On Refined Neutrosophic Vector Spaces I

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Abstract

The objective of this paper is to present the concept of a refined neutrosophic vector space. Weak(strong) refined neutrosophic vector spaces and subspaces, and, strong refined neutrosophic quotient vector spaces are studied. Several interesting results and examples are presented. It is shown that every weak (strong) refined neutrosophic vector space is a vector space and it is equally shown that every strong refined neutrosophic vector space is a weak refined neutrosophic vector space.

Keywords: Neutrosophy, neutrosophic vector space, neutrosophic vector subspace, refined neutrosophic vector space, refined neutrosophic vector subspace, refined neutrosophic quotient vector space.

1 Introduction and Preliminaries

Neutrosophy is a new branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. The concept of neutrosophic logic/set was introduced by Smarandache in [20,22] as a generalization of fuzzy log/set [29] and respectively intuitionistic fuzzy logic/set [9]. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I) and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$. In [21], Smarandache introduced the notion of refined neutrosophic components of the form $< T_1, T_2, \cdots, T_p; I_1, I_2, \cdot, I_r; F_1, F_2, \cdots, F_s >$ of the neutrosophic components $< T, I, F >$. The refinement has given rise to the introduction of refined neutrosophic set and the extension of neutrosophic numbers $a + bI$ into refined neutrosophic numbers of the form $(a + b_1I_1 + b_2I_2 + \cdots + b_nI_n)$ where $a, b_1, b_2, \cdots, b_n$ are real or complex numbers. Agboola in [4] introduced the concept of refined neutrosophic algebraic structures and studied refined neutrosophic groups with their basic and fundamental properties. Since then, several neutrosophic researchers have studied this concept and a great deal of results have been published. Recently, Adeleke et al. studied refined neutrosophic rings and refined neutrosophic subrings in [1] and also in [2], they presented several results and examples on refined neutrosophic ideals and refined neutrosophic ring homomorphisms.

The concept of neutrosophic vector space was introduced by Vasantha Kandasamy and Florentin Smarandache in [23]. Further studies on neutrosophic vector spaces were carried out by Agboola and Akinleye in [8] where they generalized some properties of vector spaces and showed that every neutrosophic vector space over a neutrosophic field (resp. field) is a vector space. A comprehensive review of neutrosophic set, neutrosophic soft set, fuzzy set, neutrosophic topological spaces, neutrosophic vector spaces and new trends in neutrosophic theory can be found in [3,5–7,10–19,23–28].

In the present paper, we present the concept of a refined neutrosophic vector space. Weak(strong) refined neutrosophic vector spaces and subspaces, and, strong refined neutrosophic quotient vector spaces are studied. Several interesting results and examples are presented. It is shown that every weak (strong) refined neutrosophic vector space is a vector space and it is equally shown that every strong refined neutrosophic vector space is a weak refined neutrosophic vector space.

For the purposes of this paper, it will be assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that:

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$$\begin{array}{lllll} I_1I_1 & = & I_1^2 & = & I_1, \\ I_2I_2 & = & I_2^2 & = & I_2, \text{ and} \\ I_1I_2 & = & I_2I_1 & = & I_1. \end{array}$$

Definition 1.1. [4] If $* : X(I_1,I_2) \times X(I_1,I_2) \mapsto X(I_1,I_2)$ is a binary operation defined on $X(I_1,I_2)$, then the couple $(X(I_1,I_2),*)$ is called a refined neutrosophic algebraic structure and it is named according to the laws (axioms) satisfied by $*$.

Definition 1.2. [4] Let $(X(I_1,I_2),+,.)$ be any refined neutrosophic algebraic structure where $+$ and . are ordinary addition and multiplication respectively.
For any two elements $(a,bI_1,cI_2),(d,eI_1,fI_2) \in X(I_1,I_2)$, we define

$$(a,bI_1,cI_2)+(d,eI_1,fI_2) = (a+d,(b+e)I_1,(c+f)I_2),$$
$$(a,bI_1,cI_2).(d,eI_1,fI_2) = (ad,(ae+bd+be+bf+ce)I_1,(af+cd+cf)I_2).$$

Definition 1.3. [4] If $''+''$ and $''.''$ are ordinary addition and multiplication, I_k with $k=1,2$ have the following properties:

1. $I_k + I_k + \cdots + I_k = nI_k$.
2. $I_k + (-I_k) = 0$.
3. $I_k \cdot I_k \cdot \cdots \cdot I_k = I_k^n = I_k$ for all positive integers $n > 1$.
4. $0 \cdot I_k = 0$.
5. I_k^{-1} is undefined and therefore does not exist.

Definition 1.4. [4] Let $(G,*)$ be any group. The couple $(G(I_1,I_2),*)$ is called a refined neutrosophic group generated by G, I_1 and I_2. $(G(I_1,I_2),*)$ is said to be commutative if for all $x,y \in G(I_1,I_2)$, we have $x*y = y*x$. Otherwise, we call $(G(I_1,I_2),*)$ a non -commutative refined neutrosophic group.

Definition 1.5. [4] If $(X(I_1,I_2),*)$ and $(Y(I_1,I_2),*')$ are two refined neutrosophic algebraic structures, the mapping

$$\phi : (X(I_1,I_2),*) \longrightarrow (Y(I_1,I_2),*')$$

is called a neutrosophic homomorphism if the following conditions hold:

1. $\phi((a,bI_1,cI_2)*(d,eI_1,fI_2)) = \phi((a,bI_1,cI_2)) *' \phi((d,eI_1,fI_2))$.
2. $\phi(I_k) = I_k$ for all $(a,bI_1,cI_2),(d,eI_1,fI_2) \in X(I_1,I_2)$ and $k=1,2$.

Example 1.6. [4] Let
$\mathbb{Z}_2(I_1,I_2) = \{(0,0,0),(1,0,0),(0,I_1,0),(0,0,I_2),(0,I_1,I_2),(1,I_1,0),(1,0,I_2),(1,I_1,I_2)\}$.
Then $(\mathbb{Z}_2(I_1,I_2),+)$ is a commutative refined neutrosophic group of integers modulo 2.
Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1,I_2),+)$ is a finite commutative refined neutrosophic group of integers modulo n.

Example 1.7. [4] Let $(G(I_1,I_2),*)$ and and $(H(I_1,I_2),*')$ be two refined neutrosophic groups.
Let $\phi : G(I_1,I_2) \times H(I_1,I_2) \to G(I_1,I_2)$ be a mapping defined by $\phi(x,y) = x$ and let
$\psi : G(I_1,I_2) \times H(I_1,I_2) \to H(I_1,I_2)$ be a mapping defined by $\psi(x,y) = y$. Then ϕ and ψ are refined neutrosophic group homomorphisms.

Definition 1.8. [1] Let $(R,+,.)$ be any ring. The abstract system $(R(I_1,I_2),+,.)$ is called a refined neutrosophic ring generated by R,I_1,I_2. $(R(I_1,I_2),+,.)$ is called a commutative refined neutrosophic ring if for all $x,y \in R(I_1,I_2)$, we have $xy = yx$. If there exists an element $e = (1,0,0) \in R(I_1,I_2)$ such that $ex = xe = x$ for all $x \in R(I_1,I_2)$, then we say that $(R(I_1,I_2),+,.)$ is a refined neutrosophic ring with unity.

Definition 1.9. [1] Let $(R(I_1,I_2),+,.)$ be a refined neutrosophic ring and let $n \in \mathbb{Z}^+$.

(i) If $nx = 0$ for all $x \in R(I_1,I_2)$, we call $(R(I_1,I_2),+,.)$ a refined neutrosophic ring of characteristic n and n is called the characteristic of $(R(I_1,I_2),+,.)$.

(ii) $(R(I_1,I_2),+,.)$ is call a refined neutrosophic ring of characteristic zero if for all $x \in R(I_1,I_2)$, $nx = 0$ is possible only if $n = 0$.

Example 1.10. [1]

(i) $\mathbb{Z}(I_1, I_2), \mathbb{Q}(I_1, I_2), \mathbb{R}(I_1, I_2), \mathbb{C}(I_1, I_2)$ are commutative refined neutrosophic rings with unity of characteristics zero.

