On Neutrosophic Soft Generalized Semi-Mappings and Their Topological Properties

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Abstract

This paper introduces and systematically studies new classes of mappings and set-theoretic structures in the context of neutrosophic soft topological structures. In particular, this study introduces and examines neutrosophic soft semi closed and semi open sets, generalized semi-mappings, and semi-continuous generalized mappings, highlighting their interrelationships and key topological properties The neutrosophic soft generalized semi-closure and semi-interior operators are also formulated, and their principal algebraic and topological characteristics are derived. These developments generalize and unify several existing notions in classical, fuzzy, and neutrosophic soft topologies. Unlike previous studies, this work provides a comprehensive mapping-based approach that clarifies how generalized semi-properties behave under neutrosophic soft transformations. The findings not only extend the theoretical foundations of NST but also open potential directions for modeling and analyzing uncertainty in advanced topological systems.

Keywords: Neutrosophic soft set; Neutrosophic soft semi closed set; Neutrosophic soft generalized semi closed set; Neutrosophic soft generalized semi closed mapping; Neutrosophic soft generalized semi-continuous mapping

1 Introduction

Since the beginning of human existence, individuals have encountered various problems in their daily lives. Resolving these problems is essential for improving quality of life. However, such problems are often characterized by inherent uncertainty and vagueness, which cannot be adequately modeled using classical set theory as formulated by Cantor.[7] To address these limitations, a number of extensions to classical set theory have been proposed, enabling more effective modeling of uncertainty in disciplines such as finance, medicine, engineering, and social sciences. Zadeh,[29] Atanassov,[2] Molodtsov,[19] and Smarandache[25] laid the foundations of fuzzy sets, intuitionistic fuzzy sets, soft sets, and neutrosophic sets, respectively, each significantly influencing the modeling of uncertainty.

Levine[17] was the first to formulate generalized closed sets in topological spaces by employing interior and closure operators. This seminal contribution inspired numerous extensions, including semi closed, α-closed, pre-closed, and b-closed sets. Subsequently, researchers[11,12] adapted these notions to neutrosophic and neutrosophic soft frameworks, thereby establishing new set-theoretic constructs such as neutrosophic semi closed,

α-closed, pre-closed, and b-closed sets. Later, Deli and Broumi[10] refined the structure of neutrosophic soft sets, enhancing its theoretical consistency and applicability. Moreover, Öztürk et al.[22] revisited several fundamental definitions in neutrosophic soft set theory and proposed novel separation axioms within neutrosophic topological settings.[13]

Salama and Alblowi[23] further developed the notion of generalized neutrosophic sets and incorporated them into the construction of neutrosophic topological space. A wide array of neutrosophic set-based structures has since been explored.[1,20] Broumi[6] later formulated generalized neutrosophic soft sets, whereas Bera and Mahapatra[5] began exploring neutrosophic soft topological spaces.

Levine[16] characterized semi open sets in the following manner: a set A is considered semi open when there exists an open set U satisfying $U \subset A \subset \mathrm{cl}(U)$, which is equivalently expressed as $A \subset \mathrm{cl}(\mathrm{int}(A))$. Building on these studies, Shanthi et al.,[24] Ebenanjar et al.,[11] and Özkan et al.[20] explored various classes of neutrosophic sets, including generalized semi closed, soft semi open, and generalized semi open sets within the context of neutrosophic soft set theory, and analyzed their fundamental properties. Several further investigations[4,8,14] have contributed to the study of semi-set structures in neutrosophic environments.

Neutrosophic soft topological structures provide flexibility beyond classical topologies by simultaneously representing uncertainty, partial membership, and truth degrees. While neutrosophic soft closed sets form the foundational elements of these spaces, their limited structure is often insufficient for modeling intermediate states and weak separation axioms. Addressing this need, neutrosophic soft generalized closed sets preserve the properties of classical neutrosophic soft closed sets while enhancing the structural diversity of neutrosophic soft spaces and providing a foundation for future weak separation axioms. Moreover, neutrosophic soft generalized closed sets enable the definition of new neutrosophic soft space types and mapping classes, such as neutrosophic soft generalized continuous, neutrosophic soft generalized open, and neutrosophic soft generalized closed mappings, offering flexible and comprehensive analysis opportunities in both theoretical and applied contexts.

This work aims to broaden the theoretical framework concerning neutrosophic soft generalized semi closed sets in the setting of neutrosophic soft topological spaces. Initially, the study presents the concepts of neutrosophic soft generalized semi interior and neutrosophic soft generalized semi closure as a basis for subsequent analysis. Subsequently, neutrosophic soft generalized semi mappings and neutrosophic soft generalized semi continuous mappings are formulated and examined, emphasizing their key characteristics and mutual relationships. Finally, we explore several fundamental properties derived from these concepts and discuss how they contribute to the broader structure of neutrosophic soft topology. It is anticipated that the findings of this research will significantly enrich the theoretical basis of neutrosophic soft topology and provide a useful framework for future theoretical developments and practical applications.

2 Preliminaries

Before presenting our main results, we first recall several definitions and propositions introduced by various authors. These foundational concepts are provided here to enhance the clarity and continuity of the subsequent sections.

Definition 2.1. [10] Consider a universal set $\mathbb{U}$ together with a nonempty set $\mathcal{P}$ of parameters characterizing the elements of $\mathbb{U}$. Let $\mathbb{Q}(\mathbb{U})$ denote the power set of $\mathbb{U}$. A pair $(\hat{\mathcal{G}}, \mathcal{P})$ is called a neutrosophic soft set (shortly, NS-set) over $\mathbb{U}$ whenever there exists an association $\hat{\mathcal{G}} : \mathcal{P} \longrightarrow \mathbb{Q}(\mathbb{U})$, assigning to each parameter $p \in \mathcal{P}$ a subset $\hat{\mathcal{G}}(p) \subset \mathbb{U}$.

In other words, the NS-set forms a parameter-dependent family of subsets of $\mathbb{U}$. Equivalently, it can be presented through ordered pairs as

$$(\hat{\mathcal{G}}, \mathcal{P}) = \Big\{ \big(p, \langle a, \mathbb{T}_{\mathcal{G}(p)}(a), \mathbb{I}_{\mathcal{G}(p)}(a), \mathbb{F}_{\mathcal{G}(p)}(a) \rangle : a \in \mathbb{U}\big) : p \in \mathcal{P} \Big\},$$

where the components

$$\mathbb{T}_{\mathcal{G}(p)}(a),\ \mathbb{I}_{\mathcal{G}(p)}(a),\ \mathbb{F}_{\mathcal{G}(p)}(a) \in [0, 1]$$

stand respectively for the degrees of truth-membership, indeterminacy-membership, and falsity-membership.

Since each of these measures lies within the unit interval, their cumulative value satisfies

$$0 \leq \mathbb{T}_{\mathcal{G}(p)}(a) + \mathbb{I}_{\mathcal{G}(p)}(a) + \mathbb{F}_{\mathcal{G}(p)}(a) \leq 3.$$

Throughout this study, for a given parameter set $\mathcal{P}$, the set of all NS-sets over $\mathbb{U}$ is represented as $NS(\mathbb{U}_{\mathcal{P}})$. For brevity, the notation $\mathcal{G}_{\mathcal{P}}$ will be employed to stand for the pair $(\hat{\mathcal{G}}, \mathcal{P})$.

Definition 2.2. [10] Let $\mathcal{G}_{\mathcal{P}}, \mathcal{H}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. If for every $p \in \mathcal{P}$ and $a \in \mathbb{U}$,

$$\mathbb{T}_{\mathcal{G}(p)}(a) \leq \mathbb{T}_{\mathcal{H}(p)}(a), \quad \mathbb{I}_{\mathcal{G}(p)}(a) \leq \mathbb{I}_{\mathcal{H}(p)}(a), \quad \mathbb{F}_{\mathcal{G}(p)}(a) \geq \mathbb{F}_{\mathcal{H}(p)}(a),$$

then $\mathcal{G}_{\mathcal{P}}$ is regarded as an NS-subset of $\mathcal{H}_{\mathcal{P}}$, symbolically written as $\mathcal{G}_{\mathcal{P}} \subset \mathcal{H}_{\mathcal{P}}$.

When both $\mathcal{G}_{\mathcal{P}} \subset \mathcal{H}_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{P}} \subset \mathcal{G}_{\mathcal{P}}$ hold, the sets $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{P}}$ are considered NS-equal, symbolically written as $\mathcal{G}_{\mathcal{P}} = \mathcal{H}_{\mathcal{P}}$.

Definition 2.3. [5] For any $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$, the complement of $\mathcal{G}_{\mathcal{P}}$, represented by $(\mathcal{G}_{\mathcal{P}})^c$, is characterized as

$$(\mathcal{G}_{\mathcal{P}})^c = \Big\{ \big(p, \langle a, \mathbb{F}_{\mathcal{G}(p)}(a),\, 1 - \mathbb{I}_{\mathcal{G}(p)}(a),\, \mathbb{T}_{\mathcal{G}(p)}(a) \rangle : a \in \mathbb{U}\big) : p \in \mathcal{P} \Big\}.$$

Clearly, the equality $\big((\mathcal{G}_{\mathcal{P}})^c\big)^c = \mathcal{G}_{\mathcal{P}}$ holds.

Definition 2.4. [21] Let $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. Then the following special NS-sets are introduced:

1. If $\mathbb{T}_{\mathcal{G}(p)}(a) = 0, \quad \mathbb{I}_{\mathcal{G}(p)}(a) = 0, \quad \mathbb{F}_{\mathcal{G}(p)}(a) = 1$ for all $p \in \mathcal{P}$ and $a \in \mathbb{U}$, the set $\mathcal{G}_{\mathcal{P}}$ is referred to as the empty NS-set, symbolically written as $\widetilde{\emptyset}_{\mathcal{P}}$.
2. If $\mathbb{T}_{\mathcal{G}(p)}(a) = 1, \quad \mathbb{I}_{\mathcal{G}(p)}(a) = 1, \quad \mathbb{F}_{\mathcal{G}(p)}(a) = 0$ for all $p \in \mathcal{P}$ and $a \in \mathbb{U}$, the set $\mathcal{G}_{\mathcal{P}}$ is termed the absolute NS-set, symbolically written as $\widetilde{1}_{\mathcal{P}}$.

It directly follows that $(\widetilde{\emptyset}_{\mathcal{P}})^c = \widetilde{1}_{\mathcal{P}} \quad$ and $\quad (\widetilde{1}_{\mathcal{P}})^c = \widetilde{\emptyset}_{\mathcal{P}}$.

