Generalized Marcinkiewicz Operators Aassociated to Twisted Surfaces

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Abstract

This paper is concerned with studying the mapping properties of generalized Marcinkiewicz integral operators along twisted surfaces. Under certain conditions on these surfaces we obtain certain L^p estimates for these operators provided that the kernel functions are in $L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)$. By an extrapolation argument, we prove that these operators are bounded on $L^p(\mathbb{R}^n\times\mathbb{R}^m)$ for $1<p<\infty$ under very weak conditions on the kernel functions. Our results extend and improve many previously known results.

Keywords: Generalized Marcinkiewicz operator; Twisted surfaces; Rough kernels

1 Introduction

For $d\geq 2$, let $\mathbb{R}^d$ ($d=n$ or m) be the Euclidean space of dimension d and $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ equipped with the normalized Lebesgue surface measure $d\sigma_d(\cdot)$.

Let Ω be an integrable function over $\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}$ such that

$$\Omega(rx,sy) \quad=\quad \Omega(x,y),\;\;\forall\, r,s>0 \tag{1}$$

and

$$\int_{\mathbb{S}^{n-1}}\Omega(x,y)d\sigma_n(x) \quad=\quad \int_{\mathbb{S}^{m-1}}\Omega(x,y)d\sigma_m(y)=0. \tag{2}$$

For an appropriate mapping $\Phi:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n\times\mathbb{R}^m$, a measurable function $g\in L^\infty(\mathbb{R}_+\times\mathbb{R}_+)$ and a function $\Omega\in L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ satisfying (1)-(2), we consider the generalized Marcinkiewicz integral operator $\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}$ along the surface $\{\Phi(u,v):(u,v)\in\mathbb{R}^n\times\mathbb{R}^m\}$ given by

$$\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}(f)(x,y)=\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|T_{r,s}(f)(x,y)\right|^{\lambda}\frac{drds}{rs}\right)^{1/\lambda}, \tag{3}$$

where $f\in C_0^\infty(\mathbb{R}^n\times\mathbb{R}^m)$, $\lambda>1$, and

$$T_{r,s}(f)(x,y)=\frac{1}{rs}\int_{|v|\leq s}\int_{|u|\leq r}f\left((x,y)-\Phi(u,v)\right)\frac{\Omega(u,v)g(|u|,|v|)}{|u|^{n-1}\,|v|^{m-1}}dudv.$$

When $\lambda = 2$, $\Phi(u,v) \equiv \mathrm{I}(u,v) = (u,v)$ and $g \equiv 1$, the operator $\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}$, denoted by $\mathcal{M}_\Omega$, is the classical Marcinkiewicz operator on product domains. Historically, the boundedness of $\mathcal{M}_\Omega$ was first studied in[1] in which the author obtained the L^2 boundedness of $\mathcal{M}_\Omega$ whenever $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. This result was later improved and extended by many authors, see,[23] and,[4] among others. Let us now recall two relevant results to our current study. The first one is the result in[4] in which the authors proved that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1,\infty)$ under the condition $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and that this condition be the weakest condition in the sense that the L^2 boundedness of $\mathcal{M}_\Omega$ will not be satisfied if this condition is replaced by any weaker condition $\Omega \in L(\log L)^\varepsilon(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ with $0 < \varepsilon < 1$. The second result is that under the condition $\Omega \in B^{(0,0)}_q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, the author of[5] established the L^p $(1 < p < \infty)$ boundedness of $\mathcal{M}_\Omega$, and proved the optimality of this condition. Here $B^{(0,0)}_q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ is a special class of block spaces introduced in.[6] For a background information,[7–10] development and applications[11–15] and recent studies.[16–18]

On the other hand, as a natural extension of $\mathcal{M}_\Omega$ is $\mathcal{G}^{(2)}_{\Phi,\Omega,g}$ for certain classes of functions Φ. If the classes of functions Φ are in the standard form $\Phi(u,v) = (\phi_1(u), \phi_2(v))$, where $\phi_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $\phi_2 : \mathbb{R}^m \to \mathbb{R}^m$ are functions satisfy some certain conditions, the L^p boundedness of $\mathcal{G}^{(2)}_{\Phi,\Omega,g}$ has been investigated by many authors. For a sampling of past studies, readers are referred to[19–22] and the references therein.

Recently, Al-Salman[23] introduced Marcinkiewicz operator $\mathcal{G}^{(2)}_{\Phi,\Omega,g}$ along the twisted surfaces

$$\{\Gamma(u,v) = \Phi(u,v) : (u,v) \in \mathbb{R}^n \times \mathbb{R}^m)\},$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ satisfies some specific conditions.