(ii) Let $\mathbb{Z}_2(I_1, I_2) = \{(0, 0, 0), (1, 0, 0), (0, I_1, 0), (0, 0, I_2), (0, I_1, I_2), (1, I_1, 0), (1, 0, I_2), (1, I_1, I_2)\}$. Then $(\mathbb{Z}_2(I_1, I_2), +, .)$ is a commutative refined neutrosophic ring of integers modulo 2 of characteristic 2. Generally for a positive integer $n \geq 2$, $(\mathbb{Z}_n(I_1, I_2), +, .)$ is a finite commutative refined neutrosophic ring of integers modulo n of characteristic n.

Example 1.11. [1] Let $M^{\mathbb{R}}_{n\times n}(I_1, I_2) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} : a_{ij} \in \mathbb{R}(I_1, I_2) \right\}$ be a refined neutrosophic set of all $n \times n$ matrix.
Then $(M^{\mathbb{R}}_{n\times n}(I_1, I_2), +, .)$ is a non-commutative refined neutrosophic ring under matrix multiplication.

Theorem 1.12. [1] *Let $(R(I_1, I_2), +, .)$ be any refined neutrosophic ring. Then $(R(I_1, I_2), +, .)$ is a ring.*

2 Formulation of a Refined Neutrosophic Vector Space

In this section, we develop the concept of refined neutrosophic vector space and its subspaces and also present some of their basic properties.

Definition 2.1. Let $(V, +, .)$ be any vector space over a field K. Let $V(I_1, I_2) =< V \cup (I_1, I_2) >$ be a refined neutrosophic set generated by V, I_1 and I_2. We call the triple $(V(I_1, I_2), +, .)$ a weak refined neutrosophic vector space over a field K, if $V(I_1, I_2)$ is a refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$, then $V(I_1, I_2)$ is called a strong refined neutrosophic vector space. The elements of $V(I_1 I_2)$ are called refined neutrosophic vectors and the elements of $K(I_1, I_2)$ are called refined neutrosophic scalars.

If $u = a + bI_1 + cI_2$, $v = d + eI_1 + fI_2 \in V(I_1, I_2)$ where a, b, c, d, e and f are vectors in V and $\alpha = k + mI_1 + nI_2 \in K(I_1, I_2)$ where k, m and n are scalars in K, we define:

$$u + v = (a + bI_1 + cI_2) + (d + eI_1 + fI_2) = (a + d) + (b + e)I_1 + (c + f)I_2,$$

and

$$\alpha u = (k + mI_1 + nI_2).(a + bI_1 + cI_2) = k.a + (k.b + m.a + m.b + m.c + n.b)I_1 + (k.c + n.a + n.c)I_2.$$

Example 2.2. Let $\mathbb{R}^2(I_1, I_2)$ denote the refined set of all ordered pairs (x, y) where x and y are refined neutrosophic real numbers given as $x = a + bI_1 + cI_2$ and $y = d + eI_1 + fI_2$.
Define addition and scalar multiplication on $\mathbb{R}^2(I_1, I_2)$ by

$$\begin{aligned} (x, y) + (x', y') &= (a + bI_1 + cI_2, d + eI_1 + fI_2) + (a' + b'I_1 + c'I_2, d' + e'I_1 + f'I_2) \\ &= (a + a' + (b + b')I_1 + (c + c')I_2, d + d' + (e + e')I_1 + (f + f')I_2) \\ &= (x + x', y + y'). \end{aligned}$$

For $\alpha = (k + mI_1 + nI_2) \in \mathbb{R}(I_1, I_2)$

$$\begin{aligned} \alpha(x, y) &= (k + mI_1 + nI_2).(a + bI_1 + cI_2, d + eI_1 + fI_2) \\ &= ((k + mI_1 + nI_2).(a + bI_1 + cI_2), (k + mI_1 + nI_2).(d + eI_1 + fI_2)) \\ &= (k.a + (k.b + m.a + m.b + m.c + n.b)I_1 + (k.c + n.a + n.c)I_2, k.d+ \\ &\quad (k.e + m.d + m.e + m.f + n.e)I_1 + (k.f + n.d + n.f)I_2,). \end{aligned}$$

Then $\mathbb{R}^2(I_1, I_2)$ is strong refined neutrosophic vector space over $\mathbb{R}(I_1, I_2)$.

And if $\alpha \in \mathbb{R}$ with scalar multiplication defined as

$$\alpha(x, y) = \alpha(a + bI_1 + cI_2, d + eI_1 + fI_2) = (\alpha.a + \alpha.bI_1 + \alpha.cI_2, \alpha.d + \alpha.eI_1 + \alpha.fI_2) = (\alpha.x, \alpha.y),$$

then $\mathbb{R}^2(I_1, I_2)$ is weak refined neutrosophic vector space over $\mathbb{R}$.

Example 2.3. $M_{m\times n}(I_1, I_2) = \{[aij] : aij \in \mathbb{Q}(I_1, I_2)\}$ is a weak refined neutrosophic vector space over a field $\mathbb{Q}$ and it is a strong refined neutrosophic vector space over a refined neutrosophic field $Q(I_1, I_2)$.

Example 2.4. Let $V = \mathbb{Q}(I_1, I_2)(\sqrt{2}) = \{a + (bI_1 + cI_2)\sqrt{2} : a, b, c \in Q\}$. Then V is a weak refined neutrosophic vector space over $\mathbb{Q}$. If $u = a + (bI_1 + cI_2)\sqrt{2}$ and $v = d + (eI_1 + fI_2)\sqrt{2}$
then $u + v = (a + d) + (b + e)I_1\sqrt{2} + (c + f)I_2\sqrt{2}$ is again in V.
Also, for $\alpha \in \mathbb{Q}, \quad \alpha u = \alpha(a + (bI_1 + cI_2)\sqrt{2}) = \alpha.a + (\alpha.bI_1 + \alpha.cI_2)\sqrt{2}$ is in V.

Proposition 2.5. *Every strong refined neutrosophic vector space is a weak refined neutrosophic vector space.*

Proof. Suppose that $V(I_1, I_2)$ is a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ say. Since $K \subseteq K(I_1, I_2)$ for every field K, we have that $V(I_1, I_2)$ is also a weak refined neutrosophic vector space. □

Proposition 2.6. *Every weak (strong) refined neutrosophic vector space is a vector space.*

Proof. Suppose that $V(I_1, I_2)$ is a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$. That $(V(I_1, I_2), +)$ is an abelian group can be established easily.
Let $u = a + bI_1 + cI_2, v = d + eI_1 + fI_2 \in V(I_1, I_2), \alpha = k + mI_1 + nI_2, \beta = p + qI_1 + rI_2 \in K(I_1, I_2)$ where $a, b, c, d, e, f \in V$ and $k, m, n, p, q, r \in K$.
Then:

1. $\alpha(u + v) = (k + mI_1 + nI_2)(a + bI_1 + cI_2 + d + eI_1 + fI_2)$
   $= (k + mI_1 + nI_2)(a + d + (b + e)I_1 + (c + f)I_2)$
   $= ka+kd+[kb+ke+ma+md+mb+me+mc+mf+nb+ne]I_1+[kc+kf+na+nd+nc+nf]I_2$
   $= [ka + (kb + ma + mb + mc + nb)I_1 + (kc + na + nc)I_2] + [kd + (ke + md + me + mf + ne)I_1 + (kf + nd + nf)I_2]$
   $= (k + mI_1 + nI_2)(a + bI_1 + cI_2) + (k + mI_1 + nI_2)(d + eI_1 + fI_2)$
   $= \alpha u + \alpha v.$

2. $(\alpha + \beta)u = (k + mI_1 + nI_2 + p + qI_1 + rI_2)(a + bI_1 + cI_2)$
   $= (k + p + (m + q)I_1 + (n + r)I_2)(a + bI_1 + cI_2)$
   $= ka + pa + [kb + pb + ma + qa + mb + qb + mc + qc + nb + rb]I_1 + [kc + pc + na + ra + nc + rc]I_2$
   $= [ka+(kb+ma+mb+mc+nb)I_1+[kc+na+nc]I_2]+[pa+(pb+qa+qb+qc+rb)I_1+[pc+ra+rc]I_2$
   $= (k + mI_1 + nI_2)(a + bI_1 + cI_2) + (p + qI_1 + rI_2)(a + bI_1 + cI_2)$
   $= \alpha u + \beta u.$

