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FULL LENGTH ARTICLE

Interval-Lattice Fixed-Point Semantics for Certified Implicit Hypergraph Neural Operators

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ABSTRACT

Implicit hypergraph models represent higher-order propagation by an equilibrium equation, but standard well-posedness arguments typically force a contraction and therefore exclude noncontractive yet order-preserving dynamics. This paper introduces an interval-lattice semantics for implicit hypergraph neural operators. On the box lattice $\mathscr{L} = \{X \in \mathbb{R}^{n\times d} : \ell \le X_{ij} \le u\}$, a normalized hypergraph propagation $P_{\mathscr{H}} \ge 0$, an entrywise nonnegative channel map $A \ge 0$, and an isotone activation generate an order-preserving operator $\mathscr{T} : \mathscr{L} \to \mathscr{L}$. The Knaster–Tarski theorem then yields a nonempty complete lattice of equilibria without requiring $\|A\| < 1$. Coupled iterations from the bottom and top elements produce certified lower and upper enclosures for every equilibrium. Under the optional metric condition $q = L_\sigma \|P_{\mathscr{H}}\|_2 \|A\|_2 < 1$, the extremal equilibria coincide; geometric convergence, a residual-to-solution certificate, and a structural perturbation bound follow. Permutation equivariance and monotone dependence on input features are also proved. Two small arithmetic tables illustrate certificate scaling rather than empirical performance. The paper concludes with seven open problems on noncontractive uniqueness, signed hypergraphs, finite certificate complexity, topology-aware perturbation metrics, expressivity, differentiable extremal selection, and asynchronous lattice iteration.

Keywords: Complete lattice ▪ Fixed-point semantics ▪ Hypergraph neural operator ▪ Monotone map ▪ Certified inference ▪ Implicit graph learning ▪ Open problems

1. INTRODUCTION

Let $\mathscr{H} = (V, \mathscr{E}, w)$ be a weighted hypergraph with incidence matrix $B \in \{0,1\}^{n\times m}$. Hypergraph neural networks replace pairwise propagation by node–hyperedge–node transport and thereby encode higher-order relations [1, 2, 3, 4]. Implicit models instead define the representation as a fixed point $X^* = \mathscr{T}(X^*)$, corresponding formally to infinite-depth weight sharing [5, 6, 7]. These two ideas were joined in recent implicit hypergraph architectures [8, 9]; their well-posedness mechanisms are metric, commonly based on projection, spectral control, or contraction.

Metric uniqueness is sufficient but not logically necessary for existence. If $\mathscr{T}$ is isotone on a complete lattice $(\mathscr{L}, \preceq)$, then

$$X \preceq Y \Longrightarrow \mathscr{T}(X) \preceq \mathscr{T}(Y), \qquad \mathrm{Fix}(\mathscr{T}) \ne \varnothing, \tag{1}$$

by the Knaster–Tarski principle [10, 11, 12]. This separates two questions:

$$\begin{aligned} \text{order layer:}\quad & \text{existence and extrema,} \\ \text{metric layer:}\quad & \text{uniqueness and rates.} \end{aligned} \tag{2}$$

The separation is useful when $q \ge 1$ prevents a contraction argument but positivity still preserves order.

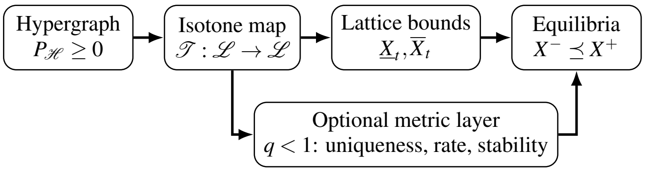

Figure 1. Order theory certifies existence and enclosures; contraction is an optional refinement.

Table 1. Fixed-point principles in adjacent frameworks.