The L^p boundedness of the generalized Marcinkiewicz operator $\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}$ has been investigated by many authors. For instance, the author of[24] confirmed L^p boundedness of $\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}$ for all $p \in (1,\infty)$ provided that $\Phi \equiv \mathrm{I}$, $\Omega \in L(\log L)^{2/\lambda}(\mathbb{S}^n \times \mathbb{S}^{m-1})$ with $\lambda > 1$. This result was extended and improved in.[25] Precisely, the authors showed that the estimate

$$\left\|\mathcal{G}^{(\lambda)}_{\Phi,\Omega,1}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_p \|f\|_{\dot{F}^{\lambda,\vec{0}}_p(\mathbb{R}^n\times\mathbb{R}^m)}$$

holds for $1 < p < \infty$ whenever $\Phi \equiv \mathrm{I}$, $\Omega \in L(\log L)^{2/\lambda}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cup B^{(0,\frac{2}{\lambda}-1)}_q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ and $\lambda > 1$. Thereafter, L^p boundedness of $\mathcal{G}^{(\lambda)}_{\Phi,\Omega,1}$ was investigated under various conditions on Ω, g and the mapping Φ is in the standard form $\Phi(u,v) = (\phi_1(u), \phi_2(v))$ as can be seen in,[26–28] among others.

Here, $\dot{F}^{\lambda,\vec{\tau}}_p(\mathbb{R}^n \times \mathbb{R}^m)$ denotes the space of homogeneous Triebel-Lizorkin functions which is defined as follows: For $\lambda, p \in (1,\infty)$ and $\vec{\tau} = (\tau_1, \tau_2) \in \mathbb{R} \times \mathbb{R}$, the space $\dot{F}^{\lambda,\vec{\tau}}_p(\mathbb{R}^n \times \mathbb{R}^m)$ is defined to be the collection of tempered distributions f on $\mathbb{R}^n \times \mathbb{R}^m$ satisfying

$$\|f\|_{\dot{F}^{\lambda,\vec{\tau}}_p(\mathbb{R}^n\times\mathbb{R}^m)} = \left\| \left(\sum_{j,k\in\mathbb{Z}} 2^{(j\tau_1+k\tau_2)\lambda} \left|(\phi_j \otimes \varphi_k) * f\right|^\lambda \right)^{1/\lambda} \right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} < \infty,$$

where $\widehat{\phi_j}(u) = \frac{1}{2^{jn}}\phi(2^{-j}u)$, $\widehat{\phi_k}(v) = \frac{1}{2^{km}}\varphi(2^{-k}v)$ and $\phi \in C^\infty_0(\mathbb{R}^n)$, $\varphi \in C^\infty_0(\mathbb{R}^m)$ are radial mappings satisfying:

(i) $0 \leq \phi,\ \varphi \leq 1$,

(ii) $supp\,(\phi) \subseteq \{u : |u| \in [\frac{1}{2}, 2]\}$, $supp\,(\varphi) \subseteq \{v : |v| \in [\frac{1}{2}, 2]\}$,

(iii) A positive constant C exists such that for all $|u|, |v| \in [\frac{3}{5}, \frac{5}{3}]$, $\phi(u) \geq C$ and $\varphi(v) \geq C$,

(iv) $\sum_{j\in\mathbb{Z}} \phi(u/2^j) = 1$ if $u \neq 0$ and $\sum_{k\in\mathbb{Z}} \varphi(v/2^k) = 1$ if $v \neq 0$.

The next properties were proved in.[24]

(i) The Schwartz space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ is dense in $\dot{F}_p^{\lambda,\overrightarrow{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)$,

(ii) When $\lambda = 2$ and $\overrightarrow{\tau} = \overrightarrow{0}$, we have $\dot{F}_p^{2,\overrightarrow{0}}(\mathbb{R}^n \times \mathbb{R}^m) = L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1, \infty)$,

(iii) If $1 < \lambda_1 \leq \lambda_2 < \infty$, then $\dot{F}_p^{\lambda_1,\overrightarrow{\tau}}(\mathbb{R}^n \times \mathbb{R}^m) \subseteq \dot{F}_p^{\lambda_2,\overrightarrow{\tau}}(\mathbb{R}^n \times \mathbb{R}^m)$.