3. $(\alpha\beta)u = ((k + mI_1 + nI_2)(p + qI_1 + rI_2))(a + bI_1 + cI_2)$
   $= (kp + (kq + mp + mq + mr + nq)I_1 + (kr + np + nr)I_2)(a + bI_1 + cI_2)$
   $= kpa + [kpb + kqa + mpa + mqa + mra + nqa + kqb + mpb + mqb + mrb + nqb + kqc + mpc + mqc + mrc + nqc + krb + npb + nrb]I_1 + [kpc + kra + npa + nra + krc + npc + nrc]I_2$
   $= (k + mI_1 + nI_2)[pa + (pb + qa + qb + qc + rb)I_1 + (pc + ra + rc)I_2]$
   $= (k + mI_1 + nI_2)((p + qI_1 + rI_2)(a + bI_1 + cI_2))$
   $= \alpha(\beta u).$

4. For $1 = 1 + 0I_1 + 0I_2 \in K(I_1I_2)$, we have
   $1u = (1 + 0I_1 + 0I_2)(a + bI_1 + cI_2)$
   $= a + (b + 0 + 0 + 0 + 0)I_1 + (c + 0 + 0)I_2$
   $= a + bI_1 + cI_2.$
   Accordingly, $V(I_1, I_2)$ is a vector space.

□

Example 2.7. Let $P_\infty(I_1, I_2)$ be the set of refined neutrosophic formal power series in variable x of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$, with $a_n \in \mathbb{R}(I_1, I_2)$ and
$a_n = p_n + q_n I_1 + r_n I_2$ for $n = 1, 2, 3 \cdots$
If addition and scalar multiplication (for $\alpha \in K(I_1, I_2)$) are defined as:

$$\begin{aligned}
(\textstyle\sum_{n=0}^{\infty} a_n x^n) + (\sum_{n=0}^{\infty} b_n x^n)) &= (\textstyle\sum_{n=0}^{\infty}(p_n + q_n I_1 + r_n I_2)x^n) + (\sum_{n=0}^{\infty}(u_n + v_n I_1 + w_n I_2)x^n) \\
&= (\textstyle\sum_{n=0}^{\infty}(p_n + u_n + (q_n + v_n)I_1 + (r_n + w_n)I_2)x^n) \\
\alpha\,(\textstyle\sum_{n=0}^{\infty}(p_n + q_n I_1 + r_n I_2)x^n) &= (\textstyle\sum_{n=0}^{\infty} \alpha(p_n + q_n I_1 + r_n I_2)x^n) \\
&= \textstyle\sum_{n=0}^{\infty}(e + fI_1 + gI_2)(p_n + q_n I_1 + r_n I_2)x^n \\
&= \textstyle\sum_{n=0}^{\infty} ep_n + (eq_n + fp_n + fq_n + fr_n + gq_n)I_1 \\
&\quad +(er_n + gp_n + gr_n)I_2.
\end{aligned}$$

Then $P_\infty(I_1, I_2)$ is a strong refined neutrosophic vector space over $K(I_1, I_2)$.

Note 1. *A refined neutrosophic formal power series can be loosely thought of as an object that is like a refined neutrosophic polynomial. Alternatively, one may think of a refined neutrosophic formal power series as a refined neutrosophic power series in which we ignore the questions of convergence by not assuming that the variable x denotes any numerical value (not even an unknown value). Thus, we do not regard these refined neutrososphic formal power series as infinite sum in P_x of refined neutrosophic monomials.*

Example 2.8. Let $P(I_1, I_2)$ be the set of all refined neutrosophic polynomial in variable x with coefficients in $R[I_1, I_2]$. Let $p, q \in P(I_1, I_2)$ and $\alpha = (k + uI_1 + vI_2) \in K(I_1, I_2)$.

$$p = (a_0 + b_0I_1 + c_0I_2) + (a_1 + b_1I_1 + c_1I_2)x + \cdots + (a_n + b_nI_1 + c_nI_2)x^n,$$

and

$$q = (a_0' + b_0'I_1 + c_0'I_2) + (a_1' + b_1'I_1 + c_1'I_2)x + \cdots + (a_m' + b_m'I_1 + c_m'I_2)x^m.$$

If $m \geq n$, the sum of p and q is given by

$$\begin{aligned} p+q &= ((a_0 + a_0') + (b_0 + b_0')I_1 + (c_0 + c_0')I_2) + ((a_1 + a_1') + (b_1 + b_1')I_1 + (c_1 + c_1')I_2)x + \cdots + \\ &\quad ((a_n + a_n') + (b_n + b_n')I_1 + (c_n + c_n')I_2) + (a_{n+1}' + b_{n+1}'I_1 + c_{n+1}'I_2)x^{n+1} + \cdots + \\ &\quad (a_m' + b_m'I_1 + c_m'I_2)x^m. \end{aligned}$$

A similar definition is given if $m < n$.
The product of p and a scalar α is given by

$$\alpha \cdot p = (k+uI_1+vI_2)(a_0+b_0I_1+c_0I_2)+(k+uI_1+vI_2)(a_1+b_1I_1+c_1I_2)x+\cdots+(k+uI_1+vI_2)(a_n+b_nI_1+c_nI_2)x^n.$$

Then $(P(I_1, I_2), +, \cdot)$ is a strong refined vector space over a refined neutrosophic field $K(I_1, 1_2)$.

Proposition 2.9. *Let $F(I_1, I_2)$ be a refined neutrosophic field and $(\mathbb{R}(I_1, I_2), \phi)$ a refined neutrosophic F-algebra, where $\mathbb{R}(I_1, I_2)$ is a refined neutrosophic ring with identity. Then $\mathbb{R}(I_1, I_2)$ is a strong refined neutrosophic vector space over $F(I_1, I_2)$ with addition being that in $\mathbb{R}(I_1, I_2)$ and scalar multiplication defined by $ar = \phi(a)r$. Here $\phi : F(I_1, I_2) \longrightarrow \mathbb{R}(I_1, I_2)$ is a neutrosophic homomorphism such that $\phi((1 + 0I_1 + 0I_2)) = (1 + 0I_1 + 0I_2)$ and $\phi(I_k) = I_k$.*

Proof. 1. That $(\mathbb{R}(I_1, I_2), +)$ is a neutrosophic abelian group can be easily established.

Let $x = a + bI_1 + cI_2, y = d + eI_1 + fI_2 \in R(I_1, I_2), \alpha = k + mI_1 + nI_2$,
$\beta = u + vI_1 + wI_2 \in F(I_1, I_2)$. Where $a, b, c, d, e, f \in \mathbb{R}$ and $k, m, n, u, v, w \in F$.

2. $\alpha(x + y) = (k + mI_1 + nI_2)((a + d + (b + e)I_1 + (c + f)I_2)$
$= ka+kd+[kb+ke+ma+md+mb+me+mc+mf+nb+ne]I_1+[kc+kf+na+nd+nc+nf]I_2$
$= [ka + (kb + ma + mb + mc + nb)I_1 + (kc + na + nc)I_2] + [kd + (ke + md + me + mf + ne)I_1 + (kf + nd + nf)I_2]$
$= (k + mI_1 + nI_2)(a + bI_1 + cI_2) + (k + mI_1 + nI_2)(d + eI_1 + fI_2)$
$= \phi((k + mI_1 + nI_2))(a + bI_1 + cI_2) + \phi((k + mI_1 + nI_2))(d + eI_1 + fI_2)$
$= \phi(\alpha)x + \phi(\alpha)y.$

3. $(\alpha + \beta)x = (k + mI_1 + nI_2 + p + qI_1 + rI_2)(a + bI_1 + cI_2)$
$= (k + p + (m + q)I_1 + (n + r)I_2)(a + bI_1 + cI_2)$
$= ka + pa + [kb + pb + ma + qa + mb + qb + mc + qc + nb + rb]I_1 + [kc + pc + na + ra + nc + rc]I_2$
$= [ka + (kb + ma + mb + mc + nb)I_1 + [kc + na + nc]I_2]$
$+ [pa + (pb + qa + qb + qc + rb)I_1 + [pc + ra + rc]I_2$
$= (k + mI_1 + nI_2)(a + bI_1 + cI_2) + (p + qI_1 + rI_2)(a + bI_1 + cI_2)$
$= \phi((k + mI_1 + nI_2))(a + bI_1 + cI_2) + \beta((p + qI_1 + rI_2))(a + bI_1 + cI_2)$
$= \phi(\alpha)x + \phi(\beta)x.$
$= (\phi(\alpha) + \phi(\beta))x = \phi(\alpha + \beta)x,$ since ϕ is a refined neutrosophic homomorphism.