Framework	Structure	Principle	Certified object
DEQ [5]	sequence	root finding	computed equilibrium
IGNN [6]	graph	Perron–Frobenius	unique state
monDEQ [7]	general	monotone operator	unique state
Monotone IGNN [19]	graph	operator splitting	well-posed state
Implicit HNN [8, 9]	hypergraph	metric control	stable state
This paper	hypergraph	Tarski + optional Banach	complete fixed-point lattice; interval enclosure

This paper contributes:

(i) an implicit hypergraph operator $\mathscr{T}_{\mathscr{H},\theta}$ on the product lattice $\mathscr{L} = [\ell, u]^{n\times d}$ and a proof that $P_{\mathscr{H}} \geq 0$, $A \geq 0$, and isotone σ imply $\mathscr{T}_{\mathscr{H},\theta}$ is isotone;

(ii) extremal equilibria $X^- \preceq X^+$ and certified iterates $\underline{X}_t \preceq X^* \preceq \overline{X}_t$ for every $X^* \in \mathrm{Fix}(\mathscr{T})$, without assuming $q < 1$;

(iii) the metric refinement $q = L_\sigma \|P_{\mathscr{H}}\|_2 \|A\|_2 < 1$, yielding $X^- = X^* = X^+$, $\|Z - X^*\|_F \leq \|Z - \mathscr{T}(Z)\|_F / (1 - q)$, and a hypergraph perturbation certificate;

(iv) permutation equivariance, monotone input dependence, compact proof-of-concept arithmetic, and seven open problems defining a theoretical research program.

2. RELATED WORK

Classical hypergraph diffusion uses the normalized incidence operator introduced for spectral learning [1]. HGNN, HyperGCN, UniGNN, AllSet, and sheaf hypergraph networks broaden the propagation and aggregation families [2, 3, 13, 4, 14]. Transferability and oversquashing have received recent theoretical treatment [15, 16]; finite-depth graph stability and transferability are also well established [17, 18]. These models are predominantly finite-depth.

Deep equilibrium models replace depth by a root or fixed-point problem [5]. IGNN uses Perron–Frobenius restrictions [6]; monDEQ and subsequent implicit GNN analyses use monotone-operator or splitting theory [7, 19, 20]. Recent implicit hypergraph proposals establish stable higher-order equilibria through metric well-posedness and then evaluate learned models [8, 9]. The present framework is not an alternative training architecture. It is a semantic layer: $\mathrm{Fix}(\mathscr{T})$ is studied first as an ordered set and only then, if desired, as a singleton in a normed space.

3. HYPERGRAPH OPERATOR ON A BOX LATTICE

Let $W = \mathrm{diag}(w_e)$, $D_e = \mathrm{diag}(|e|)$, and $D_v = \mathrm{diag}(\sum_e w_e B_{ve})$. Assume no isolated vertices and define

$$P_{\mathscr{H}} = D_v^{-1/2} B W D_e^{-1} B^\top D_v^{-1/2}. \tag{3}$$

Then $P_{\mathscr{H}} = P_{\mathscr{H}}^\top \succeq 0$, $P_{\mathscr{H}} \geq 0$ entrywise, and $\|P_{\mathscr{H}}\|_2 = 1$. For $\ell < u$, let

$$\mathscr{L} = [\ell, u]^{n\times d}, \qquad X \preceq Y \iff X_{ij} \leq Y_{ij}\ \forall (i,j). \tag{4}$$

The lattice operations are coordinatewise $\wedge$ and $\vee$. Let $U \in \mathbb{R}^{n\times p}$ be an input, $A \in \mathbb{R}^{d\times d}$, $C \in \mathbb{R}^{p\times d}$, $b \in \mathbb{R}^d$, and let σ act coordinatewise. Define

$$\mathscr{T}_{\mathscr{H},\theta}(X; U) = \mathrm{clip}_{[\ell,u]}\Big(\sigma(P_{\mathscr{H}} X A + UC + \mathbf{1} b^\top)\Big). \tag{5}$$

Definition 1 (Order-admissible parameters)**.** *The tuple $\theta = (A, C, b, \sigma, \ell, u)$ is order-admissible if $A \geq 0$ entrywise and $\sigma : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing. It is input-isotone if additionally $C \geq 0$ entrywise.*