Definition 2.5. [21] Consider two NS-sets $\mathcal{G}_{\mathcal{P}}, \mathcal{H}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. Their *union*, symbolized by $\mathcal{G}_{\mathcal{P}} \cup \mathcal{H}_{\mathcal{P}} = \mathcal{L}_{\mathcal{P}}$, is formulated as

$$\mathcal{L}_{\mathcal{P}} = \Big\{ \big(p, \langle a, \mathbb{T}_{\mathcal{L}(p)}(a), \mathbb{I}_{\mathcal{L}(p)}(a), \mathbb{F}_{\mathcal{L}(p)}(a) \rangle : a \in \mathbb{U}\big) : p \in \mathcal{P} \Big\},$$

where, for each $p \in \mathcal{P}$ and $a \in \mathbb{U}$,

$$\mathbb{T}_{\mathcal{L}(p)}(a) = \max\{\mathbb{T}_{\mathcal{G}(p)}(a), \mathbb{T}_{\mathcal{H}(p)}(a)\}, \quad \mathbb{I}_{\mathcal{L}(p)}(a) = \max\{\mathbb{I}_{\mathcal{G}(p)}(a), \mathbb{I}_{\mathcal{H}(p)}(a)\},$$
$$\mathbb{F}_{\mathcal{L}(p)}(a) = \min\{\mathbb{F}_{\mathcal{G}(p)}(a), \mathbb{F}_{\mathcal{H}(p)}(a)\}.$$

Definition 2.6. [21] Let $\mathcal{G}_{\mathcal{P}}, \mathcal{H}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. Their *intersection*, symbolized by $\mathcal{G}_{\mathcal{P}} \cap \mathcal{H}_{\mathcal{P}} = \mathcal{L}_{\mathcal{P}}$, is defined as

$$\mathcal{L}_{\mathcal{P}} = \Big\{ \big(p, \langle a, \mathbb{T}_{\mathcal{L}(p)}(a), \mathbb{I}_{\mathcal{L}(p)}(a), \mathbb{F}_{\mathcal{L}(p)}(a) \rangle : a \in \mathbb{U}\big) : p \in \mathcal{P} \Big\},$$

with

$$\mathbb{T}_{\mathcal{L}(p)}(a) = \min\{\mathbb{T}_{\mathcal{G}(p)}(a), \mathbb{T}_{\mathcal{H}(p)}(a)\}, \quad \mathbb{I}_{\mathcal{L}(p)}(a) = \min\{\mathbb{I}_{\mathcal{G}(p)}(a), \mathbb{I}_{\mathcal{H}(p)}(a)\},$$
$$\mathbb{F}_{\mathcal{L}(p)}(a) = \max\{\mathbb{F}_{\mathcal{G}(p)}(a), \mathbb{F}_{\mathcal{H}(p)}(a)\}.$$

Definition 2.7. [13] For $a^p_{(\alpha,\beta,\gamma)} \in NS(\mathbb{U}_{\mathcal{P}})$, the element $a^p_{(\alpha,\beta,\gamma)}$ is called an NS-point associated with $a \in \mathbb{U}$, parameter $p \in \mathcal{P}$, and degrees $0 < \alpha, \beta, \gamma \leq 1$. It is determined by

$$a^p_{(\alpha,\beta,\gamma)}(p')(b) = \begin{cases} (\alpha, \beta, \gamma), & \text{if } p' = p \text{ and } b = a, \\ (0, 0, 1), & \text{otherwise.} \end{cases}$$

Definition 2.8. [13] Given $a^p_{(\alpha,\beta,\gamma)}, \mathcal{G}_\mathcal{P} \in NS(\mathbb{U}_\mathcal{P})$, the NS-point $a^p_{(\alpha,\beta,\gamma)}$ belongs to $\mathcal{G}_\mathcal{P}$ whenever

$$\alpha \leq \mathbb{T}_{\mathcal{G}(p)}(a), \quad \beta \leq \mathbb{I}_{\mathcal{G}(p)}(a), \quad \gamma \geq \mathbb{F}_{\mathcal{G}(p)}(a),$$

denoted symbolically as $a^p_{(\alpha,\beta,\gamma)} \in \mathcal{G}_\mathcal{P}$.

Definition 2.9. [13] Let $x^p_{(\alpha,\beta,\gamma)}$ and $y^{p'}_{(\alpha',\beta',\gamma')}$ be two NS-points in $NS(\mathbb{U}_\mathcal{P})$. They are NS-distinct if

$$x^p_{(\alpha,\beta,\gamma)} \cap y^{p'}_{(\alpha',\beta',\gamma')} = \widetilde{\emptyset}_\mathcal{P}.$$

In particular, if either $x \neq y$ or $p' \neq p$, then the NS-points are distinct. Conversely, if two NS-points are distinct, at least one of these conditions holds.

Definition 2.10. [21] Let $\widetilde{\tau}^{NS} \subset NS(\mathbb{U}_\mathcal{P})$. If the collection $\widetilde{\tau}^{NS}$ satisfies:

1. It contains the empty NS-set $\widetilde{\emptyset}_\mathcal{P}$ and the universal NS-set $\widetilde{1}_\mathcal{P}$,
2. It is closed under arbitrary unions of its members,
3. It is closed under finite intersections of its members,

then $\widetilde{\tau}^{NS}$ is called an neutrosophic soft topology (shortly, NST) on $\mathbb{U}$.

The triple $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ is called a *neutrosophic soft topological space* (shortly, NSTS). Members of $\widetilde{\tau}^{NS}$ are termed *neutrosophic soft-open sets* (shortly, NSOSs), while their complements are *neutrosophic soft-closed sets* (shortly, NSCSs).

Definition 2.11. [10] Let $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ be an NSTS and $\mathcal{G}_\mathcal{P} \in NS(\mathbb{U}_\mathcal{P})$. An NS-point $a^p_{(\alpha,\beta,\gamma)} \in \mathcal{G}_\mathcal{P}$ has an *NS-neighborhood* $\mathcal{G}_\mathcal{P}$ if there exists an NSOS $\mathcal{O}_\mathcal{P}$ such that $a^p_{(\alpha,\beta,\gamma)} \in \mathcal{O}_\mathcal{P} \subset \mathcal{G}_\mathcal{P}$.

Definition 2.12. [5] Let $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ be an NSTS and $\mathcal{G}_\mathcal{P} \in NS(\mathbb{U}_\mathcal{P})$. Then:

1. The *NS-interior* of $\mathcal{G}_\mathcal{P}$, denoted $N\widetilde{s}int(\mathcal{G}_\mathcal{P})$, is

$$N\widetilde{s}int(\mathcal{G}_\mathcal{P}) = \bigcup \left\{\mathcal{H}_\mathcal{P} : \mathcal{H}_\mathcal{P} \text{ is an NSOS with } \mathcal{H}_\mathcal{P} \subset \mathcal{G}_\mathcal{P}\right\}.$$

2. The *NS-closure* of $\mathcal{G}_\mathcal{P}$, denoted $N\widetilde{s}cl(\mathcal{G}_\mathcal{P})$, is

$$N\widetilde{s}cl(\mathcal{G}_\mathcal{P}) = \bigcap \left\{\mathcal{H}_\mathcal{P} : \mathcal{H}_\mathcal{P} \text{ is an NSCS with } \mathcal{H}_\mathcal{P} \supset \mathcal{G}_\mathcal{P}\right\}.$$

Some fundamental properties of $N\widetilde{s}int(\mathcal{G}_\mathcal{P})$ and $N\widetilde{s}cl(\mathcal{G}_\mathcal{P})$ are given in.[5,21]

Definition 2.13. Let $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ be an NSTS and $\mathcal{G}_\mathcal{P} \in NS(\mathbb{U}_\mathcal{P})$. Then:

1. $\mathcal{G}_\mathcal{P}$ is a *neutrosophic soft semi open set* (shortly, NSsOS) if there exists an NSOS $\mathcal{O}_\mathcal{P}$ such that

$$\mathcal{O}_\mathcal{P} \subset \mathcal{G}_\mathcal{P} \subset N\widetilde{s}cl(\mathcal{O}_\mathcal{P}).$$

2. $\mathcal{G}_\mathcal{P}$ is a *neutrosophic soft semi closed set* (shortly, NSsCS) if there exists an NSCS $\mathcal{F}_\mathcal{P}$ such that

$$N\widetilde{s}int(\mathcal{F}_\mathcal{P}) \subset \mathcal{G}_\mathcal{P} \subset \mathcal{F}_\mathcal{P}.$$

Example 2.14. The NS-set $(\mathcal{F}_\mathcal{P})_1$ introduced in Example 3.11 of[22] qualifies as an NSsOS.

Remark 2.15. Every NSOS is automatically an NSsOS; however, the converse does not necessarily hold.

Example 2.16. The complement of the NS-set $(\mathcal{F}_\mathcal{P})_1$ from Example 3.11 in[22] serves as an NSsCS.

Remark 2.17. It follows that each NSCS is inherently an NSsCS, yet the converse may fail.

Lemma 2.18. *Let $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ be an NSTS and $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. The following statements are equivalent:*

(i) $\mathcal{G}_{\mathcal{P}}$ *is an NSsCS.*

(ii) $N\widetilde{s}int\big(N\widetilde{s}cl(\mathcal{G}_{\mathcal{P}})\big) \subset \mathcal{G}_{\mathcal{P}}$.

(iii) $\mathcal{G}^c_{\mathcal{P}}$ *is an NSsOS.*

Proof. (i) ⇔ (ii): Suppose $\mathcal{G}_{\mathcal{P}}$ is an NSsCS. By Definition 2.13 (ii), there exists an NSCS $\mathcal{F}_{\mathcal{P}}$ such that

$$N\widetilde{s}int(\mathcal{F}_{\mathcal{P}}) \subset \mathcal{G}_{\mathcal{P}} \subset \mathcal{F}_{\mathcal{P}}.$$

This immediately gives

$$N\widetilde{s}int\big(N\widetilde{s}cl(\mathcal{G}_{\mathcal{P}})\big) \subset N\widetilde{s}int(\mathcal{F}_{\mathcal{P}}) \subset \mathcal{G}_{\mathcal{P}}.$$

Conversely, suppose

$$N\widetilde{s}int\big(N\widetilde{s}cl(\mathcal{G}_{\mathcal{P}})\big) \subset \mathcal{G}_{\mathcal{P}}.$$

Taking $N\widetilde{s}cl(\mathcal{G}_{\mathcal{P}}) = \mathcal{F}_{\mathcal{P}}$, it follows that

$$N\widetilde{s}int(\mathcal{F}_{\mathcal{P}}) \subset \mathcal{G}_{\mathcal{P}} \subset N\widetilde{s}cl(\mathcal{G}_{\mathcal{P}}) = \mathcal{F}_{\mathcal{P}},$$

which confirms that $\mathcal{G}_{\mathcal{P}}$ is NSsCS.

(i) ⇔ (iii): Let $\mathcal{G}_{\mathcal{P}}$ be NSsCS. From Definition 2.13 (ii), there exists an NSCS $\mathcal{F}_{\mathcal{P}}$ such that

$$N\widetilde{s}int(\mathcal{F}_{\mathcal{P}}) \subset \mathcal{G}_{\mathcal{P}} \subset \mathcal{F}_{\mathcal{P}}.$$

Taking complements, we get

$$\mathcal{F}^c_{\mathcal{P}} \subset \mathcal{G}^c_{\mathcal{P}} \subset \big(N\widetilde{s}int(\mathcal{F}_{\mathcal{P}})\big)^c = N\widetilde{s}cl(\mathcal{F}^c_{\mathcal{P}}),$$

showing that $\mathcal{G}^c_{\mathcal{P}}$ is NSsOS.

Conversely, if $\mathcal{G}^c_{\mathcal{P}}$ is NSsOS, then by Definition 2.13 (i), there exists an NSOS $\mathcal{O}_{\mathcal{P}}$ such that

$$\mathcal{O}_{\mathcal{P}} \subset \mathcal{G}^c_{\mathcal{P}} \subset N\widetilde{s}cl(\mathcal{O}_{\mathcal{P}}),$$

which is equivalent to

$$\big(N\widetilde{s}cl(\mathcal{O}_{\mathcal{P}})\big)^c \subset \mathcal{G}_{\mathcal{P}} \subset \mathcal{O}^c_{\mathcal{P}}.$$

Using Theorem 3.8.6 in,[5] we get

$$N\widetilde{s}int(\mathcal{O}^c_{\mathcal{P}}) = \big(N\widetilde{s}cl(\mathcal{O}_{\mathcal{P}})\big)^c \subset \mathcal{G}_{\mathcal{P}} \subset \mathcal{O}^c_{\mathcal{P}}.$$

Setting $\mathcal{F}_{\mathcal{P}} = \mathcal{O}^c_{\mathcal{P}}$, we obtain

$$N\widetilde{s}int(\mathcal{F}_{\mathcal{P}}) \subset \mathcal{G}_{\mathcal{P}} \subset \mathcal{F}_{\mathcal{P}},$$

which establishes that $\mathcal{G}_{\mathcal{P}}$ is NSsCS. □

Definition 2.19. [11] Let $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ be an NSTS and $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. Then the following notions are introduced:

(i) The *NS-semi interior* of $\mathcal{G}_{\mathcal{P}}$, denoted $N\widetilde{s}int_s(\mathcal{G}_{\mathcal{P}})$, is defined as

$$N\widetilde{s}int_s(\mathcal{G}_{\mathcal{P}}) = \bigcup\big\{\mathcal{N}_{\mathcal{P}} : \mathcal{N}_{\mathcal{P}} \text{ is an NSsOS on } \mathbb{U} \text{ and } \mathcal{N}_{\mathcal{P}} \subset \mathcal{G}_{\mathcal{P}}\big\}.$$