In light of the results in[23] regarding the L^p boundedness of Marcinkiewicz integral $\mathcal{G}^{(2)}_{\Phi,\Omega,g}$ along twisted surfaces and of the results in[24–28] regarding the L^p boundedness of generalized Marcinkiewicz integral $\mathcal{G}^{(\lambda)}_{\mathrm{I},\Omega,g}$ along surfaces of standard form, the following question arises naturally:

Question. Is the generalized Marcinkiewicz integral $\mathcal{G}^{(\lambda)}_{\Theta,\Omega,g}$ bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $p \in (1, \infty)$ along certain classes of twisted surfaces $\{\Gamma(u, v) = \Phi(u, v) : (u, v) \in \mathbb{R}^n \times \mathbb{R}^m)\}$?

We shall answer the above question in affirmative as can be described in the following results.

Theorem 1.1. *Suppose that $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q \in (1, 2]$ satisfies (1)-(2). Suppose that $\Phi(x, y) = (Q(|y|)x, P(|x|)y)$, where $Q, P : \mathbb{R} \to \mathbb{R}$ are polynomials. Then the inequality*

$$\left\|\mathcal{G}^{(\lambda)}_{\Theta,\Omega,g}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_p \|g\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}_+)} \|\Omega\|_{L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})} \left(\frac{1}{q-1}\right)^{2/\lambda} \|f\|_{\dot{F}_p^{\lambda,\overrightarrow{0}}(\mathbb{R}^n\times\mathbb{R}^m)} \tag{4}$$

holds for all $p \in (1, \infty)$ if Φ satisfies one of the following conditions:

(a) $Q(0) = 0 = P(0)$ *and* $\lim_{r\to 0} \frac{Q(r)}{r} = 0 = \lim_{r\to 0} \frac{P(r)}{r}$.

(b) $\deg(Q) + \deg(P) = 1$.

(c) $\deg(Q) = \deg(P) = 1$, $Q(0) \neq 0$, *and* $P(0) \neq 0$.

(d) $Q(r) = r$ *and* $\deg(P) > 1$ *with* $\lim_{s\to 0} \frac{P(s)-P(0)}{s} = 0$ *or* $P(s) = s$ *and* $\deg(Q) > 1$ *with* $\lim_{r\to 0} \frac{Q(r)-Q(0)}{r} = 0$.

The constant C_p may depend on the degrees of Q and P, but it is independent of their coefficients.

The conclusion comes from Theorem 1.1 along with Yano's extrapolation argument (see[29,30]) lead to the following:

Theorem 1.2. *Suppose that g and Φ are givens as in Theorem 1.1 and that Ω belongs either to $L(\log L)^{2/\lambda}(\mathbb{S}^{n-1}\times \mathbb{S}^{m-1})$ or to $B_q^{(0,\frac{2}{\lambda}-1)}(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$ for some $q > 1$. Then the operator $\mathcal{G}^{(\lambda)}_{\Theta,\Omega,g}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1, \infty)$.*

Remarks

(1) For the case $\lambda = 2$, $\Phi \equiv \mathrm{I}$ and $g \equiv 1$, L^2 boundedness of $\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}$ was obtained in[1] under the assumption $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subsetneq L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$. Thus, the findings in Theorem 1.2 are fundamental improvement and generalization to the results in.[1]

(2) For the special case $\lambda = 2$, $\Phi \equiv \mathrm{I}$, $g \equiv 1$, and $\Omega \in L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$, Theorem 1.2 confirms that $\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}$ is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $p \in (1, \infty)$. Therefore, the main results in[2] are improved here.

(3) Our results extend the results in[3] in which the L^p $(1 < p < \infty)$ boundedness of $\mathcal{G}^{(2)}_{\mathrm{I},\Omega,1}$ is obtained whenever $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \subsetneq L(\log L)(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})$.

(4) For the special case $\lambda = 2$, $\Phi \equiv \mathrm{I}$ and $g \equiv 1$, the assumptions on Ω are optimal, see.[4,5]

(5) In Theorem 1.2, the L^p boundedness of $\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}$ was established for the full range of $p \in (1, \infty)$.

(6) The main result in[23] is acquired directly from Theorem 1.2 by taking $\lambda = 2$.