Lemma 1 (Isotonicity)**.** *For order-admissible θ, $X \preceq Y$ implies $\mathscr{T}(X; U) \preceq \mathscr{T}(Y; U)$. If θ is input-isotone, $U \preceq U'$ also implies $\mathscr{T}(X; U) \preceq \mathscr{T}(X; U')$.*

Proof. Because $P_{\mathscr{H}} \geq 0$ and $A \geq 0$, $X \preceq Y$ gives $P_{\mathscr{H}} X A \preceq P_{\mathscr{H}} Y A$. Coordinatewise σ and clipping preserve order. The input claim follows from $C \geq 0$. □

4. EXTREMAL FIXED POINTS AND ENCLOSURES

Let $\bot = \ell \mathbf{1}\mathbf{1}_d^\top$ and $\top = u \mathbf{1}\mathbf{1}_d^\top$.

Theorem 1 (Lattice well-posedness)**.** *For order-admissible θ, $\mathrm{Fix}(\mathscr{T})$ is a nonempty complete lattice. It has least and greatest elements X^- and X^+. Moreover,*

$$\underline{X}_0 = \bot, \qquad \underline{X}_{t+1} = \mathscr{T}(\underline{X}_t; U), \tag{6}$$

$$\overline{X}_0 = \top, \qquad \overline{X}_{t+1} = \mathscr{T}(\overline{X}_t; U), \tag{7}$$

converge coordinatewise to X^- and X^+, respectively.

Proof. The box $\mathscr{L}$ is a complete lattice and Lemma 1 makes $\mathscr{T}$ isotone; Tarski's theorem yields the complete fixed-point lattice [10]. Since $\bot \preceq \mathscr{T}(\bot)$ and $\mathscr{T}(\top) \preceq \top$, the two sequences are monotone and bounded. Their coordinatewise limits exist. Continuity of $\mathscr{T}$ makes the limits fixed points. Minimality and maximality follow by induction against any $X \in \mathrm{Fix}(\mathscr{T})$. □

Corollary 1 (Proof-carrying interval)**.** *For every $t \geq 0$ and $X^* \in \mathrm{Fix}(\mathscr{T})$,*

$$\underline{X}_t \preceq X^- \preceq X^* \preceq X^+ \preceq \overline{X}_t. \tag{8}$$

Hence $\mathscr{I}_t = [\underline{X}_t, \overline{X}_t]$ certifies all admissible equilibria, and $\mathrm{diam}_F\, \mathrm{Fix}(\mathscr{T}) \leq \left\|\overline{X}_t - \underline{X}_t\right\|_F$.

Corollary 2 (Monotone input semantics)**.** *For input-isotone θ and $U \preceq U'$, the extremal solutions satisfy $X^-(U) \preceq X^-(U')$ and $X^+(U) \preceq X^+(U')$.*

Proof. Apply induction to the lower and upper iterations and pass to the limits. □

Corollary 3 (Certified monotone readout)**.** *Let $r : \mathscr{L} \to \mathbb{R}^k$ be isotone. Then every $X^* \in \mathrm{Fix}(\mathscr{T})$ obeys*

$$r(\underline{X}_t) \preceq r(X^*) \preceq r(\overline{X}_t). \tag{9}$$

For a scalar threshold rule $\mathbf{1}\{r(X^*) \geq \gamma\}$, $r(\underline{X}_t) \geq \gamma$ certifies the positive decision for all equilibria, while $r(\overline{X}_t) < \gamma$ certifies the negative decision.

5. METRIC REFINEMENT AND CERTIFICATES

Assume σ is L_σ-Lipschitz and put

$$q = L_\sigma \|P_{\mathscr{H}}\|_2 \|A\|_2 = L_\sigma \|A\|_2 . \tag{10}$$

Clipping is nonexpansive in Frobenius norm.

Theorem 2 (Uniqueness and rate)**.** *If $q < 1$, then $X^- = X^+ = X^*$ and, for any $X_0 \in \mathscr{L}$,*

$$\|X_t - X^*\|_F \leq q^t \|X_0 - X^*\|_F . \tag{11}$$

In particular, $\left\|\overline{X}_t - \underline{X}_t\right\|_F \leq q^t \|\top - \bot\|_F$.