4. $(\alpha\beta)u = ((k + mI_1 + nI_2)(p + qI_1 + rI_2))(a + bI_1 + cI_2)$
$= (kp + (kq + mp + mq + mr + nq)I_1 + (kr + np + nr)I_2)(a + bI_1 + cI_2)$
$= kpa + [kpb + kqa + mpa + mqa + mra + nqa + kqb + mpb + mqb + mrb + nqb + kqc + mpc + mqc + mrc + nqc + krb + npb + nrb]I_1 + [kpc + kra + npa + nra + krc + npc + nrc]I_2$
$= (k + mI_1 + nI_2)[pa + (pb + qa + qb + qc + rb)I_1 + (pc + ra + rc)I_2]$
$= (k + mI_1 + nI_2)((p + qI_1 + rI_2)(a + bI_1 + cI_2))$
$= \phi((k + mI_1 + nI_2))(\phi((p + qI_1 + rI_2))(a + bI_1 + cI_2))$
$= \phi(\alpha)(\phi(\beta)x).$

5. For $1 = 1 + 0I_1 + 0I_2 \in F(I_1 I_2),$ we have
   $1x = (1 + 0I_1 + 0I_2)(a + bI_1 + cI_2)$
   $= \phi(1 + 0I_1 + 0I_2)(a + bI_1 + cI_2).$

Accordingly, $\mathbb{R}(I_1, I_2)$ is a strong refined neutrosophic vector space over $F(I_1, I_2)$.

□

Lemma 2.10. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $x = a+bI_1+cI_2$, $y = d+eI_1+fI_2$, $z = k+mI_1+nI_2, \in V(I_1, I_2)$, $\beta = u+vI_1+wI_2 \in K(I_1, I_2)$. Then*

1. $x + z = y + z$ *implies* $x = y$.
2. $0x = 0.$
3. $\beta 0 = 0.$
4. $(-\beta)x = \beta(-x) = -(\beta x).$

Proof.

1. $x + z = y + z$
   $(a + bI_1 + cI_2) + (k + mI_1 + nI_2) = (d + eI_1 + fI_2) + (k + mI_1 + nI_2)$
   $\Longrightarrow a + k + (b + m)I_1 + (c + n)I_2 = d + k + (e + m)I_1 + (f + n)I_2$
   $\Longleftrightarrow a + k = d + k, b + m = e + m$ and $c + n = f + n$
   $\Longleftrightarrow a = d, b = e$ and $c = f$
   $\Longrightarrow a + bI_1 + cI_2 = d + eI_1 + fI_2 \Longrightarrow x = y.$

2. Consider $\beta x = ((u + vI_1 + wI_2) + (0 + 0I_1 + 0I_2))(a + bI_1 + cI_2)$
   $= ((u + 0) + (v + 0)I_1 + (w + 0)I_2)(a + bI_1 + cI_2)$
   $= (u+0)a+((u+0)b+(v+0)a+(v+0)b+(v+0)c+(w+0)b)I_1+((u+0)c+(w+0)a+(w+0)c)I_2$
   $= ua + 0a + (ub + 0b + va + 0a + vb + 0b + vc + 0c + wb + 0b)I_1 + (uc + 0c + wa + 0a + wc + 0c)I_2$
   $= (ua+(ub+va+vb+vc+wb)I_1+(uc+wa+wc)I_2)+(0a+(0b+0a+0b+0c+0b)I_1+(0c+0a+0c)I_2)$
   $= \beta x + 0x$
   $\Longrightarrow 0x = 0.$

3. Since $\beta \in K(I_1, I_2)$,
   $\beta 0 + \beta 0 = (u0 + (u0 + v0 + v0 + v0 + w0)I_1 + (u0 + w0 + w0)I_2) + (u0 + (u0 + v0 + v0 + v0 + w0)I_1 + (u0 + w0 + w0)I_2)$
   $= (u0+u0+(u0+u0+v0+v0+v0+v0+v0+v0+w0+w0)I_1+(u0+u0+w0+w0+w0+w0)I_2)$
   $= (u(0+0)+(u(0+0)+v(0+0)+v(0+0)+v(0+0)+w(0+0))I_1+(u(0+0)+w(0+0)+w(0+0))I_2$
   $= (u0 + (u0 + v0 + v0 + v0 + w0)I_1 + (u0 + w0 + w0)I_2 = \beta 0$
   $\Longrightarrow \beta 0 = 0.$

4. Let $\beta \in K(I_1, I_2)$ and $x \in V(I_1, I_2)$.
   So $\beta x + (-\beta)x = (u + vI_1 + wI_2)(a + bI_1 + cI_2) + (-(u + vI_1 + wI_2))(a + bI_1 + cI_2)$
   $= (ua + (ub + va + vb + vc + wb)I_1 + (uc + wa + wc)I_2) + (-ua + (-ub - va - vb - vc - wb)I_1 + (-uc - wa - wc)I_2)$
   $= (ua - ua + (ub - ub + va - va + vb - vb + vc - vc + wb - wb)I_1 + (uc - uc + wa - wa + wc - wc)I_2)$
   $= ((u-u)a+((u-u)b+(v-v)a+(v-v)b+(v-v)c+(w-w)b)I_1+((u-u)c+(w-w)a+(w-w)c)I_2)$
   $= (0a + (0b + 0a + 0b + 0c + 0b)I_1 + (0c + 0a + 0c)I_2)$
   $= (0 + 0I_1 + 0I_2)(a + bI_1 + cI_2) = 0x = 0$ by 2.
   Then $\beta x + (-\beta)x = 0 \Longrightarrow (-\beta)x = -\beta x.$
   So $\beta x + \beta(-x) = (u + vI_1 + wI_2)(a + bI_1 + cI_2) + ((u + vI_1 + wI_2))(-(a + bI_1 + cI_2)) = (ua + (ub + va + vb + vc + wb)I_1 + (uc + wa + wc)I_2) + (u(-a) + (u(-b) + v(-a) + v(-b) + v(-c) + w(-b))I_1 + (u(-c) + w(-a) + w(-c))I_2)$
   $= (ua + u(-a) + (ub + u(-b) + va + v(-a) + vb + v(-b) + vc + v(-c) + wb + w(-b))I_1 + (uc + u(-c) + wa + w(-a) + wc + w(-c))I_2)$
   $= (u(a-a)+(u(b-b)+v(a-a)+v(b-b)+v(c-c)+w(b-b))I_1+(u(c-c)+w(a-a)+w(c-c))I_2)$
   $= (u0 + (u0 + v0 + v0 + v0 + w0)I_1 + (u0 + w0 + w0)I_2) = (u + vI_1 + wI_2)(0 + 0I_1 + 0I_2)$
   $= \beta 0 = 0$ by 3.
   Then $\beta x + \beta(-x) = 0 \Longrightarrow \beta(-x) = -\beta x.$

□

Definition 2.11. Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $W(I_1, I_2)$ be a nonempty subset of $V(I_1, I_2)$. $W(I_1, I_2)$ is called a strong refined neutrosophic subspace of $V(I_1, I_2)$ if $W(I_1, I_2)$ is itself a strong refined neutrosophic vector space over $K(I_1, I_2)$. It is essential that $W(I_1, I_2)$ contains a proper subset which is a vector space.

Definition 2.12. Let $V(I_1, I_2)$ be a weak refined neutrosophic vector space over a field K and let $W(I_1, I_2)$ be a nonempty subset of $V(I_1, I_2)$. $W(I_1, I_2)$ is called a weak refined neutrosophic subspace of $V(I_1, I_2)$ if $W(I_1, I_2)$ is itself a weak refined neutrosophic vector space over K. It is essential that $W(I_1, I_2)$ contains a proper subset which is a vector space.