(7) For the special case $\Phi \equiv \mathrm{I}$, our results lead to the main results in.[24,25]

2 Preliminary Lemmas

In this section, we give some subsidiary results which are needed to prove our main results. Let

$$Q(t) = \sum_{i=2}^{M} a_i t^i \quad and \quad P(t) = \sum_{k=i}^{N} b_i t^i.$$

Set $Q_1(t) = P_1(t) = 0$, and for $1 < \mu \leq M$ and $1 < \upsilon \leq N$, let

$$Q_\mu(t) = \sum_{i=2}^{\mu} a_i t^i \quad and \quad P_\upsilon(t) = \sum_{i=2}^{\upsilon} b_i t^i.$$

For $1 \leq \mu \leq M$, $1 \leq \upsilon \leq N$, and $\theta \geq 2$, we define the family of measures $\Upsilon^{(\mu,\upsilon)}_{\Omega,g,r,s} := \{\Upsilon^{(\mu,\upsilon)}_{r,s} : r, s \in \mathbb{R}_+\}$ and its corresponding maximal operators $(\Upsilon^{(\mu,\upsilon)}_g)^*$ and $\mathrm{M}_{g,\theta}$ on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$\iint_{\mathbb{R}^n \times \mathbb{R}^m} f d\Upsilon^{(\mu,\upsilon)}_{r,s} = \frac{1}{sr} \int_{\frac{1}{2}s \leq |v| \leq s} \int_{\frac{1}{2}r \leq |u| \leq r} f\left(Q_\mu(|v|)u, P_\upsilon(|u|)v\right) \frac{\Omega(u,v) g(|u|,|v|)}{|u|^{n-1} |v|^{m-1}} du dv,$$

$$(\Upsilon^{(\mu,\upsilon)}_g)^*(f) = \sup_{r,s \in \mathbb{R}_+} ||\Upsilon^{(\mu,\upsilon)}_{r,s}| * f|,$$

and

$$\mathrm{M}_{g,\theta}(f) = \sup_{j,k \in \mathbb{Z}} \int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} ||\Upsilon^{(\mu,\upsilon)}_{r,s}| * f| \frac{drds}{rs},$$

where $|\Upsilon^{(\mu,\upsilon)}_{r,s}|$ is defined similarly as $\Upsilon^{(\mu,\upsilon)}_{r,s}$ but with replacing g by $|g|$ and Ω by $|\Omega|$.

The next two lemmas due to.[23]

Lemma 2.1. *Suppose that $\Omega \in L^q\left(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}\right)$ for some $q \in (1, 2]$ and $g \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$. Then the estimates*

$$\|(\Upsilon^{(\mu,\upsilon)}_g)^*(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} C_{p,\Omega} \tag{5}$$

and

$$\|\mathrm{M}_{g,\theta}(f)\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^m)} C_{p,\Omega} (\ln \theta)^2 \tag{6}$$

hold for all $p \in (1, \infty)$, where $C_{p,\Omega} = C_p \|g\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R}_+)} \|\Omega\|_{L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1})}$.

Lemma 2.2. *Suppose that Ω and g are given as in Lemma 2.1. Then there exists $C > 0$ such that for $(\xi,\zeta) \in \mathbb{R}^n \times \mathbb{R}^m$, the following inequalities hold:*

$$\begin{aligned}
\left\|\Upsilon_{r,s}^{(\mu,\upsilon)}(\xi,\zeta)\right\| &\leq C, \\
\left|\widehat{\Upsilon}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta)\right| &\leq C\left|rs^{\mu}a_{\mu}\xi\right|^{-\frac{1}{8q'(\mu+1)}}\left|sr^{\upsilon}b_{\upsilon}\zeta\right|^{-\frac{1}{8q'(\mu+1)}}, \\
\left|\widehat{\Upsilon}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta)-\widehat{\Upsilon}_{r,s}^{(\mu-1,\upsilon)}(\xi,\zeta)\right| &\leq C\left|rs^{\mu}a_{\mu}\xi\right|^{\frac{1}{8q'(\mu+1)}}\left|sr^{\mu}b_{\upsilon}\zeta\right|^{\frac{-1}{8q'(\mu+1)}}, \\
\left|\widehat{\Upsilon}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta)-\widehat{\Upsilon}_{r,s}^{(\mu,\upsilon-1)}(\xi,\zeta)\right| &\leq C\left|rs^{\mu}a_{\mu}\xi\right|^{-\frac{1}{8q'(\mu+1)}}\left|sr^{\upsilon}b_{\upsilon}\zeta\right|^{\frac{1}{8q'(\mu+1)}}, \\
\left|\widehat{\Upsilon}_{r,s}^{(\mu,\upsilon-1)}(\xi,\zeta)-\widehat{\Upsilon}_{r,s}^{(\mu-1,\upsilon-1)}(\xi,\zeta)\right| &\leq C\left|rs^{\mu}a_{\mu}\xi\right|^{\frac{1}{4q'(\mu+1)}}, \\
\left|\widehat{\Upsilon}_{r,s}^{(\mu-1,\upsilon)}(\xi,\zeta)-\widehat{\Upsilon}_{r,s}^{(\mu-1,\upsilon-1)}(\xi,\zeta)\right| &\leq C\left|sr^{\upsilon}b_{\upsilon}\xi\right|^{\frac{1}{4q'(\mu+1)}},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\widehat{\Upsilon}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta)-\widehat{\Upsilon}_{r,s}^{(\mu,\upsilon-1)}(\xi,\zeta)-\widehat{\Upsilon}_{r,s}^{(\mu-1,\upsilon)}(\xi,\zeta)+\widehat{\Upsilon}_{r,s}^{(\mu-1,\upsilon-1)}(\xi,\zeta)\right| \\
&\qquad\qquad \leq C\left|rs^{\mu}a_{\mu}\xi\right|^{\frac{1}{4q'(\mu+1)}}\left|sr^{\upsilon}b_{\upsilon}\zeta\right|^{\frac{1}{4q'(\mu+1)}}.
\end{aligned}$$