Proof. Equation (5) gives $\|\mathscr{T}(X) - \mathscr{T}(Y)\|_F \leq q\|X - Y\|_F$. The Banach fixed-point theorem yields uniqueness and (11); the interval estimate follows by applying the contraction to both sequences [21]. □

Proposition 1 (Residual certificate)**.** *Under $q < 1$, every $Z \in \mathscr{L}$ satisfies*

$$\|Z - X^*\|_F \leq \frac{\|Z - \mathscr{T}(Z;U)\|_F}{1-q}. \tag{12}$$

Proof. Use $\|Z - X^*\| \leq \|Z - \mathscr{T}(Z)\| + q\|Z - X^*\|$. □

Proposition 2 (Damping)**.** *For $\alpha \in (0,1]$, let $\mathscr{T}_\alpha = (1-\alpha)I + \alpha\mathscr{T}$. Then $\mathrm{Fix}(\mathscr{T}_\alpha) = \mathrm{Fix}(\mathscr{T})$ and $\mathscr{T}_\alpha$ is isotone. If $q < 1$, its contraction factor is at most*

$$q_\alpha = 1 - \alpha(1-q). \tag{13}$$

Theorem 3 (Structural perturbation)**.** *Let $P, \widetilde{P}$ be two normalized nonnegative hypergraph propagations and let $X^*, \widetilde{X}^*$ be the unique fixed points of (5) with identical U, θ. If $q = L_\sigma \|A\|_2 < 1$ and $R = \max_{X \in \mathscr{L}} \|X\|_F$, then*

$$\left\|\widetilde{X}^* - X^*\right\|_F \leq \frac{L_\sigma \|A\|_2 R}{1-q} \left\|\widetilde{P} - P\right\|_2 . \tag{14}$$

For simultaneous input perturbation ΔU and bias perturbation Δb, the right side gains $\frac{L_\sigma}{1-q}(\|C\|_2 \|\Delta U\|_F + \sqrt{n}\|\Delta b\|_2)$.

Proof. Insert and subtract $\mathscr{T}_P(\widetilde{X}^*;U)$, apply Lipschitz continuity, and rearrange:

$$\left\|\widetilde{X}^* - X^*\right\|_F \leq q\left\|\widetilde{X}^* - X^*\right\|_F + L_\sigma \left\|\widetilde{P} - P\right\|_2 \left\|\widetilde{X}^* A\right\|_F .$$

Bound $\left\|\widetilde{X}^* A\right\|_F \leq R\|A\|_2$. The additional terms are analogous. □

6. EQUIVARIANCE

Theorem 4 (Relabeling equivariance)**.** *Let Q be an $n \times n$ permutation matrix. Under vertex relabeling, $P_{Q\mathscr{H}} = QP_{\mathscr{H}}Q^\top$ and $U_Q = QU$. Then*

$$\mathscr{T}_{Q\mathscr{H}}(QX;QU) = Q\mathscr{T}_{\mathscr{H}}(X;U). \tag{15}$$

Consequently, QX^- and QX^+ are the extremal equilibria of the relabeled system; when $q < 1$, $X_Q^* = QX^*$.

Proof. Substitute $QPQ^\top$, QX, and QU in (5). Rowwise activation and clipping commute with Q. Extremality is preserved because Q is an order isomorphism. □

7. PROOF-OF-CONCEPT ARITHMETIC

The following calculations instantiate the proven formulas; they are not dataset experiments. Normalize the initial interval width to one. For tolerance τ, Theorem 2 gives

$$N_\tau(q) = \left\lceil \frac{\log \tau}{\log q} \right\rceil, \qquad M(q) = \frac{1}{1-q}. \tag{16}$$

Table 2. Contraction arithmetic for $\tau = 10^{-3}$.

