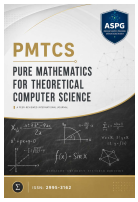
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FULL LENGTH ARTICLE

Hybrid Functorial Structure and MultiFunctorial Structure

Takaaki Fujita[1,*] **Ajoy Kanti Das**[2]

[1] Independent Researcher, Tokyo, Japan

[2] Associate Professor, Department of Mathematics, Tripura University, Agartala-799022, Tripura, India

Emails: Takaaki.fujita060@gmail.com · ajoykantidas@gmail.com

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ABSTRACT

A Functorial Structure is defined as a covariant functor $F : \mathscr{C} \to \mathbf{Set}$, assigning sets to objects and functions to morphisms, ensuring functoriality. In this paper, we introduce and formally define two new concepts: the HybridFunctorial Structure and the MultiFunctorial Structure. A HybridFunctorial Structure combines two functors on the same category, linked by a natural transformation, ensuring consistent pushforward compatibility. A MultiFunctorial Structure involves multiple functors indexed by a preorder, coherently related via natural transformations, forming compatible families with functorial consistency.

Keywords: Functorial Structure ▪ HybridFunctorial Structure ▪ MultiFunctorial Structure

1. PRELIMINARIES

In this section we collect the notation and basic notions used later. Unless explicitly stated otherwise, all underlying sets are finite.

1.1 Functorial Structure

A Functorial Structure is defined as a covariant functor $F : \mathscr{C} \to \mathbf{Set}$, assigning sets to objects and functions to morphisms, ensuring functoriality [1]. A Functorial Set can represent various frameworks such as Fuzzy Set [2, 3], Plithogenic Set [4, 5], Hesitant Fuzzy Set [6, 7], Intuitionistic Fuzzy Set [8, 9], Neutrosophic Set [10, 11], Quadripartitioned Neutrosophic Set [12, 13], Rough Set [14, 15], Soft Set [16, 17], and Probabilistic Set [18].

Definition 1.1 (Functorial Set)**.** Let $\mathscr{C}$ be a category and

$$F : \mathscr{C} \longrightarrow \mathbf{Set}$$

be a covariant functor. For any object $X \in \mathrm{Ob}(\mathscr{C})$, an F-set over X is an element

$$s \in F(X).$$

We denote the collection of all F-sets over X simply by $F(X)$. A morphism $f : X \to Y$ in $\mathscr{C}$ induces a pushforward

$$F(f) : F(X) \longrightarrow F(Y), \qquad s \longmapsto F(f)(s).$$

Example 1.2 (Real-world Functorial Set: project team selection with address-book unification)**.** Let $\mathscr{C} = \mathbf{FinSet}$ (finite sets and functions) and take the covariant functor $F = \mathscr{P}$ (powerset) defined by $F(X) = \mathscr{P}(X)$ and, for $f : X \to Y$,

$$F(f) : \mathscr{P}(X) \longrightarrow \mathscr{P}(Y),$$
$$F(f)(S) = f[S] = \{f(x) \mid x \in S\}.$$

Here, an F-set over X (Definition 1.1) is simply a chosen subset $S \subseteq X$.

Interpretation. Let

$$X = \{\text{Hiroko}, \text{Yutaka}, \text{Cara}\}$$

be employees at Site A, let $Y = \{a123, b234, c345\}$ be corpo-

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rate master IDs, and let $f : X \to Y$ be the ID-link

$$f(\text{Hiroko}) = a123, \quad f(\text{Yutaka}) = b234,$$
$$f(\text{Cara}) = c345.$$

A concrete F-set $s \in F(X)$ is a project team:

$$s = \{\text{Hiroko}, \text{Cara}\} \in \mathscr{P}(X).$$

The pushforward (image) along f is

$$F(f)(s) = \{a123, c345\} \in \mathscr{P}(Y),$$

which is the same team expressed in corporate IDs. If $g : Y \to Z$ later remaps IDs to anonymized tokens, then functoriality gives

$$F(g \circ f)(s) = F(g)\big(F(f)(s)\big),$$

i.e., anonymization after unification equals direct anonymization of the original team.

Definition 1.3 (Functorial Structure)**.** Let $\mathscr{C}$ be a category. A Functorial Structure on $\mathscr{C}$ is simply a covariant functor

$$F : \mathscr{C} \longrightarrow \mathbf{Set}.$$

For each object $X \in \mathrm{Ob}(\mathscr{C})$, an element $s \in F(X)$ is called an F-structure on X. Every morphism $f : X \to Y$ in $\mathscr{C}$ induces a pushforward

$$F(f) : F(X) \longrightarrow F(Y), \qquad s \longmapsto F(f)(s),$$

and the usual functoriality conditions

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \qquad F(g \circ f) = F(g) \circ F(f)$$

hold.

Example 1.4 (Real-world Functorial Structure: inventory aggregation from SKUs to categories (finite multiset functor))**.** Let $\mathscr{C} = \mathbf{FinSet}$ and define the covariant functor $F : \mathscr{C} \to \mathbf{Set}$ by

$$F(X) = \mathbb{N}^{(X)} = \{m : X \to \mathbb{N} \text{ with finite support}\},$$

viewed as finite multisets on X. For a function $f : X \to Y$, define

$$F(f) : \mathbb{N}^{(X)} \longrightarrow \mathbb{N}^{(Y)},$$
$$\big(F(f)(m)\big)(y) := \sum_{x \in f^{-1}(y)} m(x).$$

Thus $F(f)$ pushes forward counts by summing over fibers. One checks $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ by associativity and commutativity of addition.