Lemma 2.3. *Let g and Ω be given as in Lemma 2.1. Assume that $\{\mathcal{V}_{j,k}(\cdot,\cdot),\ j,k\in\mathbb{Z}\}$ is arbitrary functions on $\mathbb{R}^n \times \mathbb{R}^m$. Then there exists constant $C_{p,\Omega} > 0$ such that*

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\Upsilon_{r,s}^{(\mu,\upsilon)}*\mathcal{V}_{j,k}\right|^{\lambda}\frac{drds}{rs}\right)^{1/\lambda}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\
&\qquad \leq C_{p,\Omega}(\ln\theta)^{2/\lambda}\left\|\left(\sum_{j,k\in\mathbb{Z}}|\mathcal{V}_{j,k}|^{\lambda}\right)^{1/\lambda}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \qquad (7)
\end{aligned}$$

for all $p \in (1,\infty)$.

Proof. Thanks to the inequality (5), we have for all $p \in (1,\infty)$,

$$\begin{aligned}
\left\|\sup_{j,k\in\mathbb{Z}}\sup_{(r,s)\in[1,\theta]\times[1,\theta]}\left|\Upsilon_{\theta^k r,\theta^j s}^{(\mu,\upsilon)}*\mathcal{V}_{j,k}\right|\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} &\leq \left\|(\Upsilon_g^{(\mu,\upsilon)})^*\left(\sup_{j,k\in\mathbb{Z}}|\mathcal{V}_{j,k}|\right)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\
&\leq C_{p,\Omega}\left\|\sup_{j,k\in\mathbb{Z}}|\mathcal{V}_{j,k}|\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}, \qquad (8)
\end{aligned}$$

which leads to

$$\begin{aligned}
&\left\|\left\|\|\Upsilon_{\theta^k r,\theta^j s}^{(\mu,\upsilon)}*\mathcal{V}_{j,k}\|_{L^\infty([1,\theta]\times[1,\theta],\frac{drds}{rs})}\right\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\
\leq\ & C_{p,\Omega}\left\|\|\mathcal{V}_{j,k}\|_{l^\infty(\mathbb{Z}\times\mathbb{Z})}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}. \qquad (9)
\end{aligned}$$

As $p>1$, duality gives that there exists $\rho\in L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)$ with $\|\rho\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)}\leq 1$ such that

$$\begin{aligned}
&\left\|\sum_{j,k\in\mathbb{Z}}\int_1^\theta\int_1^\theta\left|\Upsilon^{(\mu,\upsilon)}_{\theta^k r,\theta^j s}*\mathcal{V}_{j,k}\right|\frac{drds}{rs}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\
=\;&\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{j,k\in\mathbb{Z}}\int_1^\theta\int_1^\theta\left|\Upsilon^{(\mu,\upsilon)}_{\theta^k r,\theta^j s}*\mathcal{V}_{j,k}\right|\frac{drds}{rs}\rho(x,y)dxdy\\
\leq\;& C\|g\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}_+)}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left(\sum_{j,k\in\mathbb{Z}}|\mathcal{V}_{j,k}(x,y)|\right)(\Upsilon^{(\mu,\upsilon)}_g)^*(\widetilde{\rho})(-x,-y)dxdy\\
\leq\;& C(\ln\theta)^2\|g\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}_+)}\left\|\sum_{j,k\in\mathbb{Z}}|\mathcal{V}_{j,k}|\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\left\|(\Upsilon^{(\mu,\upsilon)}_g)^*(\widetilde{\rho})\right\|_{L^{p'}(\mathbb{R}^n\times\mathbb{R}^m)},
\end{aligned}\tag{10}$$