(ii) The *NS-semi closure* of $\mathcal{G}_{\mathcal{P}}$, denoted $N\widetilde{s}cl_s(\mathcal{G}_{\mathcal{P}})$, is defined as

$$N\widetilde{s}cl_s(\mathcal{G}_{\mathcal{P}}) = \bigcap\big\{\mathcal{N}_{\mathcal{P}} : \mathcal{N}_{\mathcal{P}} \text{ is an NSsCS on } \mathbb{U} \text{ and } \mathcal{G}_{\mathcal{P}} \subset \mathcal{N}_{\mathcal{P}}\big\}.$$

Definition 2.20. [20] Let $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ be an NSTS and $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. Then:

(i) $\mathcal{G}_{\mathcal{P}}$ is called a *neutrosophic soft generalized closed set* (shortly, NSgCS) in $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ if for every NSOS $\mathcal{O}_{\mathcal{P}}$ such that $\mathcal{G}_{\mathcal{P}} \subset \mathcal{O}_{\mathcal{P}}$, it holds that

$$N\widetilde{s}cl(\mathcal{G}_{\mathcal{P}}) \subset \mathcal{O}_{\mathcal{P}}.$$

(ii) $\mathcal{G}_{\mathcal{P}}$ is called a *neutrosophic soft generalized semi closed set* (shortly, NSgsCS) in $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ if for every NSOS $\mathcal{O}_{\mathcal{P}}$ containing $\mathcal{G}_{\mathcal{P}}$, it holds that

$$N\widetilde{s}cl_s(\mathcal{G}_{\mathcal{P}}) \subset \mathcal{O}_{\mathcal{P}}.$$

Theorem 2.21. [27] *Let $(\mathbb{U}, \widetilde{\tau}^{NS}, \mathcal{P})$ be an NSTS. Then the following relationships hold among neutrosophic soft sets:*

(i) Every NSCS is an NSgCS.

(ii) Every NSCS is an NSgsCS.

(iii) Every NSgCS is an NSgsCS.

(iv) Every NSOS is an NSsOS.

(v) Every NSsOS is an NSgsOS.

Proof. All statements follow immediately from the definitions and the hierarchy of neutrosophic soft generalized semi-sets as presented in.[27] □

Definition 2.22. [22] Let $(\mathbb{U}, \widetilde{\tau}^{NS}_{\mathbb{U}}, \mathcal{P})$ and $(\mathbb{W}, \widetilde{\tau}^{NS}_{\mathbb{W}}, \mathcal{Q})$ be two NSTSs. A mapping $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is said to be:

(i) a *neutrosophic soft open mapping* (shortly, NSOM) if for every NSOS $\mathcal{G}_{\mathcal{P}}$ in $NS(\mathbb{U}_{\mathcal{P}})$, the image $f(\mathcal{G}_{\mathcal{P}})$ is an NSOS in $NS(\mathbb{W}_{\mathcal{Q}})$.

(ii) a *neutrosophic soft closed mapping* (shortly, NSCM) if for every NSCS $\mathcal{F}_{\mathcal{P}}$ in $NS(\mathbb{U}_{\mathcal{P}})$, the image $f(\mathcal{F}_{\mathcal{P}})$ is an NSCS in $NS(\mathbb{W}_{\mathcal{Q}})$.

Lemma 2.23. *Let $(\mathbb{U}, \widetilde{\tau}^{NS}_{\mathbb{U}}, \mathcal{P})$ and $(\mathbb{W}, \widetilde{\tau}^{NS}_{\mathbb{W}}, \mathcal{Q})$ be two NSTSs, and let $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ be an NSCM. If $\mathcal{H}_{\mathcal{Q}} \in NS(\mathbb{W}_{\mathcal{Q}})$ and $\mathcal{G}_{\mathcal{P}}$ is an NSOS in $NS(\mathbb{U}_{\mathcal{P}})$ containing $f^{-1}(\mathcal{H}_{\mathcal{Q}})$, then there exists an NSOS $\mathcal{K}_{\mathcal{Q}}$ such that $\mathcal{H}_{\mathcal{Q}} \subset \mathcal{K}_{\mathcal{Q}}$ and $f^{-1}(\mathcal{K}_{\mathcal{Q}}) \subset \mathcal{G}_{\mathcal{P}}$.*

Proof. Suppose f is an NSCM, $\mathcal{H}_{\mathcal{Q}} \in NS(\mathbb{W}_{\mathcal{Q}})$, and $\mathcal{G}_{\mathcal{P}}$ is an NSOS with $f^{-1}(\mathcal{H}_{\mathcal{Q}}) \subset \mathcal{G}_{\mathcal{P}}$. Since f preserves NSCSs, the image of the complement $\mathcal{G}^c_{\mathcal{P}}$ is an NSCS in $NS(\mathbb{W}_{\mathcal{Q}})$.

Define

$$\mathcal{K}_{\mathcal{Q}} := \big(f(\mathcal{G}^c_{\mathcal{P}})\big)^c,$$

which is an NSOS. Observe that

$$f^{-1}(\mathcal{H}_{\mathcal{Q}}) \subset \mathcal{G}_{\mathcal{P}} \implies f^{-1}(\mathcal{H}_{\mathcal{Q}}) \cap \mathcal{G}^c_{\mathcal{P}} = \widetilde{\emptyset}_{\mathcal{P}},$$

hence

$$\mathcal{H}_{\mathcal{Q}} \cap f(\mathcal{G}^c_{\mathcal{P}}) = \widetilde{\emptyset}_{\mathcal{Q}} \implies \mathcal{H}_{\mathcal{Q}} \subset \mathcal{K}_{\mathcal{Q}}.$$

Finally, by properties of inverse images and complements,

$$f^{-1}(\mathcal{K}_{\mathcal{Q}}) = f^{-1}((f(\mathcal{G}^c_{\mathcal{P}}))^c) \subset \mathcal{G}_{\mathcal{P}}.$$

This proves the lemma. □

Definition 2.24. [22] Consider two NSTSs, $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{Q})$. Let a neutrosophic soft mapping $f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ be defined by component mappings $u : \mathbb{U} \to \mathbb{W}$ and $v : \mathcal{P} \to \mathcal{Q}$.

The mapping f is said to be *neutrosophic soft continuous* (shortly, NScM) at an NS-point $x^p_{(\alpha,\beta,\gamma)}$ if, for each NS-neighborhood $\mathcal{H}_{\mathcal{Q}}$ of $f(x^p_{(\alpha,\beta,\gamma)})$, there exists an NS-neighborhood $\mathcal{G}_{\mathcal{P}}$ of $x^p_{(\alpha,\beta,\gamma)}$ such that $f(\mathcal{G}_{\mathcal{P}}) \subset \mathcal{H}_{\mathcal{Q}}$. If this condition holds for all NS-points in $NS(\mathbb{U}_{\mathcal{P}})$, then f is called an *NScM on* $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$.

Definition 2.25. [18] Let $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ be an NSTS. The space is *neutrosophic soft semi-normal* (shortly, NSs-normal) if, for any two disjoint NSCSs $\mathcal{F}^1_{\mathcal{P}}$ and $\mathcal{F}^2_{\mathcal{P}}$ in $\mathbb{U}$, there exist disjoint NSsOSs $(\mathcal{S}_{\mathcal{P}})_1$ and $(\mathcal{S}_{\mathcal{P}})_2$ such that $\mathcal{F}^1_{\mathcal{P}} \subset (\mathcal{S}_{\mathcal{P}})_1, \quad \mathcal{F}^2_{\mathcal{P}} \subset (\mathcal{S}_{\mathcal{P}})_2$.

Definition 2.26. [24] Let $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ be an NSTS. It is *neutrosophic soft semi-regular* (shortly, NSs-regular) if, for any NSCS $\mathcal{F}_{\mathcal{P}}$ and any NS-point $x^p_{(\alpha,\beta,\gamma)}$ satisfying $x^p_{(\alpha,\beta,\gamma)} \cap \mathcal{F}_{\mathcal{P}} = \widetilde{\emptyset}_{\mathcal{P}}$, there exist two disjoint NSsOSs $(\mathcal{S}_{\mathcal{P}})_1$ and $(\mathcal{S}_{\mathcal{P}})_2$ such that $x^p_{(\alpha,\beta,\gamma)} \in (\mathcal{S}_{\mathcal{P}})_1, \quad \mathcal{F}_{\mathcal{P}} \subset (\mathcal{S}_{\mathcal{P}})_2, \quad (\mathcal{S}_{\mathcal{P}})_1 \cap (\mathcal{S}_{\mathcal{P}})_2 = \widetilde{\emptyset}_{\mathcal{P}}$.

3 Neutrosophic Soft Generalized Semi Mappings

In this section, we introduce and investigate the notion of *neutrosophic soft generalized semi mappings*, which generalizes the concept of mappings within neutrosophic soft topological spaces. The discussion begins with the study of *neutrosophic soft generalized semi closed mappings*, followed by an analysis of the fundamental properties of neighborhoods of neutrosophic soft generalized semi sets. In particular, we introduce the operators $N\tilde{s}cl_{gs}$ and $N\tilde{s}int_{gs}$ for subsets of an NSTS $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ to study the structural behavior of NSgsCMs. Finally, we examine *neutrosophic soft generalized semi open mappings* to provide a deeper understanding of their structural and topological properties in neutrosophic soft environments.

3.1 Neutrosophic Soft Generalized Semi Closed Mappings

Definition 3.1. Consider two NSTSs, $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{Q})$. A mapping $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is said to be:

(i) a *neutrosophic soft semi closed mapping* (shortly, NSsCM) if the image under f of any NSCS $\mathcal{K}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$ is an NSsCS in $NS(\mathbb{W}_{\mathcal{Q}})$.

(ii) a *neutrosophic soft generalized semi closed mapping* (shortly, NSgsCM) if, for every NSCS $\mathcal{K}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$, the image $f(\mathcal{K}_{\mathcal{P}})$ belongs to the class of NSgsCS sets in $NS(\mathbb{W}_{\mathcal{Q}})$.

Theorem 3.2. *Let $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs, and let*

$$f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$$

be a neutrosophic soft mapping. Then the following inclusions of mapping types hold:

(i) Every NSCM is an NSgsCM.

(ii) Every NSsCM is an NSgsCM.

Proof. (i) Assume f is an NSCM. For each NSCS $\mathcal{K}_{\mathcal{P}} \subset NS(\mathbb{U}_{\mathcal{P}})$, the image $f(\mathcal{K}_{\mathcal{P}})$ is an NSCS in $NS(\mathbb{W}_{\mathcal{Q}})$. According to Theorem 2.3 in,[27] any NSCS is included in the class of NSgsCS sets. Therefore, $f(\mathcal{K}_{\mathcal{P}})$ is an NSgsCS, establishing that f is an NSgsCM.

(ii) Suppose f is an NSsCM. Then, for each NSCS $\mathcal{K}_{\mathcal{P}}$, the image $f(\mathcal{K}_{\mathcal{P}})$ is an NSsCS. By the same reasoning as in Theorem 2.3 in,[27] every NSsCS is contained within the class of NSgsCS sets. Consequently, $f(\mathcal{K}_{\mathcal{P}})$ is an NSgsCS, confirming that f qualifies as an NSgsCM.

□

Remark 3.3. It should be noted, as illustrated in Example 3.4, that the reverse implications of Theorem 3.2 do not hold in general.

Example 3.4. Consider the universe $\mathbb{U} = \{\eta_1, \eta_2\}$ with parameter set $\mathcal{P} = \{e_1, e_2\}$. Define a collection

$$\tau_{\mathbb{U}}^{NS} = \{\widetilde{\emptyset}_{\mathcal{P}}, \widetilde{1}_{\mathcal{P}}, \mathcal{G}_{\mathcal{P}}\},$$

where the NS-set $\mathcal{G}_{\mathcal{P}}$ over $\mathbb{U}$ is specified as

$$\mathcal{G}_{\mathcal{P}} = \begin{Bmatrix} (e_1, \{\langle \eta_1, 0.8, 0.9, 0.3\rangle, \langle \eta_2, 0.6, 0.7, 0.4\rangle\}), \\ (e_2, \{\langle \eta_1, 0.6, 0.6, 0.5\rangle, \langle \eta_2, 0.7, 0.8, 0.2\rangle\}) \end{Bmatrix}.$$

Then $\tau_{\mathbb{U}}^{NS}$ forms an NST on $\mathbb{U}$, making $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ an NSTS.