q	$N_{10^{-3}}$	q^{10}	Residual multiplier $M(q)$
0.25	5	9.54×10^{-7}	1.333
0.50	10	9.77×10^{-4}	2.000
0.75	25	5.63×10^{-2}	4.000
0.90	66	3.49×10^{-1}	10.000

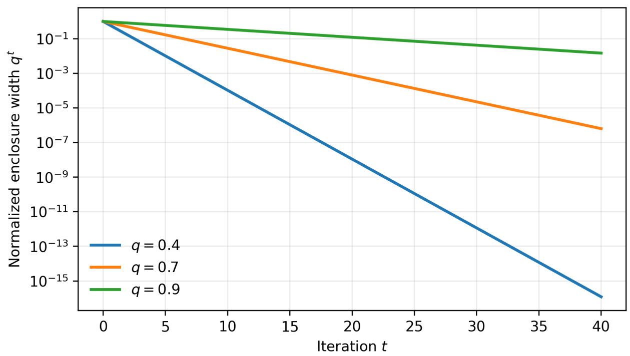

Figure 2. Geometric enclosure bound q^t from Theorem 2.

For $\left\|P - \widetilde{P}\right\|_2 = \delta_P$, $R = 1$, and $L_\sigma \|A\|_2 = q$, (14) reduces to $\left\|\widetilde{X}^* - X^*\right\|_F \leq [q/(1-q)]\delta_P$.

Table 3. Structural certificate $[q/(1-q)]\delta_P$.

q	$\delta_P = 10^{-3}$	10^{-2}	5×10^{-2}
0.40	6.67×10^{-4}	6.67×10^{-3}	3.33×10^{-2}
0.70	2.33×10^{-3}	2.33×10^{-2}	1.17×10^{-1}
0.90	9.00×10^{-3}	9.00×10^{-2}	4.50×10^{-1}

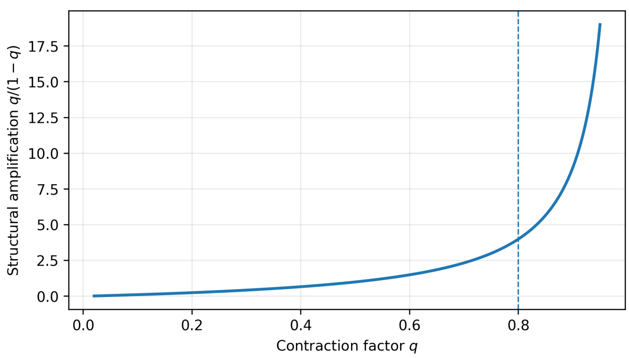

Figure 3. The factor $q/(1-q)$ exposes the stability cost of approaching the uniqueness boundary.

8. OPEN PROBLEMS

Open Problem 1 (Noncontractive uniqueness). Find verifiable conditions weaker than $q < 1$ for

$$X^- = X^+ \iff \mathrm{Fix}(\mathscr{T}) = \{X^*\}. \tag{17}$$

Candidates include strict isotonicity, order irreducibility, concavity, and nonlinear Perron–Frobenius conditions. The challenge is to obtain uniqueness without suppressing long-range propagation.

Open Problem 2 (Signed and directed hypergraphs). When $P_{\mathscr{H}}$ has negative or asymmetric entries, the standard positive cone fails. Construct a cone $K \subset \mathbb{R}^{n\times d}$ such that $X \preceq_K Y \Rightarrow \mathscr{T}(X) \preceq_K \mathscr{T}(Y)$, or characterize eventual positivity $\mathscr{T}^r(K) \subseteq K$.

Open Problem 3 (Finite certificate complexity). Without contraction, Theorem 1 gives asymptotic enclosure but no universal rate. For piecewise-linear σ, bound

$$\tau(\varepsilon) = \min\{t : \|\overline{X}_t - \underline{X}_t\|_\infty \leq \varepsilon\} \tag{18}$$

by lattice dimension, activation regions, hypergraph diameter, or a discrete height parameter.

