moreover, $\Pi_s(x)$ is the unique greatest safe vector below x.

Proof. Meet with a fixed element is isotone and reductive, and $(x \wedge p) \wedge p = x \wedge p$. Also, $\Pi_s(x) = x$ iff $x \preceq p$. If $z \preceq x, p$, then $z \preceq x \wedge p = \Pi_s(x)$. □

Theorem 17 (Target-bound stability)**.** *Assume the residual vectors are finite. If $\|s - s'\|_\infty \leq \delta$ and both backward sequences are feasible, then*

$$\left\|y_l(s) - y_l(s')\right\|_\infty \leq \delta, \qquad 0 \leq l \leq L. \tag{84}$$

If the corresponding radii are finite and $r_{\min} = \min_i r_i$, then

$$\left|\varepsilon^*(s) - \varepsilon^*(s')\right| \leq \frac{\delta}{r_{\min}}. \tag{85}$$

Proof. For every scalar λ, $W\backslash(y + \lambda\mathbf{1}) = W\backslash y + \lambda\mathbf{1}$. From $y \preceq y' + \delta\mathbf{1}$ and $y' \preceq y + \delta\mathbf{1}$, isotonicity gives $\|W\backslash y - W\backslash y'\|_\infty \leq \delta$. Backward induction proves (84). Finally, the minimum in (50) is $1/r_{\min}$-Lipschitz in p. □

14. EXACT SAFE-REGION MEASURE

Let $Q = [a, u] \subset \mathbb{R}^n$ with $a_i < u_i$, and let

$$Q_s = Q \cap \{x : \mathcal{N}(x) \preceq s\}. \tag{86}$$

Theorem 18 (Exact bounded safe region)**.** *Assume $\mathcal{N}^{-1}(\downarrow s) = \downarrow p$ and put $v = u \wedge p$. Then*

$$Q_s = \begin{cases} [a, v], & a \preceq v, \\ \varnothing, & a \npreceq v. \end{cases} \tag{87}$$

Its normalized Lebesgue measure is

$$\frac{\operatorname{vol}(Q_s)}{\operatorname{vol}(Q)} = \prod_{i=1}^{n} \left[\frac{\min\{u_i, p_i\} - a_i}{u_i - a_i}\right]_{[0,1]}, \tag{88}$$

where $[t]_{[0,1]} = \min\{1, \max\{0, t\}\}$.

Proof. Intersect $[a, u]$ with $\downarrow p$ coordinatewise. The volume of a nonempty box is the product of its side lengths; division by $\prod_i (u_i - a_i)$ gives (88). □

Corollary 10 (Product-measure safety mass)**.** *Let X_i be independent with continuous distribution functions F_i, conditioned on $a_i \leq X_i \leq u_i$. Then*

$$\Pr(\mathcal{N}(X) \preceq s) = \prod_{i=1}^{n} \frac{F_i(\min\{u_i, p_i\}) - F_i(a_i)}{F_i(u_i) - F_i(a_i)}, \tag{89}$$

with a zero factor whenever $p_i < a_i$.

Proof. Use (87) and independence. □

Proposition 10 (Interior sensitivity)**.** *If $a_i < p_i < u_i$ for $i \in I$ and $p_i \geq u_i$ otherwise, then*

$$\frac{\partial}{\partial p_i} \log \operatorname{vol}(Q_s) = \frac{1}{p_i - a_i}, \qquad i \in I. \tag{90}$$

Proof. Here $\operatorname{vol}(Q_s) = \prod_{i \in I}(p_i - a_i) \prod_{i \notin I}(u_i - a_i)$; differentiate its logarithm. □

15. LIMITS AND OPEN PROBLEMS

The calculus is exact for upper principal-ideal specifications and monotone max–plus layers. It does not convert arbitrary class-margin constraints into principal ideals, and signed affine maps destroy isotonicity.

Open Problem 1 (Signed decomposition). Characterize the smallest ordered product domain on which a signed piecewise-linear layer admits an exact residuated factorization with polynomial representation size.

Open Problem 2 (Two-sided classification margins). Given logits $z = \mathcal{N}(x)$, determine when $z_c - z_j \geq \delta$ for all $j \neq c$ has an exact residual representation without exponential disjunction.

Open Problem 3 (Residuated attention). Construct an attention operator that preserves arbitrary joins and admits a computable right adjoint.

Open Problem 4 (Complete relational domains). Find non-box Galois insertions for which monotone or tropical neural layers are forward complete, so that relational invariants remain exact under depth.

Open Problem 5 (Minimal residual realization). Given an isotone piecewise-linear map F, decide whether F admits a polynomial-size max–plus realization whose residual computes every upper preimage exactly.

Open Problem 6 (Antichain compression). Determine the optimal symbolic representation of the minimal generators in (46); characterize matrices for which the antichain size is polynomial in $m + n$.

Open Problem 7 (Support-changing stability). Extend (64) to perturbations that create or delete finite max–plus edges, using an extended metric on $\overline{\mathbb{R}}$.

16. CONCLUSION

The exact calculus is summarized by

$$\alpha_\square F([\ell, u]) = [F(\ell), F(u)], \qquad \mathcal{N}^{-1}(\downarrow s) = \downarrow p(s), \tag{91}$$

with emptiness determined by bias feasibility. It yields

$$\varepsilon^* = \min_i \frac{p_i - x_i^0}{r_i}, \qquad \Pi_s(x) = x \wedge p, \tag{92}$$

plus lattice, complexity, and perturbation guarantees. Lower bounds expose the exact boundary: upper preimages are principal; lower preimages may require exponentially many generators. The remaining problems concern residuated expressivity, compact antichains, signed maps, and support-changing perturbations.

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