where $\widetilde{\rho}(x,y)=\rho(-x,-y)$. Define the linear operator $\mathcal{T}$ on $\{\mathcal{V}_{j,k}\}$ by $\mathcal{T}(\mathcal{V}_{j,k})=\Upsilon_{\theta^k r,\theta^j s}*\mathcal{V}_{j,k}$. Hence, by combining (9) with (10), we deduce

$$\begin{aligned}
&\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\Upsilon^{(\mu,\upsilon)}_{r,s}*\mathcal{V}_{j,k}\right|^\lambda\frac{drds}{rs}\right)^{1/\lambda}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\qquad\leq C\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_1^\theta\int_1^\theta\left|\Upsilon^{(\mu,\upsilon)}_{\theta^k r,\theta^j s}*\mathcal{V}_{j,k}\right|^\lambda\frac{drds}{rs}\right)^{1/\lambda}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}\\
&\qquad\leq C_{p,\Omega}(\ln\theta)^{2/\lambda}\left\|\left(\sum_{j,k\in\mathbb{Z}}|\mathcal{V}_{j,k}|^\lambda\right)^{1/\lambda}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}
\end{aligned}$$

for all $p\in(1,\infty)$. The proof of Lemma 2.3 is complete. □

3 Proof of Theorem 1.1

We shall prove only part (i) in Theorem 1.1 since the proofs of the other parts follow by similar technique with minor modifications. Assume that $\Omega\in L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)$ with $q\in(1,2]$ and $g\in L^\infty\left(\mathbb{R}_+\times\mathbb{R}_+\right)$. By Minkowski's inequality we have

$$\begin{aligned}
\mathcal{G}^{(\lambda)}_{\Phi,\Omega,g}(f)(x,y)\;=\;&\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\sum_{j,k=0}^\infty\frac{1}{sr}\int_{2^{-j-1}s<|v|\leq 2^{-j}s}\int_{2^{-k-1}r<|u|\leq 2^{-k}r}\frac{\Omega(u,v)g(|u|,|v|)}{|u|^{n-1}|v|^{m-1}}\right.\right.\\
&\times\left.\left. f\left(x-Q_\mu(|v|)u,y-P_\upsilon(|u|)v\right)\;dudv\right|^\lambda\frac{drds}{rs}\right)^{1/\lambda}\\
\leq\;&\sum_{j,k=0}^\infty\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\frac{1}{sr}\int_{2^{-j-1}s<|v|\leq 2^{-j}s}\int_{2^{-k-1}r<|u|\leq 2^{-k}r}\frac{\Omega(u,v)g(|u|,|v|)}{|u|^{n-1}|v|^{m-1}}\right.\right.\\
&\times\left.\left. f\left(x-Q_\mu(|v|)u,y-P_\upsilon(|u|)v\right)\;dudv\right|^\lambda\frac{drds}{rs}\right)^{1/\lambda}\\
\leq\;& C\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\Upsilon^{(\mu,\upsilon)}_{r,s}*f(x,y)\right|^\lambda\frac{drds}{rs}\right)^{1/\lambda}.
\end{aligned}\tag{11}$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^m)$ be two schwartz functions satisfying that $\hat{\phi}(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\hat{\phi}(x) = 0$ if $|x| \geq 1$, and $\hat{\varphi}(y) = 1$ if $|y| \leq \frac{1}{2}$ and $\hat{\varphi}(x) = 0$ if $|y| \geq 1$. For $1 \leq \mu \leq M$, $1 \leq \upsilon \leq N$, and $r, s \in \mathbb{R}_+$, let $\widehat{\phi_{r,s}^{(\mu)}}(\xi) = \widehat{\phi}(\theta^{r+s\upsilon} a_\mu \xi)$ and $\widehat{\varphi_{r,s}^{(\upsilon)}}(\zeta) = \widehat{\varphi}(\theta^{s+r\mu} b_\upsilon \zeta)$, where $\theta = 2^{q'}$. Define

$$\widehat{\sigma}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta) = \widehat{\Upsilon}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta) \prod_{\mu<j\leq M} \widehat{\phi_{r,s}^{(j)}}(\xi) \prod_{\upsilon<k\leq N} \widehat{\varphi_{r,s}^{(k)}}(\zeta)$$

and

$$\widehat{\Theta}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta) = \widehat{\sigma}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta) - \widehat{\sigma}_{r,s}^{(\mu-1,\upsilon)}(\xi,\zeta) - \widehat{\sigma}_{r,s}^{(\mu,\upsilon-1)}(\xi,\zeta) + \widehat{\sigma}_{r,s}^{(\mu-1,\upsilon-1)}(\xi,\zeta).$$