Interpretation. Let $X = \{\text{sku}_1, \text{sku}_2, \text{sku}_3, \text{sku}_4\}$ and $Y = \{\text{CatA}, \text{CatB}\}$ with a classification $f : X \to Y$:

$$f(\text{sku}_1) = \text{CatA}, \quad f(\text{sku}_2) = \text{CatA},$$
$$f(\text{sku}_3) = \text{CatB}, \quad f(\text{sku}_4) = \text{CatB}.$$

A concrete F-structure $m \in F(X)$ is an inventory multiset (counts per SKU):

$$m(\text{sku}_1) = 5, \quad m(\text{sku}_2) = 2,$$
$$m(\text{sku}_3) = 7, \quad m(\text{sku}_4) = 3.$$

The pushforward to category-level stock is

$$\big(F(f)(m)\big)(\text{CatA}) = m(\text{sku}_1) + m(\text{sku}_2) = 7,$$
$$\big(F(f)(m)\big)(\text{CatB}) = m(\text{sku}_3) + m(\text{sku}_4) = 10.$$

Functoriality (explicit check). If $g : Y \to Z = \{\text{All}\}$ sends both categories to All, then

$$\big(F(g) \circ F(f)\big)(m)(\text{All}) = 7 + 10 = 17,$$

while pushing forward in one step yields

$$F(g \circ f)(m)(\text{All}) = \sum_{x \in X} m(x) = 5 + 2 + 7 + 3 = 17.$$

Hence $F(g \circ f) = F(g) \circ F(f)$ on m, as required.

Example 1.5 (Real-world Fuzzy Functorial Structure: product interest from SKUs to categories to global)**.** Let $\mathscr{C} = \mathbf{FinSet}$. Define a covariant functor

$$F : \mathscr{C} \longrightarrow \mathbf{Set},$$
$$F(X) = [0,1]^X = \{\mu : X \to [0,1]\},$$

whose elements are fuzzy sets (membership functions) [2, 19] on X. For a function $f : X \to Y$, define the pushforward $F(f) : F(X) \to F(Y)$ by the direct-image aggregator

$$\big(F(f)(\mu)\big)(y) := \sup\{\mu(x) \mid x \in X,\ f(x) = y\},$$
$$\sup \varnothing := 0.$$

Functoriality holds (on any $z \in Z$) because, for $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\begin{aligned}\big(F(g \circ f)(\mu)\big)(z) &= \sup_{x : g(f(x)) = z} \mu(x) \\ &= \sup_{y : g(y) = z} \ \sup_{x : f(x) = y} \mu(x) \\ &= \Big(F(g)\big(F(f)(\mu)\big)\Big)(z),\end{aligned}$$

and clearly $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Interpretation (concrete, real-life). Let $X = \{\text{sku}_1, \text{sku}_2, \text{sku}_3, \text{sku}_4\}$ be individual products, $Y = \{\text{CatA}, \text{CatB}\}$ be product categories, and $Z = \{\text{All}\}$ a single global bucket. Let $f : X \to Y$ classify SKUs to categories and $g : Y \to Z$ send both categories to All:

$$f(\text{sku}_1) = f(\text{sku}_2) = \text{CatA},$$
$$f(\text{sku}_3) = f(\text{sku}_4) = \text{CatB},$$
$$g(\text{CatA}) = g(\text{CatB}) = \text{All}.$$

A fuzzy set $\mu \in F(X)$ encodes customer interest scores for each SKU:

$$\mu(\text{sku}_1) = 0.80, \quad \mu(\text{sku}_2) = 0.30,$$
$$\mu(\text{sku}_3) = 0.60, \quad \mu(\text{sku}_4) = 0.20.$$

Pushing forward along f aggregates to category-level interest by taking the maximum on each fiber:

$$\big(F(f)(\mu)\big)(\text{CatA}) = \max\{0.80, 0.30\} = 0.80,$$
$$\big(F(f)(\mu)\big)(\text{CatB}) = \max\{0.60, 0.20\} = 0.60.$$

Pushing forward again along g yields the global interest score

$$F(g)\big(F(f)(\mu)\big)(\text{All}) = \max\{0.80, 0.60\} = 0.80.$$

Direct aggregation in one step agrees by functoriality:

$$\begin{aligned}\big(F(g\circ f)(\mu)\big)(\text{All}) &= \max\{0.80, 0.30, 0.60, 0.20\}\\ &= 0.80 = \big(F(g)\circ F(f)\big)(\mu)(\text{All}).\end{aligned}$$

Practical reading. F turns any mapping of items to coarser groupings into a consistent rule for fuzzy (graded) summaries:

$$\text{SKU scores} \xrightarrow[\text{along } f]{\text{max-aggregate}} \text{Category scores} \xrightarrow[\text{along } g]{\text{max-aggregate}} \text{Global score},$$

and the result is invariant to whether one aggregates stepwise or in a single pass.

Example 1.6 (Real-world Neutrosophic Functorial Structure: clinical triage from patients to wards to hospital)**.** Let $\mathscr{C} =$ **FinSet**. Define a covariant functor

$$\begin{aligned}F &: \mathscr{C} \longrightarrow \mathbf{Set},\\ F(X) &= \big([0,1]^3\big)^X = \{\nu : X \to [0,1]^3\},\end{aligned}$$

where for $x \in X$ we write

$$\nu(x) = \big(T_X(x), I_X(x), F_X(x)\big)$$

as the neutrosophic triple (truth/indeterminacy/falsity degrees) for a fixed predicate (e.g., "influenza suspicion"). For a function $f : X \to Y$, define the pushforward componentwise by fiberwise suprema:

$$\begin{aligned}\big(F(f)(\nu)\big)(y) = \Big(&\sup_{x:f(x)=y} T_X(x), \sup_{x:f(x)=y} I_X(x),\\ &\sup_{x:f(x)=y} F_X(x)\Big), \qquad \sup\varnothing := 0.\end{aligned}$$

Then $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ is immediate, and for $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $z \in Z$,

$$\begin{aligned}&\big(F(g\circ f)(\nu)\big)(z)\\ &= \Big(\sup_{x:g(f(x))=z} T_X(x),\\ &\qquad \sup_{x:g(f(x))=z} I_X(x),\\ &\qquad \sup_{x:g(f(x))=z} F_X(x)\Big)\\ &= \Big(\sup_{y:g(y)=z}\sup_{x:f(x)=y} T_X(x),\\ &\qquad \sup_{y:g(y)=z}\sup_{x:f(x)=y} I_X(x),\\ &\qquad \sup_{y:g(y)=z}\sup_{x:f(x)=y} F_X(x)\Big)\\ &= F(g)\big(F(f)(\nu)\big)(z),\end{aligned}$$

since $\sup_y \sup_{x\in f^{-1}(y)}(\cdot) = \sup_x(\cdot)$; hence $F(g\circ f) = F(g)\circ F(f)$.