Proposition 2.13. *Let $V(I_1, I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ and let $W(I_1, I_2)$ be a nonempty subset of $V(I_1, I_2)$. $W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$ if and only if the following conditions hold:*

1. *$u, v \in W(I_1, I_2)$ implies $u + v \in W(I_1, I_2)$.*
2. *$u \in W(I_1, I_2)$ implies $\alpha u \in W(I_1, I_2)$ for all $\alpha \in K(I_1, I_2)$.*
3. *$W(I_1, I_2)$ contains a proper subset which is a vector space.*

Example 2.14. Let $V(I_1, I_2)$ be a weak (strong) refined neutrosophic vector space. $V(I_1, I_2)$ is a weak (strong) refined neutrosophic subspace called a trivial weak (strong) refined neutrosophic subspace.

Example 2.15. Let $K(I_1, I_2) = \mathbb{R}(I_1, I_2)$ be a refined neutrosophic field and $V(I_1, I_2) = \mathbb{R}^3(I_1, I_2)$ be a strong refined neutrosophic vector space. Take $W(I_1, I_2)$ to be the set of all vectors in $V(I_1, I_2)$ whose last component is $0 = 0 + 0I_1 + 0I_2$. Then $W(I_1, I_2)$ is strong refined neutrosophic subspace of $V(I_1, I_2)$.

Proof. To see this, let

$$W(I_1, I_2) = \{(x = a + bI_1 + cI_2, y = d + eI_1 + fI_2, 0 = 0 + 0I_1 + 0I_2) \in V(I_1, I_2) : a, b, c, d, e, f \in V\}.$$

1. Given that $u, v \in W(I_1, I_2)$, where $u = (x, y, 0)$ and $v = (x', y', 0)$. Then
$u + v = (x, y, 0) + (x', y', 0) = (a + bI_1 + cI_2, d + eI_1 + fI_2, 0 + 0I_1 + 0I_2) + (a' + b'I_1 + c'I_2, d' + e'I_1 + f'I_2, 0 + 0I_1 + 0I_2)$
$= (a + a' + (b + b')I_1 + (c + c')I_2, d + d' + (e + e')I_1 + (f + f')I_2, 0 + 0 + (0 + 0)I_1 + (0 + 0)I_2)$
$= (a + a' + (b + b')I_1 + (c + c')I_2, d + d' + (e + e')I_1 + (f + f')I_2, 0 + 0I_1 + 0I_2)$.
Hence we have that $u + v \in W(I_1, I_2)$.

2. Given $u \in W(I_1, I_2)$ and scalar $\alpha \in K(I_1, I_2)$ with $\alpha = r + sI_1 + tI_2$.
Then $\alpha u = (r + sI_1 + tI_2)(a + bI_1 + cI_2, d + eI_1 + fI_2, 0 + 0I_1 + 0I_2)$
$((r + sI_1 + tI_2)(a + bI_1 + cI_2), (r + sI_1 + tI_2)(d + eI_1 + fI_2), (r + sI_1 + tI_2)(0 + 0I_1 + 0I_2))$
$= (ra + (rb + sa + sb + sc + tb)I_1 + (rc + ta + tc)I_2, \quad rd + (re + sd + se + sf + te)I_1 + (rf + td + tf)I_2, \quad r0 + (r0 + s0 + s0 + s0 + t0)I_1 + (r0 + t0 + t0)I_2)$
$= (ra + (rb + sa + sb + sc + tb)I_1 + (rc + ta + tc)I_2, \quad rd + (re + sd + se + sf + te)I_1 + (rf + td + tf)I_2, \quad 0 + 0I_1 + 0I_2) \in W(I_1, I_2)$.

3. Since $W \subset W(I_1, I_2)$ is a proper subset which is a vector space, $W(I_1, I_2)$ is strong refined neutrosophic subspace.

□

Example 2.16. Let $V(I_1, I_2) = \mathbb{R}^2(I_1, I_2)$ be a strong refined neutrosophic vectors space over a refined neutrosophic field $\mathbb{R}(I_1, I_2)$ let

$$W(I_1, I_2) = \{(x = a + bI_1 + cI_2, y = d + eI_1 + fI_2) \in V(I_1, I_2) : x = y \ \text{with} a, b, c, d, e, f \in V\}.$$

Then $W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.

Example 2.17. Let $V(I_1, I_2) = M_{n\times n}(I_1, I_2) = \{[a_{ij}] : a_{ij} \in \mathbb{R}(I_1, I_2)\}$ be a strong refined neutrosophic vector space over $R(I_1, I_2)$ and let $W(I_1, I_2) = A_{n\times n}(I_1, I_2) = \{[b_{ij}] : b_{ij} \in \mathbb{R}(I_1, I_2) \ \text{ and } \ \text{trace(A)}= 0\}$. Then $W(I_1, I_2)$ is a strong refined neutrosophic subspace of $V(I_1, I_2)$.

Example 2.18. Let $V(I_1,I_2)=\mathbb{R}^3(I_1,I_2)$ be a strong refined neutrosophic vectors space of column refined neutrosophic vectors of length 3 over a refined neutrosophic field $\mathbb{R}(I_1,I_2)$. Consider

$$W(I_1,I_2)=\left\{\begin{pmatrix} x\\ y\\ 0\end{pmatrix};x=a+bI_1+cI_2,y=d+eI_1+fI_2\in V(I_1,I_2)\ \ a,b,c,d,e,f\in V\right\}\subseteq V(I_1,I_2).$$

$W(I_1,I_2)$ consisting of all refined neutrosophic vectors with $0=0+0I_1+0I_2$ in the last entry. Then $W(I_1,I_2)$ is a strong refined neutrosophic subspace of $V(I_1,I_2)$.

Proposition 2.19. *Let $V(I_1,I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1,I_1)$ and Let $\{U_n(I_1,I_2)\}_{n\in\lambda}$ be a family of strong refined neutrosophic subspaces of $V(I_1,I_2)$. Then $\bigcap_{n\in\lambda}U_n(I_1,I_2)$ is a strong refined neutrosophic subspace of $V(I_1,I_2)$.*

Proof. Consider the collection of strong refined neutrosophic subspaces $\{U_n(I_1,I_2):n\in\lambda\}$ of $V(I_1,I_2)$. Take $u=a+bI_1+cI_2$, $v=d+eI_1+fI_2$, $\alpha=k+pI_1+qI_2$ and $\beta=r+sI_1+tI_2$.
Let $u,v\in\bigcap_{n\in\lambda}U_n(I_1,I_2)$ then $u,v\in U_n(I_1,I_2)\ \forall\ n\in\lambda$. Now for all scalars $\alpha,\beta\in K(I_1,I_2)$ we have that $\alpha u+\beta v=(k+pI_1+qI_2)(a+bI_1+cI_2)+(r+sI_1+tI_2)(d+eI_1+fI_2)$
$=(ka+(kb+pa+pb+pc+qb)I_1+(kc+qa+qc)I_2)+(rd+(re+sd+se+sf+te)I_1+(rf+td+tf)I_2)$
$=(ka+rd)+(kb+pa+pb+pc+qb+re+sd+se+sf+te)I_1+(kc+qa+qc+rf+td+tf)I_2$.
Therefore $\alpha u+\beta v\in U_n(I_1,I_2)\ \forall\ n\in\lambda\Longrightarrow\alpha u+\beta v\in\bigcap_{n\in\lambda}U_n(I_1,I_2)$.
Lastly, since $U_n(I_1,I_2)$ for all $n\in\lambda$ contains a proper subset U_n which is vector space, we have that $\bigcap_{n\in\lambda}U_n(I_1,I_2)$ is a strong refined neutrosophic subspace. ☐

Proposition 2.20. *Let $V(I_1,I_2)$ be a strong refined neutrosophic vector space over the neutrosophic field $K(I_1,I_2)$ and let $U_1(I_1,I_2),U_2(I_1,I_2)$ be any strong refined neutrosophic subspaces of $V(I_1,I_2)$. Then $U_1(I_1,I_2)\cup U_2(I_1,I_2)$ is a strong refined neutrosophic subspaces if and only if $U_1(I_1,I_2)\subseteq U_2(I_1,I_2)$ or $U_1(I_1,I_2)\supseteq U_2(I_1,I_2)$.*