Thus, by Lemma 2.1 and Lemma 2.2, we get the following:

$$\begin{aligned} \left\| \Theta_{r,s}^{(\mu,\upsilon)}(\xi,\zeta) \right\| &\leq C, \\ \Upsilon_{r,s}^{(\mu,\upsilon)}(\xi,\zeta) &= \sum_{\mu=1}^{M}\sum_{\upsilon=1}^{N} \Theta_{r,s}^{(\mu,\upsilon)}(\xi,\zeta), \\ \|(\Theta_g^{(\mu,\upsilon)})^*(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} &\leq \|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} C_{p,\Omega} \quad for\ all\ p \in (1,\infty), \\ \widehat{\Theta}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta) &\leq C\, |\theta^r \theta^{s\mu} a_\mu \xi|^{\pm\frac{1}{8q'(\mu+1)}} \, |\theta^s \theta^{r\upsilon} b_\upsilon \zeta|^{\pm\frac{1}{8q'(\mu+1)}}, \end{aligned} \tag{12}$$

where $a^{\pm b} = \min\{a^b, a^{-b}\}$. Let Λ_θ be a non-negative smooth function on $\mathbb{R}_+$ satisfying the following:

$$0 \leq \Lambda_\theta(t) \quad \leq 1, \quad \text{supp } (\Lambda_\theta) \subseteq [\frac{4}{5\theta}, \frac{5\theta}{4}], \ and \int_{\mathbb{R}_+} \Lambda_\theta(t)\frac{dt}{t} = 2\ln\theta.$$

Define the family of $C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ functions $\mathcal{U}_{\theta,r,s}^{(\mu,\upsilon)}$ by

$$\mathcal{U}_{\theta,r,s}^{(\mu,\upsilon)}(\xi,\zeta) = \Lambda_\theta\left(|\theta^r\theta^{s\mu}\xi|^2\right)\Lambda_\theta\left(|\theta^s\theta^{r\upsilon}\zeta|^2\right). \tag{13}$$

So, for any $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, we have

$$\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\Upsilon_{r,s}^{(\mu,\upsilon)} * f(x,y)\right|^\lambda \frac{drds}{rs}\right)^{1/\lambda} \leq C_{M,N} \iint_{\mathbb{R}_+\times\mathbb{R}_+} \mathcal{I}_{l,t}(f)(x,y)dldt, \tag{14}$$

where

$$\mathcal{I}_{l,t}(f)(x,y) = \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+} \left|\Theta_{r,s}^{(\mu,\upsilon)} * \mathcal{U}_{\theta,r+l,s+t}^{(\mu,\upsilon)} * f(x,y)\right|^\lambda \frac{drds}{rs}\right)^{1/\lambda}.$$

Let us estimate the norm of $\mathcal{I}_{l,t}(f)$. First, we estimate it for the case $p = \lambda = 2$; by utilizing Fubini's theorem, Plancherel's theorem and the inequality (12), we obtain

$$\begin{aligned} &\|\mathcal{I}_{l,t}(f)\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}^2 \\ \leq\ & \sum_{j,k\in\mathbb{Z}} \iint_{E_{k,j}} \left(\int_{\theta^j}^{\theta^{j+1}} \int_{\theta^k}^{\theta^{k+1}} \left|\hat{\Theta}_{r,s}^{(\mu,\upsilon)}(\xi,\zeta)\right|^2 \left|\hat{\mathcal{U}}_{\theta,r+l,s+t}^{(\mu,\upsilon)}\right|^2 \frac{drds}{rs}\right) \left|\hat{f}(\xi,\zeta)\right|^2 d\xi d\zeta \\ \leq\ & C_p(\ln\theta)^2 C_{p,\Omega}^2 \sum_{j,k\in\mathbb{Z}} \iint_{E_{k,j}} \left|\theta^{r+l}\theta^{(s+t)\mu} a_\mu \xi\right|^{\pm\frac{1}{8q'(\mu+1)}} \left|\theta^{s+t}\theta^{(r+l)\upsilon} b_\upsilon \zeta\right|^{\pm\frac{1}{8q'(\mu+1)}} \left|\hat{f}(\xi,\zeta)\right|^2 d\xi d\zeta \\ \leq\ & C_p(\ln\theta)^2 K_{t,l}^{(\mu,\upsilon)} C_{p,\Omega}^2 \sum_{j,k\in\mathbb{Z}} \iint_{E_{n,m}} \left|\hat{f}(\xi,\zeta)\right|^2 d\xi d\zeta \\ \leq\ & C_p(\ln\theta)^2 K_{t,l}^{(\mu,\upsilon)} C_{p,\Omega}^2 \|f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^m)}^2, \end{aligned} \tag{15}$$