Concrete instance (with explicit numbers). Let $X = \{p_1, p_2, p_3, p_4\}$ be patients, $Y = \{\text{WardA}, \text{WardB}\}$ wards, and $Z = \{\text{Hospital}\}$ a single node. Define

$$\begin{aligned}f : X \to Y, \quad & f(p_1) = f(p_2) = \text{WardA},\\ & f(p_3) = f(p_4) = \text{WardB},\\ g : Y \to Z, \quad & g(\text{WardA}) = g(\text{WardB}) = \text{Hospital}.\end{aligned}$$

Let $\nu \in F(X)$ encode triage assessments for "influenza suspicion" as

$$\begin{aligned}\nu(p_1) &= (T, I, F) = (0.90, 0.10, 0.00),\\ \nu(p_2) &= (0.40, 0.30, 0.20),\\ \nu(p_3) &= (0.50, 0.40, 0.10),\\ \nu(p_4) &= (0.10, 0.20, 0.80).\end{aligned}$$

Pushforward to wards (componentwise maxima over each fiber of f):

$$\begin{aligned}\big(F(f)(\nu)\big)(\text{WardA}) &= \big(\max\{0.90, 0.40\}, \max\{0.10, 0.30\},\\ &\qquad \max\{0.00, 0.20\}\big)\\ &= (0.90, 0.30, 0.20),\\ \big(F(f)(\nu)\big)(\text{WardB}) &= \big(\max\{0.50, 0.10\}, \max\{0.40, 0.20\},\\ &\qquad \max\{0.10, 0.80\}\big)\\ &= (0.50, 0.40, 0.80).\end{aligned}$$

Pushforward to the hospital via g (maximum across wards):

$$\begin{aligned}F(g)\big(F(f)(\nu)\big)(\text{Hospital}) &= \big(\max\{0.90, 0.50\},\\ &\qquad \max\{0.30, 0.40\},\\ &\qquad \max\{0.20, 0.80\}\big)\\ &= (0.90, 0.40, 0.80).\end{aligned}$$

Direct one-step pushforward agrees by functoriality:

$$\begin{aligned}&\big(F(g\circ f)(\nu)\big)(\text{Hospital})\\ &= \big(\max\{0.90, 0.40, 0.50, 0.10\},\\ &\quad \max\{0.10, 0.30, 0.40, 0.20\},\\ &\quad \max\{0.00, 0.20, 0.10, 0.80\}\big)\\ &= (0.90, 0.40, 0.80) = \big(F(g)\circ F(f)\big)(\nu)(\text{Hospital}).\end{aligned}$$

Reading. F assigns to each population a neutrosophic assessment and pushes it forward along any regrouping map (patients → wards → hospital) by fiberwise maxima, giving consistent, order-invariant summaries of truth/indeterminacy/falsity degrees.

2. MAIN RESULTS

This section presents the main results of the paper.

2.1 HybridFunctorial Structure

We couple two functors on the same base by a natural transformation and package their compatible pairs into a single functor.

Definition 2.1 (HybridFunctorial Structure)**.** Fix a category $\mathscr{C}$. A HybridFunctorial Structure on $\mathscr{C}$ is a triple (F, G, τ) where $F, G : \mathscr{C} \to \mathbf{Set}$ are covariant functors and $\tau : F \Rightarrow G$ is a natural transformation, i.e., for every morphism $f : X \to Y$

in $\mathscr{C}$,

$$G(f)\circ\tau_X=\tau_Y\circ F(f).$$

Its underlying hybrid functor $H_{F,G,\tau}:\mathscr{C}\to\mathbf{Set}$ is defined on objects by

$$H_{F,G,\tau}(X):=\{(x,y)\in F(X)\times G(X)\mid y=\tau_X(x)\},$$

and on a morphism $f:X\to Y$ by

$$H_{F,G,\tau}(f)(x,y):=\big(F(f)(x),G(f)(y)\big).$$

Example 2.2 (Concrete hybrid on $\mathscr{C}=\mathbf{FinSet}$)**.** Let $F=\mathscr{P}$ and $G=\mathscr{P}^2$ be the powerset and double powerset functors. Define $\tau_X:\mathscr{P}(X)\to\mathscr{P}^2(X)$ by $\tau_X(A)=\{A\}$. For any $f:X\to Y$,

$$\begin{aligned}G(f)\big(\tau_X(A)\big)&=\mathscr{P}^2(f)(\{A\})\\&=\{\mathscr{P}(f)(A)\}\\&=\tau_Y\big(\mathscr{P}(f)(A)\big),\end{aligned}$$

so τ is natural. Hence $(\mathscr{P},\mathscr{P}^2,\tau)$ is a HybridFunctorial Structure and

$$H_{F,G,\tau}(X)=\{(A,\{A\})\mid A\subseteq X\}.$$

Example 2.3 (Real-world HybridFunctorial Structure: E–commerce logistics)**.** Fix a category $\mathscr{C}$ whose objects are delivery zones Z, each equipped with a set of postal addresses A_Z and a fixed finite set of package types Pkg. A morphism $f:Z\to Z'$ is specified by an address remapping function $p_f:A_Z\to A_{Z'}$; composition is function composition.

Define two covariant functors $F,G:\mathscr{C}\to\mathbf{Set}$ by

$$\begin{aligned}F(Z)&=\mathscr{P}_{\text{fin}}(A_Z\times\text{Pkg}),\\F(f)(S)&=\{(p_f(a),p)\mid(a,p)\in S\},\\G(Z)&=\mathscr{P}_{\text{fin}}(A_Z\times\text{Pkg}\times\{\text{deliver}\}),\\G(f)(T)&=\{(p_f(a),p,\text{deliver})\mid(a,p,\text{deliver})\in T\}.\end{aligned}$$

Define a natural transformation $\tau:F\Rightarrow G$ by

$$\tau_Z(S)=\{(a,p,\text{deliver})\mid(a,p)\in S\}.$$

Then (F,G,τ) is a HybridFunctorial Structure. Naturality holds objectwise: for any $f:Z\to Z'$ and $S\in F(Z)$,

$$\begin{aligned}G(f)\big(\tau_Z(S)\big)&=\{(p_f(a),p,\text{deliver})\mid(a,p)\in S\}\\&=\tau_{Z'}\big(F(f)(S)\big).\end{aligned}$$