Proof. Let $U_1(I_1,I_2)$ and $U_2(I_1,I_2)$ be any strong refined neutrosophic subspaces of $V(I_1,I_2)$.
$\Longrightarrow$ Now, suppose $U_1(I_1,I_2)\subseteq U_2(I_1,I_2)$ or $U_1(I_1,I_2)\supseteq U_2(I_1,I_2)$ then we shall show the $U_1(I_1,I_2)\cup U_2(I_1,I_2)$ is a strong refined neutrosophic subspaces of $V(I_1,I_2)$.
Without loss of generality, suppose that $U_1(I_1,I_2)\subseteq U_2(I_1,I_2)$.
Then we have that $U_1(I_1,I_2)\cup U_2(I_1,I_2)=U_2(I_1,I_2)$. But $U_2(I_1,I_2)$ is defined to be a strong refined neutrosophic subspace of $V(I_1,I_2)$, so we can say that $U_1(I_1,I_2)\cup U_2(I_1,I_2)$ is a strong refined neutrosophic subspace of $V(I_1,I_2)$.
$\Longleftarrow$ We want to show that if $U_1(I_1,I_2)\cup U_2(I_1,I_2)$ is a strong refined neutrosophic subspace of $V(I_1,I_2)$ then either $U_1(I_1,I_2)\subseteq U_2(I_1,I_2)$ or $U_1(I_1,I_2)\supseteq U_2(I_1,I_2)$.
Now suppose that $U_1(I_1,I_2)\cup U_2(I_1,I_2)$ is a strong refined neutrosophic subspace of $V(I_1,I_2)$ and suppose by contradiction that $U_1(I_1,I_2)\nsubseteq U_2(I_1,I_2)$ or $U_1(I_1,I_2)\nsupseteq U_2(I_1,I_2)$.
Thus there exist elements $x_1=a_1+b_1I_1+c_1I_2\in U_1(I_1,I_2)\backslash U_2(I_1,I_2)$ and $x_2=a_2+b_2I_1+c_2I_2\in U_2(I_1,I_2)\backslash U_1(I_1,I_2)$. So we have that $x_1,x_2\in U_1(I_1,I_2)\cup U_2(I_1,I_2)$, since $U_1(I_1,I_2)\cup U_2(I_1,I_2)$ is a strong refined neutrosophic subspace, we must have that $x_1+x_2=x_3\in U_1(I_1,I_2)\cup U_2(I_1,I_2)$.
Therefore $x_1+x_2=x_3\in U_1(I_1,I_2)$ or $x_1+x_2=x_3\in U_2(I_1,I_2)$
$\Longrightarrow x_2=x_3-x_1\in U_1(I_1,I_2)$ or $x_1=x_3-x_2\in U_2(I_1,I_2)$ which is a contradiction.
Hence $U_1(I_1,I_2)\subseteq U_2(I_1,I_2)$ or $U_1(I_1,I_2)\supseteq U_2(I_1,I_2)$ as required. ☐

Remark 2.21. Let $V(I_1,I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1,I_2)$ and let $W_1(I_1,I_2)$ and $W_2(I_1,I_2)$ be two distinct strong refined neutrosophic subspaces of $V(I_1,I_2)$. $W_1(I_1,I_2)\cup W_2(I)$ is a strong refined neutrosophic subspace of $V(I_1,I_2)$ iff $W_1(I_1,I_2)\subseteq W_2(I_1,I_2)$ or $W_2(I_1,I_2)\subseteq W_1(I_1,I_2)$.

Definition 2.22. Let $U(I_1,I_2)$ and $W(I_1,I_2)$ be any two strong refined neutrosophic subspaces of a strong refined neutrosophic vector space $V(I_1,I_2)$ over a neutrosophic field $K(I_1,I_2)$.

1. The sum of $U(I_1,I_2)$ and $W(I_1,I_2)$ denoted by $U(I_1,I_2)+W(I_1,I_2)$ is defined by the set $\{u+w:u\in U(I_1,I_2),w\in W(I_1,I_2)\}$.

2. $V(I_1,I_2)$ is said to be the direct sum of $U(I_1,I_2)$ and $W(I_1,I_2)$ written $V(I_1,I_2)=U(I_1,I_2)\oplus W(I_1,I_2)$ if every element $v\in V(I_1,I_2)$ can be written uniquely as $v=u+w$ where $u\in U(I_1,I_2)$ and $w\in W(I_1,I_2)$.

Proposition 2.23. *Let $U(I_1,I_2)$ and $W(I_1,I_2)$ be any two strong refined neutrosophic subspaces of a strong refined neutrosophic vector space $V(I_1,I_2)$, over a refined neutrosophic field $K(I_1,I_2)$.*
Then $U(I_1,I_2)+W(I_1,I_2)$ is a strong refined neutrosophic subspace of $V(I_1,I_2)$.

Proof. Since $U(I_1,I_2)$ and $W(I_1,I_2)$ are nonempty strong refined neutrosophic subspaces, $U(I_1,I_2)+W(I_1,I_2)\neq\{\}$.
Obviously $U(I_1,I_2)+W(I_1,I_2)$ contains a proper subset $U+W$ which is a vector space.
Now let $x,y\in U(I_1,I_2)+W(I_1,I_2)$ and $\alpha,\beta\in K(I_1,I_2)$.
Then $x=(u_1+u_2I_1+u_3I_2)+(w_1+w_2I_1+w_3I_2)$, $y=(u_4+u_5I_1+u_6I_2)+(w_4+w_5I_1+w_6I_2)$ where $u_i\in U, w_i\in W$, with $i=1,2,3,4,5,6$. $\alpha=k+mI_1+nI_2, \beta=p+qI_1+rI_2$ where $k,m,n,p,q,r\in K(I_1,I_2)$.
Then,
$\alpha x+\beta y=(k+mI_1+nI_2)[(u_1+w_1)+(u_2+w_2)I_1+(u_3+w_3)I_2]+(p+qI_1+rI_2)[(u_4+w_4)+(u_5+w_5)I_1+(u_6+w_6)I_2]$
$=[(ku_1+kw_1)+(ku_2+kw_2+mu_1+mw_1+mu_2+mw_2+mu_3+mw_3+nu_2+nw_2)I_1+(ku_3+kw_3+nu_1+nw_1+nu_3+nw_3)I_2]+[(pu_4+pw_4)+(pu_5+pw_5+qu_4+qw_4+qu_5+qw_5+qu_6+qw_6+ru_5+rw_5)I_1+(pu_6+pw_6+ru_4+rw_4+ru_6+rw_6)I_2]$
$=[(ku_1+pu_4)+(ku_2+pu_5+mu_1+qu_4+mu_2+qu_5+mu_3+qu_6+nu_2+ru_5)I_1+(ku_3+pu_6+nu_1+ru_4+nu_3+ru_6)I_2]+[(kw_1+pw_4)+(kw_2+pw_5+mw_1+qw_4+mw_2+qw_5+mw_3+qw_6+nw_2+rw_5)I_1+(kw_3+pw_6+nw_1+rw_4+nw_3+rw_6)I_2]\in U(I_1,I_2)+W(I_1,I_2)$.
Accordingly $U(I_1,I_2)+W(I_1,I_2)$ is a strong refined neutrosophic subspace of $V(I_1,I_2)$. ☐

Proposition 2.24. *Let $U(I_1,I_2)$ and $W(I_1,I_2)$ be strong refined neutrosophic subspaces of a strong refined neutrosophic vector space $V(I_1,I_2)$ over a refined neutrosophic field $K(I_1,I_2)$.*
$V(I_1,I_2)=U(I_1,I_2)\oplus W(I_1,I_2)$ if and only if the following conditions hold:

1. *$V(I_1,I_2)=U(I_1,I_2)+W(I_1,I_2)$ and*
2. *$U(I_1,I_2)\cap W(I_1,I_2)=\{0\}$.*

Proof. The proof is similar to the proof in classical case. ☐

Example 2.25. Let $V(I_1,I_2)=\mathbb{R}^3(I_1,I_2)$ be a strong refined neutrosophic vector space over a refined neutrosophic field $R(I_1,I_2)$ and let
$U(I_1,I_2)=\{(u,0,w):u=a+bI_1+cI_2, w=g+hI_1+kI_2\in R(I_1,I_2)\}$ and
$W(I_1,I_2)=\{(0,v,0):v=d+eI_1+fI_2\in R(I_1,I_2)\}$,
be strong refined neutrosophic subspaces of $V(I_1,I_2)$. Then $V(I_1,I_2)=U(I_1,I_2)\oplus W(I_1,I_2)$.