Next, let $\mathbb{W} = \{b_1, b_2\}$ with parameters $\mathcal{Q} = \{e_1', e_2'\}$, and define

$$\tau_{\mathbb{W}}^{NS} = \{\widetilde{\emptyset}_{\mathcal{Q}}, \widetilde{1}_{\mathcal{Q}}, (\mathcal{V}_{\mathcal{Q}})_1, (\mathcal{V}_{\mathcal{Q}})_2\},$$

where the NS-sets $(\mathcal{V}_{\mathcal{Q}})_1$ and $(\mathcal{V}_{\mathcal{Q}})_2$ are given by

$$(\mathcal{V}_{\mathcal{Q}})_1 = \begin{Bmatrix} (e_1', \{\langle b_1, 0.5, 0.6, 0.4\rangle, \langle b_2, 0.6, 0.5, 0.3\rangle\}), \\ (e_2', \{\langle b_1, 0.7, 0.5, 0.2\rangle, \langle b_2, 0.5, 0.5, 0.5\rangle\}) \end{Bmatrix}, \quad (\mathcal{V}_{\mathcal{Q}})_2 = \begin{Bmatrix} (e_1', \{\langle b_1, 0.4, 0.4, 0.5\rangle, \langle b_2, 0.3, 0.2, 0.7\rangle\}), \\ (e_2', \{\langle b_1, 0.2, 0.3, 0.7\rangle, \langle b_2, 0.5, 0.5, 0.5\rangle\}) \end{Bmatrix}.$$

Clearly, $\tau_{\mathbb{W}}^{NS}$ defines an NST on $\mathbb{W}$, so $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ is also an NSTS.

Define a neutrosophic soft mapping $f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ by

$$u(\eta_1) = b_2, \quad u(\eta_2) = b_1, \quad v(e_1) = e_1', \quad v(e_2) = e_2'.$$

Considering the image of the complement of $\mathcal{G}_{\mathcal{P}}$ under f, we obtain

$$f(\mathcal{G}_{\mathcal{P}}^c) = \begin{Bmatrix} (e_1, \{\langle \eta_1, 0.4, 0.3, 0.6\rangle, \langle \eta_2, 0.3, 0.1, 0.8\rangle\}), \\ (e_2, \{\langle \eta_1, 0.2, 0.2, 0.7\rangle, \langle \eta_2, 0.5, 0.4, 0.6\rangle\}) \end{Bmatrix}.$$

It is evident that $f(\mathcal{G}_{\mathcal{P}}^c)$ is neither an NSCS nor an NSsCS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$, yet it satisfies the conditions of an NSgsCS. Consequently, f is an NSgsCM while being neither an NSCM nor an NSsCM.

Theorem 3.5 characterizes the structure of an NSgsCM by establishing its behavior on NSCSs and their images under the mapping. It thereby facilitates the identification and verification of such mappings within the framework of NSTSs.

Theorem 3.5. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. A neutrosophic soft mapping*

$$f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$$

is an NSgsCM if and only if for every NS-subset $\mathcal{G}_{\mathcal{Q}}$ of $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ and every NSOS $\mathcal{O}_{\mathcal{P}}$ in $NS(\mathbb{U}_{\mathcal{P}})$ satisfying

$$f^{-1}(\mathcal{G}_{\mathcal{Q}}) \subset \mathcal{G}_{\mathcal{P}},$$

there exists an NSgsOS $\mathcal{S}_{\mathcal{Q}}$ of $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ such that

$$\mathcal{G}_{\mathcal{Q}} \subset \mathcal{S}_{\mathcal{Q}} \quad \text{and} \quad f^{-1}(\mathcal{S}_{\mathcal{Q}}) \subset \mathcal{O}_{\mathcal{P}}.$$

Proof. $(\Rightarrow)$ Assume that f is an NSgsCM. Let $\mathcal{G}_{\mathcal{Q}}$ be any NS-subset of $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ and $\mathcal{O}_{\mathcal{P}}$ an NSOS in $NS(\mathbb{U}_{\mathcal{P}})$ with $f^{-1}(\mathcal{G}_{\mathcal{Q}}) \subset \mathcal{O}_{\mathcal{P}}$. Define

$$\mathcal{S}_{\mathcal{Q}} := \big(f(\mathcal{O}_{\mathcal{P}}^c)\big)^c.$$

Since $\mathcal{O}_{\mathcal{P}}^c$ is an NSCS and f is an NSgsCM, the image $f(\mathcal{O}_{\mathcal{P}}^c)$ is an NSgsCS. Consequently, its complement $\mathcal{S}_{\mathcal{Q}}$ is an NSgsOS. By construction, we have $\mathcal{G}_{\mathcal{Q}} \subset \mathcal{S}_{\mathcal{Q}}$ and $f^{-1}(\mathcal{S}_{\mathcal{Q}}) \subset \mathcal{O}_{\mathcal{P}}$, as required.

($\Leftarrow$) Conversely, suppose the stated condition holds. To prove that f is an NSgsCM, let $\mathcal{O}_{\mathcal{P}}$ be any NSCS in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$, and let $\mathcal{V}_{\mathcal{Q}}$ be any NSOS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ such that

$$f(\mathcal{O}_{\mathcal{P}}) \subset \mathcal{V}_{\mathcal{Q}}.$$

Observe that $\mathcal{O}_{\mathcal{P}}^c$ is an NSOS in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$, and we have

$$f^{-1}((\mathcal{V}_{\mathcal{Q}})^c) \subset \mathcal{O}_{\mathcal{P}}^c.$$

Applying the hypothesis with $\mathcal{G}_{\mathcal{Q}} = \mathcal{V}_{\mathcal{Q}}^c$ and $\mathcal{O}_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}}^c$, there exists an NSgsOS $\mathcal{S}_{\mathcal{Q}}$ such that

$$\mathcal{V}_{\mathcal{Q}}^c \subset \mathcal{S}_{\mathcal{Q}} \quad \text{and} \quad f^{-1}(\mathcal{S}_{\mathcal{Q}}) \subset \mathcal{O}_{\mathcal{P}}^c.$$

Taking complements, we obtain

$$\mathcal{S}_{\mathcal{Q}}^c \subset \mathcal{V}_{\mathcal{Q}} \quad \text{and} \quad \mathcal{O}_{\mathcal{P}} \subset \left(f^{-1}(\mathcal{S}_{\mathcal{Q}})\right)^c.$$

Since $\mathcal{S}_{\mathcal{Q}}$ is an NSgsOS, its complement $\mathcal{S}_{\mathcal{Q}}^c$ is NSgsCS. Applying f gives

$$f(\mathcal{O}_{\mathcal{P}}) \subset f\Big(\left(f^{-1}(\mathcal{S}_{\mathcal{Q}})\right)^c\Big) \subset \mathcal{S}_{\mathcal{Q}}^c \subset \mathcal{V}_{\mathcal{Q}}.$$

Finally, by properties of NSgsCS, we conclude

$$N\widetilde{s}cl_s(f(\mathcal{O}_{\mathcal{P}})) \subset N\widetilde{s}cl_s(\mathcal{S}_{\mathcal{Q}}^c) = \mathcal{S}_{\mathcal{Q}}^c \subset \mathcal{V}_{\mathcal{Q}},$$

showing that $f(\mathcal{O}_{\mathcal{P}})$ is an NSgsCS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$. Hence, f is an NSgsCM. □

Theorem 3.6. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. If the mapping $f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is both an NScM and an NSCM, then for every NSgCS $\mathcal{G}_{\mathcal{P}}$ in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$, the image $f(\mathcal{G}_{\mathcal{P}})$ is an NSgCS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$.*

Proof. Straightforward. □

Theorem 3.7 strengthens and extends Theorem 3.6 by considering mappings that are both NScM and NSgsCM, thereby providing a more comprehensive characterization.

Theorem 3.7. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. If $f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is both an NScM and an NSgsCM, and if $\mathcal{G}_{\mathcal{P}}$ is an NSgsCS in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$, then $f(\mathcal{G}_{\mathcal{P}})$ is an NSgsCS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$.*

Proof. Take any NSOS $\mathcal{O}_{\mathcal{Q}}$ in $NS(\mathbb{W}_{\mathcal{Q}})$ with

$$f(\mathcal{G}_{\mathcal{P}}) \subset \mathcal{O}_{\mathcal{Q}}.$$

By the definition of NSgsCS, we have

$$N\widetilde{s}cl_s(\mathcal{G}_{\mathcal{P}}) \subset f^{-1}(\mathcal{O}_{\mathcal{Q}}).$$

Since f is an NScM, $f^{-1}(\mathcal{O}_{\mathcal{Q}})$ is an NSOS in $NS(\mathbb{U}_{\mathcal{P}})$.

Applying f to both sides gives

$$f(N\widetilde{s}cl_s(\mathcal{G}_{\mathcal{P}})) \subset \mathcal{O}_{\mathcal{Q}}.$$

Using that f is an NSgsCM, the image $f(N\widetilde{s}cl_s(\mathcal{G}_{\mathcal{P}}))$ is itself an NSgsCS in $NS(\mathbb{W}_{\mathcal{Q}})$.

Finally, the monotonicity of the NSgs-semi-closure operator yields

$$N\widetilde{s}cl_s(f(\mathcal{G}_{\mathcal{P}})) \subset N\widetilde{s}cl_s(f(N\widetilde{s}cl_s(\mathcal{G}_{\mathcal{P}}))) \subset \mathcal{O}_{\mathcal{Q}}.$$

Hence, by the definition of NSgsCS, $f(\mathcal{G}_{\mathcal{P}})$ is an NSgsCS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$. □

This result extends the preservation properties of neutrosophic soft closed mappings to the broader class of neutrosophic soft generalized semi-closed mappings, indicating that the latter class naturally inherits the stronger topological structure under such mappings.

In the sequel, we present the invariance properties of NSgsCMs.

Theorem 3.8. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. If $f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is both an NScM and an NSgsCM, and if f maps an NSs-normal space $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ onto $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$, then $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ is an NSs-normal.*

Proof. Let $(\mathcal{G}_{\mathcal{Q}})_1$ and $(\mathcal{G}_{\mathcal{Q}})_2$ be two disjoint NSCSs in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$. Since f is an NScM, Theorem 3.3 in[22] implies that

$$f^{-1}((\mathcal{G}_{\mathcal{Q}})_1), \quad f^{-1}((\mathcal{G}_{\mathcal{Q}})_2)$$

are disjoint NSCSs in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$.