Concrete instance. Let $A_Z=\{\alpha,\beta\}$, $A_{Z'}=\{\gamma,\delta\}$, Pkg $=\{S,M\}$, and $p_f(\alpha)=\gamma$, $p_f(\beta)=\delta$. For $S=\{(\alpha,S),(\beta,M)\}$,

$$\tau_Z(S)=\{(\alpha,S,\text{deliver}),(\beta,M,\text{deliver})\},$$

and

$$\begin{aligned}G(f)\big(\tau_Z(S)\big)&=\{(\gamma,S,\text{deliver}),(\delta,M,\text{deliver})\}\\&=\tau_{Z'}\big(\{(\gamma,S),(\delta,M)\}\big)\\&=\tau_{Z'}\big(F(f)(S)\big).\end{aligned}$$

Example 2.4 (HybridFunctorial Structure: list-to-multiset tally is natural)**.** Let $\mathscr{C}=\mathbf{FinSet}$. Define two functors $F,G:\mathscr{C}\to\mathbf{Set}$ by

$$F(X)=\text{List}(X):=\bigsqcup_{n\ge0}X^n,\qquad G(X)=\mathbb{N}^{(X)}.$$

For $f:X\to Y$ and a list $L=[x_1,\dots,x_n]\in F(X)$, set

$$F(f)(L):=[f(x_1),\dots,f(x_n)],$$

and, for $m\in G(X)$, define the multiset pushforward by summing on fibers,

$$\big(G(f)(m)\big)(y):=\sum_{x\in f^{-1}(y)}m(x),\qquad y\in Y.$$

Define a natural transformation $\tau:F\Rightarrow G$ by the tally map

$$\big(\tau_X(L)\big)(x):=\#\{i\in\{1,\dots,n\}\mid x_i=x\}.$$

Naturality. For $f:X\to Y$ and $y\in Y$,

$$\begin{aligned}\big(G(f)\circ\tau_X\big)(L)(y)&=\sum_{x\in f^{-1}(y)}\#\{i\mid x_i=x\}\\&=\#\{i\mid f(x_i)=y\}\\&=\big(\tau_Y\circ F(f)\big)(L)(y),\end{aligned}$$

so $G(f)\circ\tau_X=\tau_Y\circ F(f)$.

Concrete numbers. Let $X=\{a,b,c\}$, $Y=\{u,v\}$ with $f(a)=u$, $f(b)=f(c)=v$, and $L=[a,b,b,c]$. Then

$$\tau_X(L)(a,b,c)=(1,2,1).$$

Pushing forward the multiset,

$$\begin{aligned}\big(G(f)\circ\tau_X\big)(L)(u)&=1,\\\big(G(f)\circ\tau_X\big)(L)(v)&=2+1=3.\end{aligned}$$

On the other hand,

$$F(f)(L)=[u,v,v,v]\quad\Longrightarrow\quad\tau_Y\big(F(f)(L)\big)(u,v)=(1,3),$$

which matches $(1,3)$ above. Therefore (F,G,τ) is a HybridFunctorial Structure.

Example 2.5 (HybridFunctorial Structure: fuzzy set to θ-cut is natural)**.** Fix a threshold $\theta\in(0,1]$ and take $\mathscr{C}=\mathbf{FinSet}$. Define

$$\begin{aligned}&F(X)=[0,1]^X,\quad\big(F(f)(\mu)\big)(y):=\sup_{x:f(x)=y}\mu(x),\\&G(X)=\mathscr{P}(X),\qquad\qquad G(f)(S)=f[S]=\{f(x)\mid x\in S\}.\end{aligned}$$

Let $\tau:F\Rightarrow G$ map each membership $\mu\in[0,1]^X$ to its θ-cut:

$$\tau_X(\mu):=\{x\in X\mid\mu(x)\ge\theta\}.$$

Naturality. For $f:X\to Y$,

$$\begin{aligned}G(f)\big(\tau_X(\mu)\big)&=\{f(x)\mid\mu(x)\ge\theta\}\\&=\{y\in Y\mid\exists x(f(x)=y\wedge\mu(x)\ge\theta)\}\\&=\left\{y\in Y\;\middle|\;\sup_{x:f(x)=y}\mu(x)\ge\theta\right\}\\&=\tau_Y\big(F(f)(\mu)\big).\end{aligned}$$

Concrete numbers. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ with $f(x_1) = y_1$, $f(x_2) = f(x_3) = y_2$, and

$$\mu(x_1) = 0.40, \quad \mu(x_2) = 0.85,$$
$$\mu(x_3) = 0.60, \qquad \theta = 0.60.$$

Then $\tau_X(\mu) = \{x_2, x_3\}$ and thus $G(f)(\tau_X(\mu)) = \{y_2\}$. Also

$$\big(F(f)(\mu)\big)(y_1) = 0.40,$$
$$\big(F(f)(\mu)\big)(y_2) = 0.85,$$

so $\tau_Y(F(f)(\mu)) = \{y_2\}$, which coincides with $G(f)(\tau_X(\mu))$.

Proposition 2.6 (Well-definedness and functoriality of $H_{F,G,\tau}$)**.** With notation as above, $H_{F,G,\tau}$ is a well-defined functor $\mathscr{C} \to \mathbf{Set}$.