Open Problem 4 (Topology-aware perturbation). Relate the analytic distance $\|P_{\mathscr{H}} - P_{\widetilde{\mathscr{H}}}\|_2$ in (14) to an edit metric on weighted hyperedges. A desired inequality is

$$\|P_{\mathscr{H}} - P_{\widetilde{\mathscr{H}}}\|_2 \leq \Phi(d_{\mathrm{edit}}(\mathscr{H}, \widetilde{\mathscr{H}}), d_{\min}, r_{\max}), \tag{19}$$

with explicit dependence on minimum degree and maximum hyperedge rank.

Open Problem 5 (Expressivity versus order). Entrywise $A \geq 0$ ensures isotonicity but may restrict signed feature interactions. Determine the distinguishing power of order-admissible implicit operators relative to hypergraph Weisfeiler–Leman hierarchies and multiset architectures [4]. The central tension is between order preservation, depth–width limitations, and long-range bottlenecks [22, 23, 24]. Is there a minimal cone-preserving lift that recovers signed expressivity?

Open Problem 6 (Differentiable extremal selection). When $X^- \neq X^+$, learning requires a selection rule. Study the regularity of $\theta \mapsto X^-(\theta)$ and $\theta \mapsto X^+(\theta)$, generalized derivatives at bifurcations, and whether interval-valued gradients can preserve the enclosure guarantee.

Open Problem 7 (Asynchronous lattice iteration). Let I_t be a fair block schedule and let delayed coordinates satisfy $0 \leq t - \tau_{ij}(t) \leq D$. For updates

$$X_{I_t}^{t+1} = \mathscr{T}_{I_t}(X^{\tau(t)}), \qquad X_{I_t^c}^{t+1} = X_{I_t^c}^{t}, \tag{20}$$

derive computable lower–upper enclosures and convergence conditions under bounded delay. Contraction-based asynchronous theory is classical [25]; the noncontractive complete-lattice case remains open.

9. DISCUSSION

The framework draws a strict boundary between three guarantees. First, isotonicity gives existence and extremal semantics through Theorem 1. Second, the interval $\mathscr{I}_t$ gives an auditable certificate even if the equilibrium is nonunique. Third, $q < 1$ collapses the interval to a unique state and supplies quantitative rates and perturbation bounds. Thus, contraction is a certificate-strengthening assumption rather than the definition of well-posedness.

The price is positivity. Standard nonnegative hypergraph diffusion fits the order axioms, but signed attention, subtractive channels, and antisymmetric operators generally do not. Cone orders, decompositions $A = A^+ - A^-$, and lifted states (X^+, X^-) are possible routes, but each changes the meaning and complexity of the certificate. The open problems therefore concern not only tighter bounds but the correct mathematical semantics for implicit higher-order computation.

10. CONCLUSION

An interval-lattice semantics was developed for implicit hypergraph neural operators. Order-admissible parameters yield a complete lattice of fixed points and computable extremal enclosures without contraction. The metric condition $q < 1$ adds uniqueness, geometric convergence, residual certification, and structural stability. Relabeling equivariance and monotone input dependence are preserved. The resulting theory is a proof-of-concept framework rather than an experimental model; its main output is a set of verifiable statements and open mathematical questions for certified higher-order neural computation.

REFERENCES

[1] D. Zhou, J. Huang, and B. Schölkopf, "Learning with hypergraphs: Clustering, classification, and embedding," *Advances in Neural Information Processing Systems*, vol. 19, pp. 1601–1608, 2006.

[2] Y. Feng, H. You, Z. Zhang, R. Ji, and Y. Gao, "Hypergraph neural networks," *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 33, no. 1, pp. 3558–3565, 2019.

[3] N. Yadati, M. Nimishakavi, P. Yadav, V. Nitin, A. Louis, and P. Talukdar, "HyperGCN: A new method for training graph convolutional networks on hypergraphs," in *Advances in Neural Information Processing Systems 32*, 2019.

[4] E. Chien, C. Pan, J. Peng, and O. Milenkovic, "You are allset: A multiset function framework for hypergraph neural networks," in *International Conference on Learning Representations*, 2022.

[5] S. Bai, J. Z. Kolter, and V. Koltun, "Deep equilibrium models," in *Advances in Neural Information Processing Systems 32*, 2019.