Because $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ is an NSs-normal (Definition 2.25), there exist disjoint NSOSs $(\mathcal{O}_{\mathcal{P}})_1$ and $(\mathcal{G}_{\mathcal{Q}})_2$ such that

$$f^{-1}((\mathcal{G}_{\mathcal{Q}})_1) \subset (\mathcal{G}_{\mathcal{Q}})_1, \quad f^{-1}((\mathcal{G}_{\mathcal{Q}})_2) \subset (\mathcal{G}_{\mathcal{Q}})_2.$$

Since f is an NSgsCM, applying Theorem 3.5 guarantees the existence of NSgsOSs $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$ in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ satisfying

$$(\mathcal{G}_{\mathcal{Q}})_1 \subset G_{\mathcal{Q}}, \quad (\mathcal{G}_{\mathcal{Q}})_2 \subset H_{\mathcal{Q}}, \quad f^{-1}(G_{\mathcal{Q}})(\subset \mathcal{G}_{\mathcal{Q}})_1, \quad f^{-1}(H_{\mathcal{Q}}) \subset (\mathcal{G}_{\mathcal{Q}})_2.$$

Disjointness of $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$: Because $(\mathcal{G}_{\mathcal{Q}})_1$ and $(\mathcal{G}_{\mathcal{Q}})_2$ are disjoint NSOSs, we have

$$(\mathcal{G}_{\mathcal{Q}})_1 \cap (\mathcal{G}_{\mathcal{Q}})_2 = \widetilde{\emptyset}_{\mathcal{P}}.$$

Then, from the preimages, it follows that

$$f^{-1}(G_{\mathcal{Q}}) \cap f^{-1}(H_{\mathcal{Q}}) \subset (\mathcal{G}_{\mathcal{Q}})_1 \cap (\mathcal{G}_{\mathcal{Q}})_2 = \widetilde{\emptyset}_{\mathcal{P}}.$$

Since f is an surjective onto $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$, the disjointness of preimages ensures

$$G_{\mathcal{Q}} \cap H_{\mathcal{Q}} = \widetilde{\emptyset}_{\mathcal{Q}}.$$

Thus, the obtained NSgsOSs $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$ are disjoint, which shows that $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ is an NSs-normal. □

Theorem 3.9. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. Suppose $f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is an NSOM, NScM, and NSgsCM mapping from an NSs-regular space $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ onto $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$. Then $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ is an NSs-regular.*

Proof. Let $\mathcal{G}_{\mathcal{Q}}$ be an arbitrary NSCS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$, and let $y^q_{(\alpha,\beta,\gamma)} \notin \mathcal{G}_{\mathcal{Q}}$. Since f is an NScM, Theorem 3.3 in[22] implies that

$$f^{-1}(\mathcal{G}_{\mathcal{Q}}) \text{ is an NSCS in } (\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P}).$$

Moreover, because f is an NScM and NSOM, for any NS-neighborhood $\mathcal{V}_{\mathcal{Q}}$ of $y^q_{(\alpha,\beta,\gamma)}$, there exists an NSOS $\mathcal{U}_{\mathcal{P}}$ in $NS(\mathbb{U}_{\mathcal{P}})$ such that

$$x^p_{(\alpha,\beta,\gamma)} \in \mathcal{U}_{\mathcal{P}} \subset f^{-1}(\mathcal{V}_{\mathcal{Q}}), \quad \text{where } f(x^p_{(\alpha,\beta,\gamma)}) = y^q_{(\alpha,\beta,\gamma)}.$$

Since $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ is an NSs-regular, by Definition 2.26, there exist disjoint NSOSs $(\mathcal{O}_{\mathcal{P}})_1$ and $(\mathcal{O}_{\mathcal{P}})_2$ satisfying

$$f^{-1}(\mathcal{G}_{\mathcal{Q}}) \subset (\mathcal{O}_{\mathcal{P}})_1, \quad f^{-1}(x^p_{(\alpha,\beta,\gamma)}) \subset (\mathcal{O}_{\mathcal{P}})_2.$$

Since f is an NSgsCM, Theorem 3.5 guarantees the existence of NSgsOSs $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$ in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ such that

$$\mathcal{G}_{\mathcal{Q}} \subset G_{\mathcal{Q}}, \quad y^q_{(\alpha,\beta,\gamma)} \in H_{\mathcal{Q}}, \quad f^{-1}(G_{\mathcal{Q}}) \subset (\mathcal{O}_{\mathcal{P}})_1, \quad f^{-1}(H_{\mathcal{Q}}) \subset (\mathcal{O}_{\mathcal{P}})_2.$$

Disjointness of $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$: Since $(\mathcal{O}_{\mathcal{P}})_1$ and $(\mathcal{O}_{\mathcal{P}})_2$ are disjoint NSOSs,

$$(\mathcal{O}_{\mathcal{P}})_1 \cap (\mathcal{O}_{\mathcal{P}})_2 = \widetilde{\emptyset}_{\mathcal{P}}.$$

Thus, taking preimages under f, we obtain

$$f^{-1}(G_{\mathcal{Q}}) \cap f^{-1}(H_{\mathcal{Q}}) \subset (\mathcal{O}_{\mathcal{P}})_1 \cap (\mathcal{O}_{\mathcal{P}})_2 = \widetilde{\emptyset}_{\mathcal{P}}.$$

Since f is an surjective onto $\mathbb{W}$, it follows that

$$G_{\mathcal{Q}} \cap H_{\mathcal{Q}} = \widetilde{\emptyset}_{\mathcal{Q}}.$$

Hence, the NSgsOSs $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$ separate $\mathcal{G}_{\mathcal{Q}}$ and $y^q_{(\alpha,\beta,\gamma)}$, confirming that $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ is an NSs-regular. □

3.2 Some Properties of Neighborhoods of Neutrosophic Soft Generalized Semi Sets

Definition 3.10. Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ be an NSTS over $\mathbb{U}$. The *neutrosophic soft generalized semi-closure* of an NS-set $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$, denoted by $N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}})$, is defined as the intersection of all NSgsCSs in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ that include $\mathcal{G}_{\mathcal{P}}$.

Equivalently, this can be written as

$$N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}}) = \bigcap \{\mathcal{S}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}}) : \mathcal{S}_{\mathcal{P}} \text{ is an NSgsCS on } \mathbb{U} \text{ and } \mathcal{S}_{\mathcal{P}} \supset \mathcal{G}_{\mathcal{P}}\}.$$

This definition guarantees that $N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}})$ represents the smallest NSgsCS containing $\mathcal{G}_{\mathcal{P}}$, thereby maintaining the minimality and closure-like properties within the generalized semi framework of NST.

Theorem 3.11. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ be an NSTS over $\mathbb{U}$ and $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. Then*

$$\mathcal{G}_{\mathcal{P}} \subset N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}}) \subset N\tilde{s}cl_{s}(\mathcal{G}_{\mathcal{P}}) \subset N\tilde{s}cl(\mathcal{G}_{\mathcal{P}}).$$

Proof. The statement follows directly from Remarks 2.15 and 2.17, Definitions 2.19 and 3.10, and Theorem 2.21. □

Remark 3.12. If $\mathcal{G}_{\mathcal{P}}$ is already an NSgsCS, then $\mathcal{G}_{\mathcal{P}} = N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}})$. However, the converse does not necessarily hold; that is, $N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}})$ may not be an NSgsCS, as demonstrated in Example 3.13.

Example 3.13. Let $\mathbb{U} = \{\eta_1, \eta_2\}$, $\mathcal{P} = \{e_1, e_2\}$, and $\tau_{\mathbb{U}}^{NS} = \{\widetilde{\emptyset}_{\mathcal{P}}, \widetilde{1}_{\mathcal{P}}, \mathcal{G}_{\mathcal{P}}\}$, where $\mathcal{G}_{\mathcal{P}}$ is an NS-set over $\mathbb{U}$ given by:

$$\mathcal{G}_{\mathcal{P}} = \left\{ \begin{array}{l} (e_1, \{\langle \eta_1, 0.7, 0.6, 0.3\rangle, \langle \eta_2, 0.5, 0.7, 0.4\rangle\}), \\ (e_2, \{\langle \eta_1, 0.6, 0.7, 0.4\rangle, \langle \eta_2, 0.6, 0.4, 0.6\rangle\}) \end{array} \right\}.$$

Consider the NS-sets $\mathcal{V}_{\mathcal{P}}$ and $\mathcal{K}_{\mathcal{P}}$ over $\mathbb{U}$ specified by:

$$\mathcal{V}_{\mathcal{P}} = \left\{ \begin{array}{l} (e_1, \{\langle \eta_1, 0.3, 0.4, 0.7\rangle, \langle \eta_2, 0.4, 0.8, 0.5\rangle\}), \\ (e_2, \{\langle \eta_1, 0.6, 0.3, 0.6\rangle, \langle \eta_2, 0.6, 0.6, 0.6\rangle\}) \end{array} \right\}, \quad \mathcal{K}_{\mathcal{P}} = \left\{ \begin{array}{l} (e_1, \{\langle \eta_1, 0.4, 0.7, 0.2\rangle, \langle \eta_2, 0.6, 0.5, 0.5\rangle\}), \\ (e_2, \{\langle \eta_1, 0.5, 0.8, 0.6\rangle, \langle \eta_2, 0.3, 0.5, 0.6\rangle\}) \end{array} \right\}.$$

By applying Definition 3.10, both $\mathcal{V}_{\mathcal{P}}$ and $\mathcal{K}_{\mathcal{P}}$ qualify as NSgsCSs on $\mathbb{U}$.

However, note that

$$\mathcal{V}_{\mathcal{P}} \cap \mathcal{K}_{\mathcal{P}} \subset \mathcal{G}_{\mathcal{P}} \quad \text{while} \quad N\tilde{s}cl_s\left(\mathcal{V}_{\mathcal{P}} \cap \mathcal{K}_{\mathcal{P}}\right) = \widetilde{1}_{\mathcal{P}} \nsubseteq \mathcal{G}_{\mathcal{P}}.$$

Therefore, the intersection $\mathcal{V}_{\mathcal{P}} \cap \mathcal{K}_{\mathcal{P}}$ does not form an NSgsCS on $\mathbb{U}$.

This example demonstrates that intersecting two NSgsCSs may result in a set that is not an NSgsCS. As a result, the neutrosophic soft generalized semi-closure $N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}})$ need not itself be an NSgsCS. This distinction underscores the non-standard nature of NSgsCS sets compared to classical closure concepts.

Definition 3.14. Let $\left(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P}\right)$ be an NSTS over $\mathbb{U}$. The *neutrosophic soft generalized semi-interior* of a neutrosophic soft set $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$, denoted $N\tilde{s}int_{gs}(\mathcal{G}_{\mathcal{P}})$, is obtained by taking the union of all NSgsOSs that lie within $\mathcal{G}_{\mathcal{P}}$.

Equivalently,

$$N\tilde{s}int_{gs}\left(\mathcal{G}_{\mathcal{P}}\right) = \bigcup \left\{\mathcal{O}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}}) : \mathcal{O}_{\mathcal{P}} \text{ is an NSgsOS on } \mathbb{U} \text{ and } \mathcal{O}_{\mathcal{P}} \subset \mathcal{G}_{\mathcal{P}}\right\}.$$

This formulation guarantees that $N\tilde{s}int_{gs}(\mathcal{G}_{\mathcal{P}})$ represents the largest NSgsOS fully contained in $\mathcal{G}_{\mathcal{P}}$, serving as a generalized interior operator within the neutrosophic soft topological framework.

Theorem 3.15. *Let $\left(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P}\right)$ be an NSTS over $\mathbb{U}$, and let $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. Then the following chain of inclusions holds:*

$$N\tilde{s}int(\mathcal{G}_{\mathcal{P}}) \subset N\tilde{s}int_s(\mathcal{G}_{\mathcal{P}}) \subset N\tilde{s}int_{gs}(\mathcal{G}_{\mathcal{P}}).$$

Proof. The result follows directly from Remarks 2.15 and 2.17, Definitions 2.19 and 3.14, as well as Theorem 2.21. □

Theorem 3.16. *Let $\left(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P}\right)$ be an NSTS over $\mathbb{U}$, and let $\mathcal{G}_{\mathcal{P}} \in NS(\mathbb{U}_{\mathcal{P}})$. Then the following duality relations are valid:*

1. *The complement of the NSgs-interior equals the NSgs-closure of the complement:*

$$\left(N\tilde{s}\,\mathrm{int}_{gs}(\mathcal{G}_{\mathcal{P}})\right)^c = N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{P}}^c),$$

2. *The complement of the NSgs-closure equals the NSgs-interior of the complement:*

$$\left(N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{P}})\right)^c = N\tilde{s}\,\mathrm{int}_{gs}(\mathcal{G}_{\mathcal{P}}^c).$$

Proof. We prove each statement separately.