Proof. For well-definedness, take $(x, y) \in H_{F,G,\tau}(X)$, so $y = \tau_X(x)$. Then

$$G(f)(y) = G(f)\big(\tau_X(x)\big) = \tau_Y\big(F(f)(x)\big),$$

where the last equality is naturality. Hence

$$H_{F,G,\tau}(f)(x, y) = \big(F(f)(x), \tau_Y(F(f)(x))\big) \in H_{F,G,\tau}(Y).$$

For functoriality, for $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$\begin{aligned} H_{F,G,\tau}(g \circ f)(x, y) &= \big(F(g \circ f)(x), G(g \circ f)(y)\big) \\ &= \big(F(g)(F(f)(x)), G(g)(G(f)(y))\big) \\ &= H_{F,G,\tau}(g)\big(F(f)(x), G(f)(y)\big) \\ &= \big(H_{F,G,\tau}(g) \circ H_{F,G,\tau}(f)\big)(x, y), \end{aligned}$$

and $H_{F,G,\tau}(\mathrm{id}_X) = \mathrm{id}_{H_{F,G,\tau}(X)}$ is immediate. □

Theorem 2.7 (Hybrid generalizes Functorial)**.** There is a fully faithful embedding

$$\mathrm{Fun}(\mathscr{C}, \mathbf{Set}) \hookrightarrow \mathrm{HyFun}(\mathscr{C}, \mathbf{Set}), \qquad F \longmapsto (F, F, \mathrm{id}_F),$$

where $\mathrm{HyFun}(\mathscr{C}, \mathbf{Set})$ is the category whose objects are (F, G, τ) and whose morphisms

$$(\alpha, \beta) : (F, G, \tau) \longrightarrow (F', G', \tau')$$

are pairs of natural transformations $\alpha : F \Rightarrow F'$ and $\beta : G \Rightarrow G'$ satisfying the hybrid compatibility

$$\tau'_X \circ \alpha_X = \beta_X \circ \tau_X \qquad \text{for all } X \in \mathrm{Ob}(\mathscr{C}).$$

Proof. Define $J : \mathrm{Fun}(\mathscr{C}, \mathbf{Set}) \to \mathrm{HyFun}(\mathscr{C}, \mathbf{Set})$ by $J(F) = (F, F, \mathrm{id}_F)$ on objects and $J(\eta) = (\eta, \eta)$ on morphisms $\eta : F \Rightarrow F'$. For compatibility,

$$\tau'_X \circ \eta_X = \mathrm{id}_{F'(X)} \circ \eta_X = \eta_X = \eta_X \circ \mathrm{id}_{F(X)} = \eta_X \circ \tau_X.$$

Thus J is a functor. Fullness and faithfulness are immediate since

$$\mathrm{Nat}(F, F') \ni \eta \longleftrightarrow (\eta, \eta) \in \mathrm{Hom}_{\mathrm{HyFun}}(J(F), J(F'))$$

is a bijection. □

2.2 MultiFunctorial Structure

We now allow finitely many functors with coherent natural transformations among them.

Definition 2.8 (MultiFunctorial Structure)**.** Fix a category $\mathscr{C}$ and a finite nonempty index set I equipped with a reflexive and transitive relation $\leq$ (a finite preorder). A MultiFunctorial Structure on $\mathscr{C}$ indexed by $(I, \leq)$ is:

- a family of covariant functors $F_i : \mathscr{C} \to \mathbf{Set}$ for $i \in I$;
- for every $i \leq j$ in I, a natural transformation $\tau_{ij} : F_i \Rightarrow F_j$,

such that the coherence axioms hold:

$$\tau_{ii} = \mathrm{id}_{F_i}, \qquad \tau_{jk} \circ \tau_{ij} = \tau_{ik} \quad \text{whenever } i \leq j \leq k.$$

Define the underlying multi-functor $M : \mathscr{C} \to \mathbf{Set}$ by

$$M(X) := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} F_i(X) \;\middle|\; \forall i \leq j : \; x_j = \tau_{ij,X}(x_i) \right\},$$

and for $f : X \to Y$,

$$M(f)\big((x_i)_{i \in I}\big) := \big(F_i(f)(x_i)\big)_{i \in I}.$$

Example 2.9 (Concrete multi on $\mathscr{C} = \mathbf{FinSet}$)**.** Let $I = \{1, 2, 3\}$ with $1 \leq 2 \leq 3$. Take $F_1 = \mathrm{id}$, $F_2 = \mathscr{P}$, and $F_3 = \mathscr{P}^2$. Define

$$\tau_{12,X}(x) = \{x\}, \qquad \tau_{23,X}(A) = \{A\},$$
$$\tau_{13,X} = \tau_{23,X} \circ \tau_{12,X}.$$

i.e., $\tau_{13,X}(x) = \{\{x\}\}$. Coherence holds since

$$\tau_{23,X}\big(\tau_{12,X}(x)\big) = \{\{x\}\} = \tau_{13,X}(x), \qquad \tau_{ii} = \mathrm{id}.$$

For $f : X \to Y$ and $(x, A, B) \in M(X)$ with $A = \{x\}$ and $B = \{A\}$,

$$\begin{aligned} M(f)(x, A, B) &= \big(f(x), \mathscr{P}(f)(A), \mathscr{P}^2(f)(B)\big) \\ &= \big(f(x), \{f(x)\}, \{\{f(x)\}\}\big), \end{aligned}$$

and the compatibility equalities are preserved because $\mathscr{P}(f)(\{x\}) = \{f(x)\}$ and $\mathscr{P}^2(f)(\{A\}) = \{\mathscr{P}(f)(A)\}$.

Example 2.10 (Real-world MultiFunctorial Structure: healthcare cohort to buckets to counts)**.** Let $\mathscr{C}$ be the category whose objects are hospitals H with a patient set P_H and fixed label sets: age brackets B and diagnosis codes D. A morphism $f : H \to H'$ is given by a referral map $r_f : P_H \to P_{H'}$ that preserves labels: for every $p \in P_H$,

$$\mathrm{Age}_{H'}\big(r_f(p)\big) = \mathrm{Age}_H(p) \in B,$$
$$\mathrm{Diag}_{H'}\big(r_f(p)\big) = \mathrm{Diag}_H(p) \in D.$$

Fix the preorder $I = \{1, 2, 3\}$ with $1 \leq 2 \leq 3$ and define three functors:

$$F_1(H) = \mathscr{P}_{\mathrm{fin}}(P_H) \qquad \text{(finite patient cohorts)},$$
$$F_2(H) = \mathscr{P}_{\mathrm{fin}}(B \times D) \qquad \text{(finite label buckets)},$$
$$F_3(H) = \mathbb{N}^{B \times D} \qquad \text{(count tables on } B \times D).$$