To see this, let $x=(u,v,w)\in V(I_1,I_2)$, then $x=(u,0,w)+(0,v,0)$, so $x\in U(I_1,I_2)+W(I_1,I_2)$.
Hence $V(I_1,I_2)=U(I_1,I_2)+W(I_1,I_2)$.
To show that $U(I_1,I_2)\cap W(I_1,I_2)=\{0\}$, let $x=(u,v,w)\in U(I_1,I_2)\cap W(I_1,I_2)$.
Then $v=0$, i.e $d+eI_1+fI_2=0+0I_1+0I_2$ because x lies in $U(I_1,I_2)$, and $u=w=0$ i.e $a+bI_1+cI_2=g+hI_1+kI_2=0+0I_1+0I_2$ because x lies in $W(I_1,I_2)$.
Thus $x=(0,0,0)=0$, so $0=0+0I_1+0I_2$ is the only refined neutrosophic vector in $U(I_1,I_2)\cap W(I_1,I_2)$.
So $U(I_1,I_2)\cap W(I_1,I_2)=\{0+0I_1+0I_2\}=\{0\}$.
Hence, $V(I_1,I_2)=U(I_1,I_2)\oplus W(I_1,I_2)$.

Example 2.26. In the strong refined neutrosophic vector space $V(I_1,I_2)=\mathbb{R}^5(I_1,I_2)$, consider the strong refined neutrosophic subspaces

$$U(I_1,I_2)=\{(a,b,c,0,0)|a=x_1+y_1I_1+z_1I_2, b=x_2+y_2I_1+z_2I_2, \text{and}\quad c=x_3+y_3I_1+z_3I_2\in V(I_1,I_2)\}$$

and

$$W=\{0,0,0,d,e)|d=x_4+y_4I_1+z_4I_2,\;\; e=x_5+y_5I_1+z_5I_2\in V(I_1,I_2)\}.$$

Then $V(I_1,I_2)=U(I_1,I_2)\oplus W(I_1,I_2)$.

To see this, let $x=(a,b,c,d,e)$ be any refined neutrosophic vector in $V(I_1,I_2)$, then $x=(a,b,c,0,0)+(0,0,0,d,e)$, so x lies in $U(I_1,I_2)+W(I_1,I_2)$.
Hence $V(I_1,I_2)=U(I_1,I_2)+W(I_1,I_2)$. To show that $U(I_1,I_2)\cap W(I_1,I_2)=\{0\}$, let $x=(a,b,c,d,e)$ lie in $U(I_1,I_2)\cap W(I_1,I_2)$.
Then $d=e=0$ i.e $x_4+y_4I_1+z_4I_2=x_5+y_5I_1+z_5I_2=0+0I_1+0I_2$ because x lies in $U(I_1,I_2)$, and

$a=b=c=0$ i.e, $x_1+y_1I_1+z_1I_2=x_2+y_2I_1+z_2I_2=x_3+y_3I_1+z_3I_2=0+0I_1+0I_2$ because x lies in $W(I_1,I_2)$.
Thus $x=(0,0,0,0,0)=0$, so $0=0+0I_1+0I_2$ is the only refined neutrosophic vector in $U(I_1,I_2)\cap W(I_1,I_2)$. So $U(I_1,I_2)\cap W(I_1,I_2)=\{0+0I_1+0I_2\}$.
Hence, $V(I_1,I_2)=U(I_1,I_2)\oplus W(I_1,I_2)$.

Example 2.27. Let $V(I_1,I_2)=P_{2n}(I_1,I_2)$ be the strong refined neutrosophic vector space over a neutrosophic field $K(I_1,I_2)$. Then let

$$U_1(I_1,I_2)=\{p(t)\in P_{2n}:a_0+a_2t^2+\cdots+a_{2n}t^{2n},\text{with}\quad a_0,a_2\cdots a_{2n}\in\mathbb{R}(I_1,I_2)\}$$

$$U_2(I_1,I_2)=\{p(t)\in P_{2n}:a_1+a_3t^3+\cdots+a_{2n-1}t^{2n-1},\text{with}\quad a_1,a_3\cdots a_{2n-1}\in\mathbb{R}(I_1,I_2)\}$$

be strong refined neutrosophic subspaces of $P_{2n}(I_1,I_2)$.
Then $P_{2n}(I_1,I_2)=U_1(I_1,I_2)\oplus U_2(I_1,I_2)$.

Proposition 2.28. *Let $U(I_1,I_2)$ and $V(I_1,I_2)$ be strong refined neutrosophic vector spaces over a refined neutrosophic field $K(I_1,I_2)$. Then*

$$U(I_1,I_2)\times V(I_1,I_2)=\{(u,v):u\in U(I_1,I_2),v\in V(I_1,I_2)\}$$

is a strong refined neutrosophic vector space over $K(I_1,I_2)$ where addition and multiplication are defined by:

$$(u,v)+(u',v')=(u+u',v+v'),$$

$$\alpha(u,v)=(\alpha u,\alpha v).$$

Proof. 1. We want to show that $(U(I_1,I_2)\times V(I_1,I_2),+)$ is refined neutrosophic abelian group .

(a) Clearly $(U(I_1,I_2)\times V(I_1,I_2,+)$ is closed, since for
$(u,v),(u',v')\in(U(I_1,I_2)\times V(I_1,I_2))$ where $(u,v)=((a+bI_1+cI_2),(d+eI_1+fI_2))$ we have that
$(u,v)+(u',v')=[((a+bI_1+cI_2),(d+eI_1+fI_2))+((a'+b'I_1+c'I_2),(d'+e'I_1+f'I_2))]$
$=[(a+a'+(b+b')I_1+(c+c')I_2),(d+d'+(e+e')I_1+(f+f')I_2]$
$=(u+u',v+v')\in U(I_1,I_2)\times V(I_1,I_2)$.

(b) Let $(u,v),(u',v')$ and $(u'',v'')\in U(I_1,I_2)\times V(I_1,I_2)$. Then
$[(u,v)+((u,'v')+(u'',v''))]=[((a+bI_1+cI_2),(d+eI_1+fI_2))$
$+((a'+a''+(b'+b'')I_1+(c'+c'')I_2),(d'+d''+(e'+e'')I_1+(f'+f'')I_2)]$
$=(a+a'+a''+(b+b'+b'')I_1+(c+c'+c'')I_2),(d+d'+d''+(e+e'+e'')I_1+(f+f'+f'')I_2)$
$=[(a+a'+(b+b')I_1+(c+c')I_2),(d+d'+(e+e')I_1+(f+f')I_2)]+$
$((a''+b''I_1+c''I_2),(d''+e''I_1+f''I_2))$
$=[(u,v)+(u',v')]+(u'',v'')$.
Then we say that $''+''$ is associative.

(c) The identity in $U(I_1,I_2)\times V(I_1,I_2))$ is $(0_{U(I_1,I_2)},0_{V(I_1,I_2)})$ where
$0_{U(I_1,I_2)}$ is the identity in $U(I_1,I_2)$ and $0_{V(I_1,I_2)}$ is the identity in $V(I_1,I_2)$ then
$(u,v)+(0_{U(I_1,I_2)},0_{V(I_1,I_2)})$
$=[(a+bI_1+cI_2),(d+eI_1+fI_2)]+[(0+0I_1+0I_2),(0+0I_1+0I_2)]$
$=(a+0+(b+0)I_1+(c+0)I_2),(d+0+(e+0)I_1+(f+0)I_2)$
$=(0+a+(0+b)I_1+(0+c)I_2),(0+d,(0+e)I_1+(0+f)I_2)$
$=(a+bI_1+cI_2),(d+eI_1+fI_2)=(u,v)$.

(d) For each $(u,v)\in U(I_1,I_2)\times V(I_1,I_2))$ the inverse is $(-u,-v)$ where
$-u\in U(I_1,I_2)$ and $-v\in V(I_1,I_2)$ is the inverse of u and v respectively. Then
$(u,v)+(-u,-v)=[((a+bI_1+cI_2),(d+eI_1+fI_2))+((-a-bI_1-cI_2),(-d-eI_1-fI_2))]$
$=(a-a+(b-b)I_1+(c-c)I_2),((d-d)+(e-e)I_1+(f-f)I_2)$
$=(-a+a+(-b+b)I_1+(-c+c)I_2),((-d+d)+(-e+e)I_1+(-f+f)I_2)$
$=(0+0I_1+0I_2),(0+0I_1+0I_2)=(0_{U(I_1,I_2)},0_{V(I_1,I_2)})$.