(i) Consider the neutrosophic soft generalized semi-interior $N\tilde{s}\,\mathrm{int}_{gs}(\mathcal{G}_{\mathcal{P}})$. According to Definition 3.14, this set represents the largest NSgsOS contained in $\mathcal{G}_{\mathcal{P}}$. Taking its complement yields the smallest NSgsCS that contains $\mathcal{G}_{\mathcal{P}}^c$, since the complement of a largest open set inside a given set corresponds to the minimal closed set containing the complement. Hence, we obtain

$$\left(N\tilde{s}\,\mathrm{int}_{gs}(\mathcal{G}_{\mathcal{P}})\right)^c = N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{P}}^c).$$

(ii) Similarly, let $N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{P}})$ denote the smallest NSgsCS containing $\mathcal{G}_{\mathcal{P}}$. Its complement is therefore the largest NSgsOS contained in $\mathcal{G}_{\mathcal{P}}^c$, which yields

$$\left(N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{P}})\right)^c = N\tilde{s}\,\mathrm{int}_{gs}(\mathcal{G}_{\mathcal{P}}^c).$$

Consequently, both duality properties follow naturally from the complementary relationship between closure and interior operators in the neutrosophic soft generalized semi-topological framework. □

Theorem 3.17. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. If $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is an NSgsCM, then for every NS-subset $\mathcal{G}_{\mathcal{P}}$ of $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$, the following inclusion holds:*

$$N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}}) \subset f\left(N\tilde{s}cl(\mathcal{G}_{\mathcal{P}})\right).$$

Proof. Let $\mathcal{G}_{\mathcal{P}}$ be an arbitrary NS-subset of $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$. Since $N\tilde{s}cl(\mathcal{G}_{\mathcal{P}})$ represents the smallest NSCS containing $\mathcal{G}_{\mathcal{P}}$ and f preserves NSgsCSs by assumption, the image $f(N\tilde{s}cl(\mathcal{G}_{\mathcal{P}}))$ forms an NSgsCS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ that contains $f(\mathcal{G}_{\mathcal{P}})$. By Definition 3.10, $N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}})$ is the minimal NSgsCS containing $f(\mathcal{G}_{\mathcal{P}})$. Consequently, we obtain

$$N\tilde{s}cl_{gs}(\mathcal{G}_{\mathcal{P}}) \subset f\left(N\tilde{s}cl(\mathcal{G}_{\mathcal{P}})\right),$$

establishing the claimed inclusion. □

3.3 Neutrosophic Soft Generalized Semi Open Mappings

Definition 3.18. Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs, and let $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ be a neutrosophic soft mapping. The mapping f is called a *neutrosophic soft generalized semi open mapping* (shortly, NSgsOM) if for every NSOS $\mathcal{O}_{\mathcal{P}} \in \tau_{\mathbb{U}}^{NS}$, the image $f(\mathcal{O}_{\mathcal{P}})$ belongs to the family of NSgsOSs in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$.

Example 3.19. Consider $\mathbb{U} = \{\eta_1, \eta_2\}$, $\mathcal{P} = \{p_1, p_2\}$, and the NST $\tau_{\mathbb{U}}^{NS} = \{\widetilde{\emptyset}_{\mathcal{P}}, \widetilde{1}_{\mathcal{P}}, \mathcal{G}_{\mathcal{P}}\}$ on $\mathbb{U}$, where

$$\mathcal{G}_{\mathcal{P}} = \left\{ \begin{array}{l} (p_1, \{\langle \eta_1, 0.5, 0.5, 0.8\rangle, \langle \eta_2, 0.2, 0.4, 0.5\rangle\}), \\ (p_2, \{\langle \eta_1, 0.3, 0.5, 0.6\rangle, \langle \eta_2, 0.2, 0.3, 0.4\rangle\}) \end{array} \right\}.$$

Then $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ is an NSTS.

Similarly, let $\mathbb{W} = \{w_1, w_2\}$, $\mathcal{Q} = \{q_1, q_2\}$, and $\tau_{\mathbb{W}}^{NS} = \{\widetilde{\emptyset}_{\mathcal{Q}}, \widetilde{1}_{\mathcal{Q}}, (\widetilde{\mathbb{W}}_{\mathcal{Q}})_1, (\widetilde{\mathbb{W}}_{\mathcal{Q}})_2\}$, where

$$(\widetilde{\mathbb{W}}_{\mathcal{Q}})_1 = \left\{ \begin{array}{l} (q_1, \{\langle w_1, 0.5, 0.6, 0.4\rangle, \langle w_2, 0.6, 0.5, 0.3\rangle\}), \\ (q_2, \{\langle w_1, 0.7, 0.5, 0.2\rangle, \langle w_2, 0.5, 0.5, 0.5\rangle\}) \end{array} \right\},$$

$$(\widetilde{\mathbb{W}}_{\mathcal{Q}})_2 = \left\{ \begin{array}{l} (q_1, \{\langle w_1, 0.4, 0.4, 0.5\rangle, \langle w_2, 0.3, 0.2, 0.7\rangle\}), \\ (q_2, \{\langle w_1, 0.2, 0.3, 0.7\rangle, \langle w_2, 0.5, 0.5, 0.5\rangle\}) \end{array} \right\}.$$

Then $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ is also an NSTS.

Define a mapping $f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ by

$$u(\eta_1) = w_2, \quad u(\eta_2) = w_1, \quad v(p_1) = q_1, \quad v(p_2) = q_2.$$

Then

$$f(\mathcal{G}_{\mathcal{P}}) = \left\{ \begin{array}{l} (q_1, \{\langle w_1, 0.2, 0.4, 0.5\rangle, \langle w_2, 0.5, 0.5, 0.8\rangle\}), \\ (q_2, \{\langle w_1, 0.2, 0.3, 0.4\rangle, \langle w_2, 0.3, 0.5, 0.6\rangle\}) \end{array} \right\},$$

which is an NSgsOS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$. Hence, f is an NSgsOM.

Theorem 3.20. *Let $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ be a neutrosophic soft mapping between two NSTSs. If f is an NSOM, then it is also an NSgsOM.*

Proof. Let $\mathcal{O}_{\mathcal{P}} \in \tau_{\mathbb{U}}^{NS}$ be an arbitrary NSOS. Since f is an NSOM, $f(\mathcal{O}_{\mathcal{P}}) \in \tau_{\mathbb{W}}^{NS}$, i.e., it is an NSOS in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$.

By Theorem 2.3 in,[27] every NSOS is also an NSgsOS. Therefore, $f(\mathcal{O}_{\mathcal{P}}) \in NSgsOS(\mathbb{W})$, which implies that f is an NSgsOM. □

Theorem 3.21. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs, and let $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ be a neutrosophic soft mapping. Then the following statements are equivalent:*

(i) *f is an NSgsOM,*

(ii) *For every NS-subset $\mathcal{G}_{\mathcal{P}}$ of $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$, the inclusion $f(N\tilde{s}\,\mathrm{int}(\mathcal{G}_{\mathcal{P}})) \subset N\tilde{s}\,\mathrm{int}_{gs}(f(\mathcal{G}_{\mathcal{P}}))$ holds,*

(iii) *For each $x^{p}_{(\alpha,\beta,\gamma)} \in NS(\mathbb{U}_{\mathcal{P}})$ and each NSOS $\mathcal{O}_{\mathcal{P}}$ containing $x^{p}_{(\alpha,\beta,\gamma)}$, there exists an NSgsOS $\mathcal{V}_{\mathcal{Q}}$ containing $f(x^{p}_{(\alpha,\beta,\gamma)})$ such that $\mathcal{V}_{\mathcal{Q}} \subset f(\mathcal{O}_{\mathcal{P}})$,*

(iv) *For every NS-subset $\mathcal{G}_{\mathcal{Q}}$ of $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$, the preimage satisfies $f^{-1}(N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{Q}})) \subset N\tilde{s}\,\mathrm{cl}(f^{-1}(\mathcal{G}_{\mathcal{Q}}))$.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) follow directly from the definitions. We provide a detailed argument for (iii) ⇒ (iv):

Let $\mathcal{G}_{\mathcal{Q}}$ be an arbitrary NS-subset and suppose

$$x^{p}_{(\alpha,\beta,\gamma)} \in f^{-1}(N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{Q}})).$$

Assume, for the sake of contradiction, that

$$x^{p}_{(\alpha,\beta,\gamma)} \notin N\tilde{s}\,\mathrm{cl}(f^{-1}(\mathcal{G}_{\mathcal{Q}})).$$

Then the NSOS

$$\mathcal{O}_{\mathcal{P}} = \left(N\tilde{s}\,\mathrm{cl}(f^{-1}(\mathcal{G}_{\mathcal{Q}}))\right)^{c}$$

contains $x^{p}_{(\alpha,\beta,\gamma)}$. By (iii), there exists an NSgsOS $\mathcal{V}_{\mathcal{Q}}$ such that

$$f(x^{p}_{(\alpha,\beta,\gamma)}) \in \mathcal{V}_{\mathcal{Q}} \subset f(\mathcal{O}_{\mathcal{P}}).$$

Observe that

$$f(\mathcal{O}_{\mathcal{P}}) \subset f(f^{-1}(\mathcal{G}_{\mathcal{Q}})^{c}) \subset (\mathcal{G}_{\mathcal{Q}})^{c},$$

which implies $\mathcal{V}_{\mathcal{Q}} \subset (\mathcal{G}_{\mathcal{Q}})^{c}$. Since $(\mathcal{V}_{\mathcal{Q}})^{c}$ is an NSgsCS, we have

$$N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{Q}}) \subset (\mathcal{V}_{\mathcal{Q}})^{c}.$$

However, this contradicts the fact that $f(x^{p}_{(\alpha,\beta,\gamma)}) \in N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{Q}})$, as $f(x^{p}_{(\alpha,\beta,\gamma)}) \in \mathcal{V}_{\mathcal{Q}}$ simultaneously.

Hence, the assumption is false, and we conclude

$$x^{p}_{(\alpha,\beta,\gamma)} \in N\tilde{s}\,\mathrm{cl}(f^{-1}(\mathcal{G}_{\mathcal{Q}})).$$

As $x^{p}_{(\alpha,\beta,\gamma)}$ was arbitrary, it follows that

$$f^{-1}(N\tilde{s}\,\mathrm{cl}_{gs}(\mathcal{G}_{\mathcal{Q}})) \subset N\tilde{s}\,\mathrm{cl}(f^{-1}(\mathcal{G}_{\mathcal{Q}})),$$

as required. □ □

Theorem 3.22. *Let $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. Suppose $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is an NSgsOM, $\mathcal{G}_{\mathcal{Q}} \in NS(\mathbb{W}_{\mathcal{Q}})$, and $\mathcal{F}_{\mathcal{P}}$ is an NSCS containing $f^{-1}(\mathcal{G}_{\mathcal{Q}})$. Then, there exists an NSgsCS $\mathcal{V}_{\mathcal{Q}}$ such that*

$$\mathcal{G}_{\mathcal{Q}} \subset \mathcal{V}_{\mathcal{Q}} \quad \textit{and} \quad f^{-1}(\mathcal{V}_{\mathcal{Q}}) \subset \mathcal{F}_{\mathcal{P}}.$$

Proof. Let $\mathcal{G}_{\mathcal{Q}} \in NS(\mathbb{W}_{\mathcal{Q}})$ and let $\mathcal{F}_{\mathcal{P}}$ be an NSCS satisfying $f^{-1}(\mathcal{G}_{\mathcal{Q}}) \subset \mathcal{F}_{\mathcal{P}}$. Define

$$\mathcal{V}_{\mathcal{Q}} := \left(f(\mathcal{F}_{\mathcal{P}}^{c})\right)^{c}.$$

Since f is an NSgsOM, the set $f(\mathcal{F}_{\mathcal{P}}^{c})$ is NSgsOS, and thus its complement $\mathcal{V}_{\mathcal{Q}}$ is an NSgsCS in $NS(\mathbb{W}_{\mathcal{Q}})$. By construction, $\mathcal{G}_{\mathcal{Q}} \subset \mathcal{V}_{\mathcal{Q}}$ and

$$f^{-1}(\mathcal{V}_{\mathcal{Q}}) = f^{-1}\left((f(\mathcal{F}_{\mathcal{P}}^{c}))^{c}\right) \subset \mathcal{F}_{\mathcal{P}},$$

as required. □

4 Neutrosophic Soft Generalized Semi Continuous Mappings

In this section, neutrosophic soft semi-continuous, generalized continuous, and generalized semi-continuous mappings are introduced. Their fundamental properties and relationships with other mappings are examined through theorems and counterexamples.

Definition 4.1. Let $\left(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P}\right)$ and $\left(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q}\right)$ be two NSTSs. A mapping $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is called:

(i) **Neutrosophic soft semi-continuous mapping (NSscM)** if for every NSCS $\mathcal{V}_{\mathcal{Q}}$ in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$, the preimage $f^{-1}(\mathcal{V}_{\mathcal{Q}})$ belongs to NSsCS in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$.