Their actions on $f : H \to H'$ are

$$F_1(f)(S) = \{r_f(p) \mid p \in S\},$$
$$F_2(f) = \mathrm{id}_{\mathscr{P}_{\mathrm{fin}}(B\times D)},$$
$$F_3(f) = \mathrm{id}_{\mathbb{N}^{B\times D}}.$$

Define natural transformations $\tau_{12} : F_1 \Rightarrow F_2$ and $\tau_{23} : F_2 \Rightarrow F_3$ by

$$\tau_{12,H}(S) = \{(\mathrm{Age}_H(p), \mathrm{Diag}_H(p)) \mid p \in S\},$$

and

$$\tau_{23,H}(T) = c_T \in \mathbb{N}^{B\times D},$$
$$c_T(b,d) = \#\{(b',d') \in T \mid (b',d') = (b,d)\}.$$

Coherence holds with $\tau_{11} = \mathrm{id}_{F_1}$, $\tau_{22} = \mathrm{id}_{F_2}$, $\tau_{33} = \mathrm{id}_{F_3}$, and $\tau_{13} = \tau_{23} \circ \tau_{12}$. Naturality is immediate: for $S \in F_1(H)$,

$$\begin{aligned} F_2(f)\big(\tau_{12,H}(S)\big) &= \tau_{12,H}(S) \\ &= \{(\mathrm{Age}_H(p), \mathrm{Diag}_H(p)) \mid p \in S\} \\ &= \{(\mathrm{Age}_{H'}(r_f(p)), \mathrm{Diag}_{H'}(r_f(p))) \mid p \in S\} \\ &= \tau_{12,H'}\big(F_1(f)(S)\big), \end{aligned}$$

and $F_3(f) \circ \tau_{23,H} = \tau_{23,H'} \circ F_2(f)$ since both sides are the identity on $\mathbb{N}^{B\times D}$.

Concrete instance. Let $B = \{30\mathrm{s}, 50\mathrm{s}\}$, $D = \{A, B\}$, and in H take three patients p_1, p_2, p_3 with

$$(\mathrm{Age}_H, \mathrm{Diag}_H)(p_1) = (30\mathrm{s}, A),$$
$$(\mathrm{Age}_H, \mathrm{Diag}_H)(p_2) = (30\mathrm{s}, B),$$
$$(\mathrm{Age}_H, \mathrm{Diag}_H)(p_3) = (50\mathrm{s}, A).$$

For the cohort $S = \{p_1, p_2, p_3\}$,

$$\tau_{12,H}(S) = \{(30\mathrm{s}, A), (30\mathrm{s}, B), (50\mathrm{s}, A)\},$$

and

$$\tau_{23,H}\big(\tau_{12,H}(S)\big) = c \in \mathbb{N}^{B\times D},$$
$$c(30\mathrm{s}, A) = 1, \quad c(30\mathrm{s}, B) = 1, \quad c(50\mathrm{s}, A) = 1,$$

with $c(b,d) = 0$ for all other $(b,d) \in B \times D$. If $f : H \to H'$ merely relabels patient identifiers via r_f while preserving $(\mathrm{Age}, \mathrm{Diag})$, then the above equalities are unchanged in H' by naturality.

Example 2.11 (MultiFunctorial Structure: fuzzy ladder to α-cut to non-emptiness)**.** Let $\mathscr{C} = \mathbf{FinSet}$ and fix a threshold $\theta \in (0,1]$. Take the preorder $I = \{1,2,3\}$ with $1 \le 2 \le 3$ and define

$$F_1(X) = [0,1]^X, \quad F_2(X) = \mathscr{P}(X),$$
$$F_3(X) = \{0,1\}.$$

For $f : X \to Y$, set

$$\big(F_1(f)(\mu)\big)(y) := \sup_{x : f(x) = y} \mu(x),$$
$$F_2(f)(S) = f[S],$$
$$F_3(f) = \mathrm{id}_{\{0,1\}}.$$

Define natural transformations

$$\tau_{12,X}(\mu) = \{x \in X \mid \mu(x) \ge \theta\},$$

and

$$\tau_{23,X}(S) = \begin{cases} 1, & S \neq \varnothing, \\ 0, & S = \varnothing. \end{cases}$$

Set $\tau_{11} = \mathrm{id}_{F_1}$, $\tau_{22} = \mathrm{id}_{F_2}$, $\tau_{33} = \mathrm{id}_{F_3}$, and $\tau_{13} := \tau_{23} \circ \tau_{12}$. Coherence holds by construction. Naturality is immediate from Example 2.5 and the fact that $F_3(f)$ is the identity.

Concrete numbers. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ with $f(x_1) = y_1$, $f(x_2) = f(x_3) = y_2$, and

$$\mu(x_1) = 0.40, \quad \mu(x_2) = 0.85,$$
$$\mu(x_3) = 0.60, \qquad \theta = 0.60.$$

Then $\tau_{12,X}(\mu) = \{x_2, x_3\}$, so

$$\tau_{23,X}\big(\tau_{12,X}(\mu)\big) = 1.$$

Pushing forward,

$$\big(F_1(f)(\mu)\big)(y_1) = 0.40,$$
$$\big(F_1(f)(\mu)\big)(y_2) = 0.85,$$

so $\tau_{12,Y}(F_1(f)(\mu)) = \{y_2\}$ and $\tau_{23,Y}$ of that set is again 1. Thus $\tau_{23} \circ \tau_{12}$ commutes with f, and M is well defined.