where $E_{k,j} = \left\{(\xi,\zeta) \in \mathbb{R}^n \times \mathbb{R}^m : (|\xi|,|\zeta|) \in [\theta^{-1-k}, \theta^{1-k}] \times [\theta^{-1-j}, \theta^{1-j}]\right\}$ and

$$K_{t,l}^{(\mu,\upsilon)} \leq (\frac{5}{4})^{\frac{1}{4(\mu+1)}} 2^{\pm\frac{1}{4(\mu+1)}} 2^{\pm\frac{|t+l\upsilon|}{8(\mu+1)}} 2^{\pm\frac{|l+t\mu|}{8(\mu+1)}}.$$

For the other cases of p, we estimate the L^p-norm of $\mathcal{I}_{l,t}(f)$ by employing an argument as that found in.[31] Precisely, by Lemma 2.3 and Littlewood-Paley theory, we have

$$\begin{aligned}
&\|\mathcal{I}_{l,t}(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\
\leq\ & C\left\|\left(\sum_{j,k\in\mathbb{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\Theta_{r,s}^{(\mu,\upsilon)} * \mathcal{U}_{\theta,r+l,s+t}^{(\mu,\upsilon)} * f\right|^{\lambda}\frac{drds}{rs}\right)^{1/\lambda}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \\
\leq\ & (\ln\theta)^{2/\lambda}\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{U}_{\theta,r+l,s+t}^{(\mu,\upsilon)} * f\right|^{\lambda}\right)^{1/\lambda}\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} C_{p,\Omega} \\
\leq\ & \left(\frac{1}{q-1}\right)^{2/\lambda}\|f\|_{\dot{F}_p^{\lambda,\vec{0}}(\mathbb{R}^n\times\mathbb{R}^m)} C_{p,\Omega} \qquad (16)
\end{aligned}$$

for all $p \in (1,\infty)$. Thus, by interpolating (15) with (16), we conclude that the estimate

$$\|\mathcal{I}_{l,t}(f)\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq \left(\frac{1}{q-1}\right)^{2/\lambda}\|f\|_{\dot{F}_p^{\lambda,\vec{0}}(\mathbb{R}^n\times\mathbb{R}^m)} C_{p,\Omega}\, 2^{-\varepsilon(|l|+|t|)} \qquad (17)$$

holds for all $p \in (1,\infty)$, where $\varepsilon \in (0,1)$. Consequently, the inequalities (11) and (14) along with (17) lead to

$$\left\|\mathcal{G}_{\Phi,\Omega,g}^{(\lambda)}(f)\right\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)} \leq C_{p,M,N}\left(\frac{1}{q-1}\right)^{2/\lambda}\|f\|_{\dot{F}_p^{\lambda,\vec{0}}(\mathbb{R}^n\times\mathbb{R}^m)}. \qquad (18)$$

The proof of Theorem 1.1 is complete.

4 Conclusion

In this paper, we introduced the generalized Marcinkiewicz integrals $\mathcal{G}_{\Phi,\Omega,g}^{(\lambda)}$ along certain twisted surfaces $\{\Gamma(u,v) = \Phi(u,v) : (u,v) \in \mathbb{R}^n \times \mathbb{R}^m)\}$. Under certain conditions on Φ, we obtained certain appropriate L^p bounds for $\mathcal{G}_{\Phi,\Omega,g}^{(\lambda)}$ provided that $\Omega \in L^q\left(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}\right)$ and $g \in L^\infty(\mathbb{R}_+\times\mathbb{R}_+)$. By using these bounds along with an extrapolation argument we proved the L^p $(1 < p < \infty)$ boundedness of $\mathcal{G}_{\Phi,\Omega,g}^{(\lambda)}$ under the weak conditions $\Omega \in L(\log L)^{2/\lambda}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1}) \cup B_q^{(0,\frac{2}{\lambda}-1)}(\mathbb{S}^{n-1}\times\mathbb{S}^{m-1})$ with $\lambda, q > 1$. Several previously known results are extended and improved in this paper, see.[1–5, 23–25]

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