(ii) **Neutrosophic soft generalized continuous mapping (NSgcM)** if for each NSCS $\mathcal{V}_{\mathcal{Q}}$ in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$, the set $f^{-1}(\mathcal{V}_{\mathcal{Q}})$ is an NSgCS in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$.

(iii) **Neutrosophic soft generalized semi-continuous mapping (NSgscM)** if for every NSCS $\mathcal{V}_{\mathcal{Q}}$ in $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$, the preimage $f^{-1}(\mathcal{V}_{\mathcal{Q}})$ lies in NSgsCS in $(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P})$.

Theorem 4.2. *Let $\left(\mathbb{U}, \tau_{\mathbb{U}}^{NS}, \mathcal{P}\right)$ and $\left(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q}\right)$ be two NSTSs, and let $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ be a neutrosophic soft mapping. Then the following implications hold:*

(i) Every NScM is an NSgcM.

(ii) Every NSgcM is an NSgscM.

(iii) Every NSscM is an NSgscM.

Proof. Let $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ be a neutrosophic soft mapping.

(i) By Definition 4.1, an NScM preserves the preimage of any NSCS as an NSCS. Since every NSCS is also an NSgCS (Theorem 2.21), the inverse image of NSCS under f is necessarily NSgCS. Hence, f is an NSgcM.

(ii) Every NSgCS is inherently an NSgsCS (Definition 4.1 and Theorem 2.21). Therefore, if f preserves inverse images of NSgCS sets, it also preserves inverse images of NSgsCS sets. Thus, f is an NSgscM.

(iii) An NSscM preserves inverse images of NSsCS sets, and since NSsCS sets are a subset of NSgsCS sets, f also preserves inverse images of NSgsCS sets. Hence, f is an NSgscM.

Consequently, the chain of implications is

$$\text{NScM} \implies \text{NSgcM} \implies \text{NSgscM}, \quad \text{and} \quad \text{NSscM} \implies \text{NSgscM}.$$

□

Remark 4.3. It follows from Example 4.4 that the converse of Theorem 4.2 does not always hold. This shows that NSgscMs form a strictly larger class than NScMs or NSscMs.

Example 4.4. An NSgscM need not be an NScM or an NSscM. Consider $\mathbb{U} = \{\eta_1, \eta_2\}$, $\mathcal{P} = \{e_1, e_2\}$, and define the NST

$$\tilde{\tau}_{\mathbb{U}}^{NS} = \{\widetilde{\emptyset}_{\mathcal{P}}, \widetilde{1}_{\mathcal{P}}, (\mathcal{F}_{\mathcal{P}})_1, (\mathcal{F}_{\mathcal{P}})_2\}$$

over $\mathbb{U}$, where $(\mathcal{F}_{\mathcal{P}})_1$ and $(\mathcal{F}_{\mathcal{P}})_2$ are NS-sets defined by:

$$(\mathcal{F}_{\mathcal{P}})_1 = \{(e_1, \{\langle \eta_1, 0.5, 0.6, 0.4\rangle, \langle \eta_2, 0.6, 0.5, 0.3\rangle\}),\ (e_2, \{\langle \eta_1, 0.7, 0.5, 0.2\rangle, \langle \eta_2, 0.5, 0.5, 0.5\rangle\})\},$$

$$(\mathcal{F}_{\mathcal{P}})_2 = \{(e_1, \{\langle \eta_1, 0.4, 0.4, 0.5\rangle, \langle \eta_2, 0.3, 0.2, 0.7\rangle\}),\ (e_2, \{\langle \eta_1, 0.2, 0.3, 0.7\rangle, \langle \eta_2, 0.5, 0.5, 0.5\rangle\})\}.$$

Then $(\mathbb{U}, \tilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ is an NSTS.

Now, let $\mathbb{W} = \{b_1, b_2\}$, $\mathcal{Q} = \{e'_1, e'_2\}$, and define

$$\tau_{\mathbb{W}}^{NS} = \{\widetilde{\emptyset}_{\mathcal{Q}}, \widetilde{1}_{\mathcal{Q}}, \mathcal{G}_{\mathcal{Q}}\},$$

with

$$\mathcal{G}_{\mathcal{Q}} = \{(e'_1, \{\langle b_1, 0.8, 0.9, 0.3\rangle, \langle b_2, 0.6, 0.7, 0.4\rangle\}),\ (e'_2, \{\langle b_1, 0.6, 0.6, 0.5\rangle, \langle b_2, 0.7, 0.8, 0.2\rangle\})\}.$$

Thus, $(\mathbb{W}, \tau_{\mathbb{W}}^{NS}, \mathcal{Q})$ is an NSTS.

Define the mapping $f = (u, v) : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ by

$$u(\eta_1) = b_2, \quad u(\eta_2) = b_1, \quad v(e_1) = e'_1, \quad v(e_2) = e'_2.$$

The inverse image

$$f^{-1}(\mathcal{G}_{\mathcal{Q}})^c = \{(e_1, \{\langle \eta_1, 0.4, 0.3, 0.6\rangle, \langle \eta_2, 0.3, 0.1, 0.8\rangle\}),\ (e_2, \{\langle \eta_1, 0.2, 0.2, 0.7\rangle, \langle \eta_2, 0.5, 0.4, 0.6\rangle\})\}$$

is not an NSCS nor NSsCS, but it is NSgsCS in $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$. Therefore, f is an NSgscM but not an NScM or NSscM.

4.1 Characterization of Neutrosophic Soft Generalized Semi Continuous Mappings

We now characterize NSgscMs through the following equivalence theorem.

Theorem 4.5. *Let $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. For a neutrosophic soft mapping $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$, the following statements are equivalent:*

- *(i) f is an NSgscM,*
- *(ii) For each $x^e_{(\alpha,\beta,\gamma)} \in NS(\mathbb{U}_{\mathcal{P}})$ and for each NSOS $\widetilde{V}_{\mathcal{Q}}$ containing $f(x^e_{(\alpha,\beta,\gamma)})$, there exists an NSgsOS $\widetilde{U}_{\mathcal{P}}$ containing $x^e_{(\alpha,\beta,\gamma)}$ such that $f(\widetilde{U}_{\mathcal{P}}) \subset \widetilde{V}_{\mathcal{Q}}$,*
- *(iii) For each NS-subset $\widetilde{F}_{\mathcal{P}}$ of $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$, $f\Big(N\widetilde{s}cl_{gs}(\widetilde{F}_{\mathcal{P}})\Big) \subset N\widetilde{s}cl\Big(f(\widetilde{F}_{\mathcal{P}})\Big)$,*
- *(iv) For each NS-subset $\widetilde{G}_{\mathcal{Q}}$ of $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{Q})$, $N\widetilde{s}cl_{gs}\Big(f^{-1}(\widetilde{G}_{\mathcal{Q}})\Big) \subset f^{-1}\Big(N\widetilde{s}cl(\widetilde{G}_{\mathcal{Q}})\Big)$.*

Proof. (i)⇒(ii). Assume that f is an NSgscM. Let $x^e_{(\alpha,\beta,\gamma)} \in NS(\mathbb{U}_{\mathcal{P}})$ and let $\widetilde{V}_{\mathcal{Q}}$ be an NSOS such that $f(x^e_{(\alpha,\beta,\gamma)}) \in \widetilde{V}_{\mathcal{Q}}$. Set $\widetilde{W}_{\mathcal{Q}} := \widetilde{V}^c_{\mathcal{Q}}$. Then $\widetilde{W}_{\mathcal{Q}}$ is an NSCS, so $f^{-1}(\widetilde{W}_{\mathcal{Q}})$ is an NSgsCS. Hence

$$\widetilde{U}_{\mathcal{P}} := \big(f^{-1}(\widetilde{W}_{\mathcal{Q}})\big)^c$$

is an NSgsOS containing $x^e_{(\alpha,\beta,\gamma)}$ and $f(\widetilde{U}_{\mathcal{P}}) \subset \widetilde{V}_{\mathcal{Q}}$, as required.

(ii)⇒(iii). Let $\widetilde{F}_{\mathcal{P}} \subset NS(\mathbb{U}_{\mathcal{P}})$ be arbitrary, and take $y \in f\big(N\widetilde{s}cl_{gs}(\widetilde{F}_{\mathcal{P}})\big).n$ Then $y = f(x)$ for some $x \in N\widetilde{s}cl_{gs}(\widetilde{F}_{\mathcal{P}})$. If $y \notin N\widetilde{s}cl\big(f(\widetilde{F}_{\mathcal{P}})\big)$, there exists an NSOS $\widetilde{V}_{\mathcal{Q}}$ with $y \in \widetilde{V}_{\mathcal{Q}}$ and $\widetilde{V}_{\mathcal{Q}} \cap f(\widetilde{F}_{\mathcal{P}}) = \varnothing$. By (ii) there is an NSgsOS $\widetilde{U}_{\mathcal{P}}$ containing x with $f(\widetilde{U}_{\mathcal{P}}) \subset \widetilde{V}_{\mathcal{Q}}$. But $x \in N\widetilde{s}cl_{gs}(\widetilde{F}_{\mathcal{P}})$ implies $\widetilde{U}_{\mathcal{P}} \cap \widetilde{F}_{\mathcal{P}} \neq \varnothing$, hence $f(\widetilde{U}_{\mathcal{P}}) \cap f(\widetilde{F}_{\mathcal{P}}) \neq \varnothing$, contradicting $\widetilde{V}_{\mathcal{Q}} \cap f(\widetilde{F}_{\mathcal{P}}) = \varnothing$. Therefore $y \in N\widetilde{s}cl\big(f(\widetilde{F}_{\mathcal{P}})\big)$, and the inclusion

$$f\big(N\widetilde{s}cl_{gs}(\widetilde{F}_{\mathcal{P}})\big) \subset N\widetilde{s}cl\big(f(\widetilde{F}_{\mathcal{P}})\big)$$

holds.

(iii)⇒(iv). Let $\widetilde{G}_{\mathcal{Q}} \subset NS(\mathbb{W}_{\mathcal{Q}})$ be arbitrary and set $\widetilde{F}_{\mathcal{P}} := f^{-1}(\widetilde{G}_{\mathcal{Q}})$. By (iii),

$$f\Big(N\widetilde{s}cl_{gs}\big(f^{-1}(\widetilde{G}_{\mathcal{Q}})\big)\Big) \subset N\widetilde{s}cl\big(f(f^{-1}(\widetilde{G}_{\mathcal{Q}}))\big) \subset N\widetilde{s}cl(\widetilde{G}_{\mathcal{Q}}).$$

Applying the preimage argument ($A \subset f^{-1}(B)$ iff $f(A) \subset B$) yields

$$N\widetilde{s}cl_{gs}\big(f^{-1}(\widetilde{G}_{\mathcal{Q}})\big) \subset f^{-1}\big(N\widetilde{s}cl(\widetilde{G}_{\mathcal{Q}})\big),$$

which is (iv).

(iv)⇒(i). Let $\widetilde{H}_{\mathcal{Q}}$ be an arbitrary NSCS. Since $N\widetilde{s}cl(\widetilde{H}_{\mathcal{Q}}) = \widetilde{H}_{\mathcal{Q}}$, (iv) gives

$$N\widetilde{s}cl_{gs}\big(f^{-1}(\widetilde{H}_{\mathcal{Q}})\big) \subset f^{-1}(\widetilde{H}_{\mathcal{Q}}).$$

On the other hand, $f^{-1}(\widetilde{H}_{\mathcal{Q}}) \subset N\widetilde{s}cl_{gs}\big(f^{-1}(\widetilde{H}_{\mathcal{Q}})\big)$ always holds, so equality follows, showing that $f^{-1}(\widetilde{H}_{\mathcal{Q}})$ is an NSgsCS. Hence, f is an NSgscM.