Example 2.12 (MultiFunctorial Structure: list to multiset to labeled counts)**.** Fix a finite label set C. Let $\mathscr{C}$ be the category whose objects are pairs (X, ℓ_X) with a labeling map $\ell_X : X \to C$, and whose morphisms $f : (X, \ell_X) \to (Y, \ell_Y)$ are functions $f : X \to Y$ preserving labels:

$$\ell_Y \circ f = \ell_X.$$

Take the preorder $I = \{1,2,3\}$ with $1 \le 2 \le 3$ and define functors

$$F_1(X, \ell_X) = \mathrm{List}(X) = \bigsqcup_{n \ge 0} X^n,$$
$$F_2(X, \ell_X) = \mathbb{N}^{(X)},$$
$$F_3(X, \ell_X) = \mathbb{N}^C.$$

For $f : (X, \ell_X) \to (Y, \ell_Y)$ set

$$F_1(f)([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)],$$
$$\big(F_2(f)(m)\big)(y) = \sum_{x \in f^{-1}(y)} m(x),$$
$$F_3(f) = \mathrm{id}_{\mathbb{N}^C}.$$

Define natural transformations by

$$\big(\tau_{12,X}(L)\big)(x) = \#\{i \mid x_i = x\},$$

and

$$\big(\tau_{23,X}(m)\big)(c) = \sum_{x : \ell_X(x) = c} m(x), \qquad c \in C.$$

Set $\tau_{11} = \mathrm{id}_{F_1}$, $\tau_{22} = \mathrm{id}_{F_2}$, $\tau_{33} = \mathrm{id}_{F_3}$, and $\tau_{13} = \tau_{23} \circ \tau_{12}$. Coherence is automatic. Naturality of τ_{12} is Example 2.4;

naturality of τ_{23} follows from $\ell_Y \circ f = \ell_X$:

$$\begin{aligned}\big(F_3(f)\circ\tau_{23,X}\big)(m)(c) &= \sum_{x:\ell_X(x)=c} m(x)\\ &= \sum_{y:\ell_Y(y)=c}\ \sum_{x:f(x)=y} m(x)\\ &= \sum_{y:\ell_Y(y)=c} \big(F_2(f)(m)\big)(y)\\ &= \big(\tau_{23,Y}\circ F_2(f)\big)(m)(c).\end{aligned}$$

Concrete numbers. Let $C=\{A,B\}$ and (X,ℓ_X) with

$$\begin{aligned}X &= \{x_1,x_2,x_3\},\\ \ell_X(x_1)=A,\quad \ell_X(x_2)&=A,\quad \ell_X(x_3)=B.\end{aligned}$$

Take the list $L=[x_1,x_2,x_2,x_3]$. Then

$$\tau_{12,X}(L)(x_1,x_2,x_3)=(1,2,1).$$

Aggregate by labels:

$$\begin{aligned}\tau_{23,X}(\tau_{12,X}(L))(A)&=1+2=3,\\ \tau_{23,X}(\tau_{12,X}(L))(B)&=1.\end{aligned}$$

Now let (Y,ℓ_Y) with $Y=\{y_1,y_2,y_3\}$ and

$$\ell_Y(y_1)=A,\quad \ell_Y(y_2)=B,\quad \ell_Y(y_3)=A,$$

and define the label-preserving map $f:X\to Y$ by

$$\begin{aligned}f(x_1)&=y_3 \quad (A\to A),\\ f(x_2)&=y_1 \quad (A\to A),\\ f(x_3)&=y_2 \quad (B\to B).\end{aligned}$$

Then

$$\begin{aligned}F_1(f)(L)&=[y_3,y_1,y_1,y_2],\\ \tau_{12,Y}(F_1(f)(L))(y_1,y_2,y_3)&=(2,1,1),\end{aligned}$$

and aggregating on labels gives

$$\begin{aligned}\tau_{23,Y}(\tau_{12,Y}(F_1(f)(L)))(A)&=2+1=3,\\ \tau_{23,Y}(\tau_{12,Y}(F_1(f)(L)))(B)&=1,\end{aligned}$$

which coincides with the previous $(3,1)$. Hence M is well defined and functorial.

Proposition 2.13 (Well-definedness and functoriality of M)**.** With notation as above, M is a well-defined functor $\mathscr{C}\to\mathbf{Set}$.

Proof. For well-definedness, suppose $(x_i)_i\in M(X)$, so for all $i\le j$, $x_j=\tau_{ij,X}(x_i)$. Then, for any $i\le j$,

$$F_j(f)(x_j)=F_j(f)\big(\tau_{ij,X}(x_i)\big)=\tau_{ij,Y}\big(F_i(f)(x_i)\big)$$

by naturality of τ_{ij}. Hence $(F_i(f)(x_i))_i\in M(Y)$. Functoriality follows componentwise:

$$\begin{aligned}M(g\circ f)((x_i)) &= \big(F_i(g\circ f)(x_i)\big)_i\\ &= \big(F_i(g)F_i(f)(x_i)\big)_i\\ &= M(g)\big(M(f)((x_i))\big),\end{aligned}$$

and $M(\mathrm{id}_X)=\mathrm{id}_{M(X)}$ holds componentwise. □

Theorem 2.14 (Multi generalizes Hybrid and Functorial)**.** Let $\mathscr{C}$ be a category.

(i) Every Functorial Structure $F:\mathscr{C}\to\mathbf{Set}$ yields a MultiFunctorial Structure by taking $I=\{1\}$, $F_1=F$, and $M(X)=F(X)$.

(ii) Every HybridFunctorial Structure (F,G,τ) yields a MultiFunctorial Structure by taking $I=\{1,2\}$ with order $1\le 2$, setting $F_1=F$, $F_2=G$, $\tau_{11}=\mathrm{id}_F$, $\tau_{22}=\mathrm{id}_G$, and $\tau_{12}=\tau$.

Conversely, any MultiFunctorial Structure with $|I|=1$ is precisely a Functorial Structure, and any MultiFunctorial Structure with $I=\{1,2\}$ and $1\le 2$ is precisely a HybridFunctorial Structure.

Proof. (i) The only coherence map is $\tau_{11}=\mathrm{id}_F$, and

$$M(X)=\{x_1 \mid x_1\in F(X)\}\cong F(X).$$

(ii) With $I=\{1,2\}$ and $1\le 2$, the coherence data are exactly $\tau_{11}=\mathrm{id}_F$, $\tau_{22}=\mathrm{id}_G$, and $\tau_{12}=\tau$. Then

$$\begin{aligned}M(X)&=\{(x_1,x_2)\in F(X)\times G(X)\mid x_2=\tau_X(x_1)\}\\ &=H_{F,G,\tau}(X),\end{aligned}$$

and $M(f)=H_{F,G,\tau}(f)$ by Definition 2.1.