(e) For $(u,v),(u',v')\in U(I_1,I_2)\times V(I_1,I_2)$ we have that
$(u,v)+(u',v')=[((a+bI_1+cI_2),(d+eI_1+fI_2))+((a'+b'I_1+c'I_2),(d'+e'I_1+f'I_2))]$
$=[(a+a'+(b+b')I_1+(c+c')I_2),((d+d')+(e+e')I_1+(f+f')I_2)]$
$=[(a'+a+(b'+b)I_1+(c'+c)I_2),((d'+d)+(e'+e)I_1+(f'+f)I_2)]$
$=[((a'+b'I_1+c'I_2),(d'+e'I_1+f'I_2))+((a+bI_1+cI_2),(d+eI_1+fI_2))]=(u',v')+(u,v)$.
Then we say that $''+''$ is commutative.

Let $\alpha = k + mI_1 + nI_2, \beta = r + sI_1 + tI_2 \in K(I_1, I_2)$, then

2. $\alpha((u, v) + (u', v')) = \alpha[((a + bI_1 + cI_2), (d + eI_1 + fI_2)) + ((a' + b'I_1 + c'I_2), (d' + e'I_1 + f'I_2))]$
   $= \alpha[(a + a' + (b + b')I_1 + (c + c')I_2), ((d + d') + (e + e')I_1 + (f + f')I_2)]$
   $= (\alpha a + \alpha a' + (\alpha b + \alpha b')I_1 + (\alpha c + \alpha c')I_2), ((\alpha d + \alpha d') + (\alpha e + \alpha e')I_1 + (\alpha f + \alpha f')I_2)$
   $= ((\alpha a + \alpha bI_1 + \alpha cI_2), (\alpha d + \alpha eI_1 + \alpha fI_2)) + ((\alpha a' + \alpha b'I_1 + \alpha c'I_2), (\alpha d' + \alpha e'I_1 + \alpha f'I_2))$
   $= (\alpha u, \alpha v) + (\alpha u', \alpha v')$
   $= \alpha(u, v) + \alpha(u', v')$.

3. $(\alpha + \beta)(u, v) = (k + r + (m + s)I_1 + (n + t)I_2)((a + bI_1 + cI_2), (d + eI_1 + fI_2))$
   $= ((k + r + (m + s)I_1 + (n + t)I_2)(a + bI_1 + cI_2), \quad (k + r + (m + s)I_1 + (n + t)I_2)(d + eI_1 + fI_2))$
   $= [(ka + ra + (ma + sa)I_1 + (na + ta)I_2 + (kb + rb + mb + sb + nb + tb)I_1 + ((mc + sc)I_1 + (kc + rc + nc + tc)I_2)), \quad (kd + rd + (md + sd)I_1 + (nd + td)I_2 + (ke + re + me + se + ne + te)I_1 + ((mf + sf)I_1 + (kf + rf + nf + tf)I_2))]$
   $= [(ka + maI_1 + naI_2) + (kb + mb + nb)I_1 + (mcI_1 + (kc + nc)I_2), \quad (kd + mdI_1 + ndI_2) + (ke + me + ne)I_1 + (mfI_1 + (kf + nf)I_2)] + [(ra + saI_1 + taI_2 + (rb + sb + tb)I_1 + (scI_1 + (rc + tc)I_2)), \quad (rd + sdI_1 + tdI_2) + (re + se + te)I_1 + (sfI_1 + (rf + tf)I_2))]$
   $= [(k + mI_1 + n_2)((a + bI_1 + cI_2), (d + eI_1 + fI_2))] + [(r + sI_1 + tI_2)((a + bI_1 + cI_2), (d + eI_1 + fI_2))]$
   $= \alpha(u, v) + \beta(u, v)$.

4. $(\alpha\beta)(u, v) = ((k + mI_1 + nI_2)(r + sI_1 + tI_2))(u, v)$
   $= (kr + (ks + mr + ms + mt + ns)I_1 + (kt + nr + nt)I_2)((a + bI_1 + cI_2), (d + eI_1 + fI_2))$
   $= (kra + (krb + ksa + mra + msa + mta + nsa + ksb + mrb + msb + mtb + nsb + ksc + mrc + msc + mtc + nsc + ktb + nrb + ntb)I_1 + (krc + kta + nra + nta + ktc + nrc + ntc)I_2, \quad krd + (kre + ksd + mrd + msd + mtd + nsd + kse + mre + mse + mte + nse + ksf + mrf + msf + mtf + nsf + kte + nre + nte)I_1 + (krf + ktd + nrd + ntd + ktf + nrf + ntf)I_2)$
   $= ((k + mI_1 + nI_2)(ra + (rb + sa + sb + sc + tb)I_1 + (rc + ta + tc)I_2) \quad (k + mI_1 + nI_2)(rd + (re + sd + se + sf + te)I_1 + (rf + td + tf)I_2))$
   $= \alpha(\beta u, \beta v)$.

5. For $(1 + 1I_1 + 1I_2) \in K(I_1, I_2)$, we have
   $(1 + 1I_1 + 1I_2) \cdot (u, v) = (1 + 1I_1 + 1I_2)(a + bI_1 + cI_2, d + eI_1 + fI_2)$
   $= 1a + (1b + 1a + 1b + 1c + 1b)I_1 + (1c + 1a + 1c)I_2 \quad 1d + (1e + 1d + 1e + 1f + 1e)I_1 + (1f + 1d + 1f)I_2$
   $= ((1 + 1I_1 + 1I_2)(a + bI_1 + cI_2), (1 + 1I_1 + 1I_2)(d + eI_1 + fI_2))$
   $= (a + bI_1 + cI_2, d + eI_1 + fI_2) = (u, v)$.

□

Proposition 2.29. *Let $U(I_1, I_2)$ be weak refined neutrosophic vector spaces over a field and V be a vector space over a field K. Then*

$$U(I_1, I_2) \times V = \{(u, v) : u = (a + bI_1 + cI_2) \in U(I_1, I_2), v \in V\}$$

is a weak refined neutrosophic vector space over K where addition and multiplication are defined by:

$$(u, v) + (u', v') = (u + u', v + v'),$$

$$\alpha(u, v) = (\alpha u, \alpha v).$$

Proof. The proof follows the same approach as in the proof of Proposition 2.28 □

Definition 2.30. Let $W(I_1, I_2)$ be a strong refined neutrosophic subspace of a strong refined neutrosophic vector space $V(I_1, I_2)$ over a refined neutrosophic field $K(I_1, I_2)$. The quotient $V(I_1, I_2)/W(I_1, I_2)$ is defined by the set

$$\{v + W(I_1, I_2) : v \in V(I_1, I_2)\}.$$

Proposition 2.31. *The quotient $V(I_1, I_2)/W(I_1, I_2)$ is a strong refined neutrosophic vector space over a refined neutrosophic field $K(I_1, I_2)$ if addition and multiplication are defined for all $u + W(I_1, I_2), v + W(I_1, I_2) \in V(I_1, I_2)/W(I_1, I_2)$ and $\alpha \in K(I_1, I_2)$ as follows:*

$$(u + W(I_1, I_2)) + (v + W(I_1, I_2)) = (u + v) + W(I_1, I_2),$$

$$\alpha(u + W(I_1, I_2)) = \alpha u + W(I_1, I_2).$$

This strong refined neutrosophic vector space $(V(I_1, I_2)/W(I_1, I_2), +, .)$ over a neutrosophic field $K(I_1, I_2)$ is called a strong refined neutrosophic quotient space.

3 Conclusion

In this paper, we have presented the concept of refined neutrosophic vector spaces. Weak(strong) refined neutrosophic vector spaces and subspaces, and, strong refined neutrosophic quotient vector spaces were studied. Several interesting results and examples were presented. It was shown that every weak (strong) refined neutrosophic vector space is a vector space and it was equally shown that every strong refined neutrosophic vector space is a weak refined neutrosophic vector space. This work will be continued in our next paper titled "On Refined Neutrosophic Vector Spaces II".

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