Thus the proof is complete. □

Theorem 4.6. *Let $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ and $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{Q})$ be two NSTSs. If $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{Q}})$ is both NSCM and NSgscM, and if $\widetilde{G}_{\mathcal{Q}}$ is an NSgsCS in $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{Q})$, then the inverse image $f^{-1}(\widetilde{G}_{\mathcal{Q}})$ is an NSgsCS in $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$.*

Proof. Let $\widetilde{G}_{\mathcal{Q}}$ be an NSgsCS in $NS(\mathbb{W}_{\mathcal{Q}})$, and let $\widetilde{U}_{\mathcal{P}}$ be any NSOS in $NS(\mathbb{U}_{\mathcal{P}})$ such that

$$f^{-1}(\widetilde{G}_{\mathcal{Q}}) \subset \widetilde{U}_{\mathcal{P}}.$$

Since f is an NSCM, by Lemma 2.23, there exists an NSOS $\widetilde{V}_{\mathcal{Q}}$ in $NS(\mathbb{W}_{\mathcal{Q}})$ satisfying

$$\widetilde{G}_{\mathcal{Q}} \subset \widetilde{V}_{\mathcal{Q}} \quad \text{and} \quad f^{-1}(\widetilde{V}_{\mathcal{Q}}) \subset \widetilde{U}_{\mathcal{P}}.$$

Because $\widetilde{G}_{\mathcal{Q}}$ is an NSgsCS, by Definition 3.14, we have

$$N\widetilde{s}cl_{gs}(\widetilde{G}_{\mathcal{Q}}) \subset \widetilde{V}_{\mathcal{Q}}.$$

Taking the inverse image under f, we obtain

$$f^{-1}(N\widetilde{s}cl_{gs}(\widetilde{G}_{\mathcal{Q}})) \subset f^{-1}(\widetilde{V}_{\mathcal{Q}}) \subset \widetilde{U}_{\mathcal{P}}.$$

Since f is an NSgscM, $f^{-1}(N\widetilde{s}cl_{gs}(\widetilde{G}_{\mathcal{Q}}))$ is an NSgsCS in $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$. Moreover, by monotonicity of the generalized semi-closure operator,

$$N\widetilde{s}cl_{gs}(f^{-1}(\widetilde{G}_{\mathcal{Q}})) \subset N\widetilde{s}cl_{gs}(f^{-1}(N\widetilde{s}cl_{gs}(\widetilde{G}_{\mathcal{Q}}))) = f^{-1}(N\widetilde{s}cl_{gs}(\widetilde{G}_{\mathcal{Q}})) \subset \widetilde{U}_{\mathcal{P}}.$$

Thus, the generalized semi-closure of the inverse image is contained in any NSOS $\widetilde{U}_{\mathcal{P}}$ containing $f^{-1}(\widetilde{G}_{\mathcal{Q}})$, which, by Definition 3.14, confirms that $f^{-1}(\widetilde{G}_{\mathcal{Q}})$ is an NSgsCS in $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$. □ □

We have the following decompositions.

Theorem 4.7. *Let $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$, $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$, and $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{P}'')$ be three NSTSs. If $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{V}_{\mathcal{P}'})$ is both an NSgscM and an NSCM, and $g : NS(\mathbb{V}_{\mathcal{P}'}) \to NS(\mathbb{W}_{\mathcal{P}''})$ is an NSgcM, then $g \circ f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{P}''})$ is an NSgscM.*

Proof. Let $\mathcal{O}_{\mathcal{P}''}$ be an arbitrary NSgsOS set in $(\mathbb{W}, \widetilde{\tau}_{\mathbb{W}}^{NS}, \mathcal{P}'')$. Since g is an NSgcM, $g^{-1}(\mathcal{O}_{\mathcal{P}''})$ is an NSgsCS set in $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$.

By Theorem 4.6, the preimage of $g^{-1}(\mathcal{O}_{\mathcal{P}''})$ under f is an NSgsCS set in $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ because f is both an NSCM and an NSgscM. That is,

$$(f^{-1} \circ g^{-1})(\mathcal{O}_{\mathcal{P}''}) = (g \circ f)^{-1}(\mathcal{O}_{\mathcal{P}''}) \in \widetilde{\tau}_{\mathbb{U}}^{NS}.$$

Since $\mathcal{O}_{\mathcal{P}''}$ was arbitrary, $(g \circ f)^{-1}$ maps NSgsOS sets to NSgsCS sets, hence $g \circ f$ is an NSgscM. □

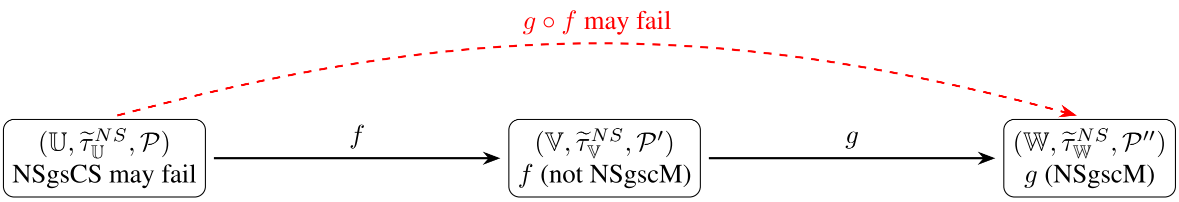
Remark 4.8. It is important to note that the composition of two NSgscMs is not guaranteed to be NSgscM if one of the mappings fails to preserve NSgsCS sets. Specifically, if f is not an NSgscM but g is, then $g \circ f$ may fail to be NSgscM.

Illustrative Scenario: Let $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{V}_{\mathcal{P}'})$ be such that there exists an NSgsCS set $\mathcal{B}_{\mathcal{P}'}$ with $f^{-1}(\mathcal{B}_{\mathcal{P}'})$ not NSgsCS. Let $g : NS(\mathbb{V}_{\mathcal{P}'}) \to NS(\mathbb{W}_{\mathcal{P}''})$ be NSgscM. Then

$$(g \circ f)^{-1}(\mathcal{K}_{\mathcal{P}''}) = f^{-1}\big(g^{-1}(\mathcal{K}_{\mathcal{P}''})\big)$$

may fail to be NSgsCS for some $\mathcal{K}_{\mathcal{P}''} \in \widetilde{\tau}_{\mathbb{W}}^{NS}$. Thus, $g \circ f$ is not an NSgscM. This highlights the necessity of additional closure-preserving properties for compositionality.

Conceptual Illustration:



Theorem 4.9. *Let* $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ *and* $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$ *be two NSTSs. Suppose* $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{V}_{\mathcal{P}'})$ *is both an NSOM and an NSgscM. Then for every NSsOS* $\mathcal{B}_{\mathcal{P}'}$ *in* $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$*,* $f^{-1}(\mathcal{B}_{\mathcal{P}'})$ *is an NSgsOS in* $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$*.*

Proof. Let $\mathcal{H}_{\mathcal{P}'}$ be an arbitrary NSsOS in $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$. By the definition of NSsOS, there exists an NSOS $\mathcal{V}_{\mathcal{P}'}$ such that

$$\mathcal{V}_{\mathcal{P}'} \subset \mathcal{H}_{\mathcal{P}'} \subset N\widetilde{s}cl(\mathcal{V}_{\mathcal{P}'}).$$

Since f is an NSOM, we have

$$f^{-1}(\mathcal{V}_{\mathcal{P}'}) \subset f^{-1}(\mathcal{H}_{\mathcal{P}'}) \subset f^{-1}(N\widetilde{s}cl(\mathcal{V}_{\mathcal{P}'})).$$

Furthermore, because f is an NSgscM and $\mathcal{V}_{\mathcal{P}'}$ is an NSOS, $f^{-1}(\mathcal{V}_{\mathcal{P}'})$ is NSgsOS in $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$.

Hence, by the standard property of NSsOS sets (see Theorem 4.4.5 in[27]), it follows that

$$f^{-1}(\mathcal{H}_{\mathcal{P}'}) \in NSgsOS(\mathbb{U}_{\mathcal{P}}).$$

□

Corollary 4.10. *Let* $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$ *and* $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$ *be two NSTSs. If* $f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{V}_{\mathcal{P}'})$ *is both NSOM and NSgscM, then for every NSsCS* $\mathcal{B}_{\mathcal{P}'}$ *in* $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$*,* $f^{-1}(\mathcal{B}_{\mathcal{P}'})$ *is an NSgsCS in* $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$*.*

Proof. This statement follows directly from Theorem 4.9 and Lemma 2.18, since the preimage of an NSsCS under an NSOM that is also NSgscM is NSgsCS. □

Theorem 4.11. *Let* $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$, $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$, *and* $(\mathbb{W}, \widetilde{\tau}_{3}^{NS}, \mathcal{P}'')$ *be three NSTSs. Let*

$$f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{V}_{\mathcal{P}'}), \quad g : NS(\mathbb{V}_{\mathcal{P}'}) \to NS(\mathbb{W}_{\mathcal{P}''})$$

be neutrosophic soft mappings, and assume that $g \circ f : NS(\mathbb{U}_{\mathcal{P}}) \to NS(\mathbb{W}_{\mathcal{P}''})$ *is an NSgsCM. Then:*

(i) If f *is an NSgscM and NS-surjective, then* g *is an NSgsCM.*

(ii) If g *is an NSOM, NSgscM, and NS-injective, then* f *is an NSgsCM.*

Proof. **(i)** Let $\mathcal{H}_{\mathcal{P}'}$ be an arbitrary NSCS in $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$. Since f is an NSgscM, we have

$$f^{-1}(\mathcal{H}_{\mathcal{P}'}) \in NSCS(\mathbb{U}_{\mathcal{P}}).$$

Because $g \circ f$ is an NSgsCM and f is an NS-surjective, we obtain

$$(g \circ f)(f^{-1}(\mathcal{H}_{\mathcal{P}'})) = g(f(f^{-1}(\mathcal{H}_{\mathcal{P}'}))) = g(\mathcal{H}_{\mathcal{P}'}) \in NSgsCS(\mathbb{W}_{\mathcal{P}''}),$$

which shows that g maps every NSCS to an NSgsCS. Hence, g is an NSgsCM.

(ii) Let $\mathcal{G}_{\mathcal{P}}$ be an arbitrary NSCS in $(\mathbb{U}, \widetilde{\tau}_{\mathbb{U}}^{NS}, \mathcal{P})$. Since $g \circ f$ is an NSgsCM, it follows that

$$(g \circ f)(\mathcal{G}_{\mathcal{P}}) \in NSgsCS(\mathbb{W}_{\mathcal{P}''}).$$

Using the NS-injectivity of g, we have

$$g^{-1}((g \circ f)(\mathcal{G}_{\mathcal{P}})) = f(\mathcal{G}_{\mathcal{P}}).$$

From Corollary 4.10, since g is an NSOM and NSgscM, it follows that $f(\mathcal{G}_{\mathcal{P}})$ is an NSgsCS in $(\mathbb{V}, \widetilde{\tau}_{\mathbb{V}}^{NS}, \mathcal{P}')$. Thus, f is an NSgsCM. □

5 Conclusion

In this study, we systematically introduced and investigated the concept of neutrosophic soft generalized semi-mappings within neutrosophic soft topological spaces. Related structures, including neutrosophic soft generalized semi closed and semi open sets, semi-continuity, and semi-regularity, were also examined. We established a clear hierarchy among various classes of mappings and demonstrated their interrelations through rigorous theorems and illustrative examples.

The results indicate that neutrosophic soft generalized semi closed sets provide a flexible generalization of classical neutrosophic soft mappings by relaxing closure conditions, thus enhancing their applicability in uncertain and indeterminate environments. The introduction of neutrosophic soft generalized semi-closure and semi-interior operators further increases the expressive power of NST, allowing for more nuanced modeling of uncertainty.

These contributions extend both classical and neutrosophic soft topological theories and provide a robust theoretical framework for modeling uncertainty. This framework holds promising potential for practical applications in fields such as uncertain decision-making, pattern recognition, and intelligent system design.

Based on the results of this study, several research directions are suggested. First, the relationships between neutrosophic soft generalized semi-sets and other separation axioms can be explored to develop a more comprehensive topological framework. Second, practical models could be constructed in applications such as multi-criteria decision making, medical diagnosis, and information systems, leveraging the flexibility of generalized semi-structures. Finally, new concepts such as compactness, connectedness, and continuity could be defined within this generalized semi-framework, providing further contributions to both theoretical and applied research.

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