Conversely, if $|I|=1$, say $I=\{1\}$, then $M(X)=F_1(X)$ and $M(f)=F_1(f)$. If $I=\{1,2\}$ with $1\le 2$, the only nontrivial coherence is a single natural transformation $\tau_{12}:F_1\Rightarrow F_2$, yielding precisely a hybrid triple (F_1,F_2,τ_{12}). □

Proposition 2.15 (Reconstruction from a root index)**.** Let $(F_i,\tau_{ij})_{i\le j}$ be a MultiFunctorial Structure indexed by a finite preorder $(I,\le)$. Assume that $r\in I$ is a root index, that is,

$$r\le i \qquad \text{for all } i\in I.$$

Then for each object $X\in\mathrm{Ob}(\mathscr{C})$, define

$$\begin{aligned}\alpha_r(X) &: M(X)\longrightarrow F_r(X),\\ \alpha_r(X)((x_i)_{i\in I}) &:= x_r,\end{aligned}$$

and

$$\begin{aligned}\beta_r(X) &: F_r(X)\longrightarrow M(X),\\ \beta_r(X)(s) &:= (\tau_{ri,X}(s))_{i\in I}.\end{aligned}$$

Then:

1. $\alpha_r(X)$ and $\beta_r(X)$ are well defined for every object X.
2. $\alpha_r=(\alpha_r(X))_X$ and $\beta_r=(\beta_r(X))_X$ are natural transformations
$$\begin{aligned}\alpha_r &: M\Rightarrow F_r,\\ \beta_r &: F_r\Rightarrow M.\end{aligned}$$
3. For every object X,
$$\alpha_r(X)\circ\beta_r(X)=\mathrm{id}_{F_r(X)},\qquad \beta_r(X)\circ\alpha_r(X)=\mathrm{id}_{M(X)}.$$

Hence M and F_r are naturally isomorphic.

Proof. Since $r \le i$ for all $i \in I$, each natural transformation $\tau_{ri} : F_r \Rightarrow F_i$ is defined.

First we show that $\beta_r(X)$ is well defined. Let $s \in F_r(X)$, and set

$$\beta_r(X)(s) := (\tau_{ri,X}(s))_{i\in I}.$$

We must verify that this family lies in $M(X)$, namely that for all $i \le j$,

$$\tau_{ij,X}\big(\tau_{ri,X}(s)\big) = \tau_{rj,X}(s).$$

This follows immediately from the coherence axiom

$$\tau_{ij} \circ \tau_{ri} = \tau_{rj}, \qquad r \le i \le j.$$

Hence $\beta_r(X)(s) \in M(X)$. The map $\alpha_r(X)$ is clearly well defined, since if $(x_i)_{i\in I} \in M(X)$, then by definition $x_r \in F_r(X)$.

Next we prove the identities. For any $s \in F_r(X)$,

$$\begin{aligned}\alpha_r(X)\big(\beta_r(X)(s)\big) &= \alpha_r(X)\big((\tau_{ri,X}(s))_{i\in I}\big)\\ &= \tau_{rr,X}(s) = \mathrm{id}_{F_r(X)}(s) = s.\end{aligned}$$

Thus

$$\alpha_r(X) \circ \beta_r(X) = \mathrm{id}_{F_r(X)}.$$

Now let $(x_i)_{i\in I} \in M(X)$. Since $(x_i)_{i\in I} \in M(X)$, the compatibility condition gives, for every $i \in I$,

$$x_i = \tau_{ri,X}(x_r),$$

because $r \le i$. Therefore,

$$\begin{aligned}\beta_r(X)\big(\alpha_r(X)((x_i)_{i\in I})\big) &= \beta_r(X)(x_r)\\ &= (\tau_{ri,X}(x_r))_{i\in I} = (x_i)_{i\in I}.\end{aligned}$$

Hence

$$\beta_r(X) \circ \alpha_r(X) = \mathrm{id}_{M(X)}.$$

It remains to prove naturality. For α_r, let $f : X \to Y$ be a morphism in $\mathscr{C}$ and let $(x_i)_{i\in I} \in M(X)$. Then

$$\begin{aligned}\alpha_r(Y)\big(M(f)((x_i)_{i\in I})\big) &= \alpha_r(Y)\big((F_i(f)(x_i))_{i\in I}\big)\\ &= F_r(f)(x_r),\end{aligned}$$

while

$$F_r(f)\big(\alpha_r(X)((x_i)_{i\in I})\big) = F_r(f)(x_r).$$

Thus

$$\alpha_r(Y) \circ M(f) = F_r(f) \circ \alpha_r(X).$$

For β_r, let $s \in F_r(X)$. Then

$$\begin{aligned}M(f)\big(\beta_r(X)(s)\big) &= M(f)\big((\tau_{ri,X}(s))_{i\in I}\big)\\ &= \big(F_i(f)(\tau_{ri,X}(s))\big)_{i\in I}.\end{aligned}$$

By naturality of τ_{ri},

$$F_i(f) \circ \tau_{ri,X} = \tau_{ri,Y} \circ F_r(f),$$

so

$$\begin{aligned}\big(F_i(f)(\tau_{ri,X}(s))\big)_{i\in I} &= \big(\tau_{ri,Y}(F_r(f)(s))\big)_{i\in I}\\ &= \beta_r(Y)(F_r(f)(s)).\end{aligned}$$

Hence

$$M(f) \circ \beta_r(X) = \beta_r(Y) \circ F_r(f).$$

Therefore $\alpha_r : M \Rightarrow F_r$ and $\beta_r : F_r \Rightarrow M$ are natural inverses, and so

$$M \cong F_r$$

naturally. □

3. CONCLUSION

We introduced HybridFunctorial and MultiFunctorial Structures on a fixed base category $\mathscr{C}$, verified the induced functors $H_{F,G,\tau}$ and M are well defined by explicit naturality equations, and proved precise embeddings:

$$\begin{aligned}&\mathrm{Fun}(\mathscr{C}, \mathbf{Set}) \xrightarrow{F\mapsto(F,F,\mathrm{id}_F)} \mathrm{HyFun}(\mathscr{C}, \mathbf{Set})\\ &\xrightarrow{(F,G,\tau)\mapsto(I=\{1,2\},\tau_{12}=\tau)} \mathrm{MultiFun}(\mathscr{C}, \mathbf{Set}).\end{aligned}$$

Thus MultiFunctorial Structures strictly generalize HybridFunctorial and Functorial Structures.

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