[6] F. Gu, H. Chang, W. Zhu, S. Sojoudi, and L. E. Ghaoui, "Implicit graph neural networks," in *Advances in Neural Information Processing Systems 33*, 2020.

[7] E. Winston and J. Z. Kolter, "Monotone operator equilibrium networks," in *Advances in Neural Information Processing Systems 33*, 2020.

[8] X. Li, G. Tang, and J. Jiang, "Implicit hypergraph neural networks: A stable framework for higher-order

relational learning with provable guarantees," *arXiv preprint arXiv:2508.09427*, 2025.

[9] A. Choudhuri, Y. Zhong, and B. Adhikari, "Implicit hypergraph neural network," *arXiv preprint arXiv:2508.14101*, 2025.

[10] A. Tarski, "A lattice-theoretical fixpoint theorem and its applications," *Pacific Journal of Mathematics*, vol. 5, no. 2, pp. 285–309, 1955.

[11] G. Birkhoff, *Lattice Theory*, 3rd ed., ser. American Mathematical Society Colloquium Publications. Providence, RI: American Mathematical Society, 1967, vol. 25.

[12] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, 2nd ed. Cambridge, UK: Cambridge University Press, 2002.

[13] J. Huang and J. Yang, "UniGNN: A unified framework for graph and hypergraph neural networks," in *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence*, 2021, pp. 2274–2280.

[14] I. Duta, G. Cassarà, F. Silvestri, and P. Liò, "Sheaf hypergraph networks," in *Advances in Neural Information Processing Systems 36*, 2023.

[15] M. Hayhoe, H. M. Riess, M. M. Zavlanos, V. M. Preciado, and A. Ribeiro, "Transferable hypergraph neural networks via spectral similarity," in *Proceedings of the Second Learning on Graphs Conference*, ser. Proceedings of Machine Learning Research, vol. 231, 2024, pp. 18:1–18:23.

[16] N. Yadati, "Oversquashing in hypergraph neural networks," in *Proceedings of the Third Learning on Graphs Conference*, ser. Proceedings of Machine Learning Research, vol. 269, 2025, pp. 30:1–30:16.

[17] F. Gama, J. Bruna, and A. Ribeiro, "Stability properties of graph neural networks," *IEEE Transactions on Signal Processing*, vol. 68, pp. 5680–5695, 2020.

[18] L. Ruiz, F. Gama, and A. Ribeiro, "Graph neural networks: Architectures, stability, and transferability," *Proceedings of the IEEE*, vol. 109, no. 5, pp. 660–682, 2021.

[19] J. Baker, Q. Wang, C. D. Hauck, and B. Wang, "Implicit graph neural networks: A monotone operator viewpoint," in *Proceedings of the 40th International Conference on Machine Learning*, ser. Proceedings of Machine Learning Research, vol. 202, 2023, pp. 1521–1548.

[20] J. M. Baker, Q. Wang, M. Berzins, T. Strohmer, and B. Wang, "Monotone operator theory-inspired message passing for learning long-range interaction on graphs," in *Proceedings of the 27th International Conference on Artificial Intelligence and Statistics*, ser. Proceedings of Machine Learning Research, vol. 238, 2024, pp. 2233–2241.

[21] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 2nd ed. Cham, Switzerland: Springer, 2017.

[22] A. Loukas, "What graph neural networks cannot learn: Depth vs width," in *International Conference on Learning Representations*, 2020.

[23] J. Topping, F. D. Giovanni, B. P. Chamberlain, X. Dong, and M. M. Bronstein, "Understanding over-squashing and bottlenecks on graphs via curvature," in *International Conference on Learning Representations*, 2022.

[24] M. M. Bronstein, J. Bruna, T. Cohen, and P. Veličković, "Geometric deep learning: Grids, groups, graphs, geodesics, and gauges," *IEEE Signal Processing Magazine*, vol. 38, no. 4, pp. 18–42, 2021.

[25] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Englewood Cliffs, NJ: Prentice-Hall, 1989.