Neutrosophic Signed Domination Function of Graphs

Duraisamy Kumar[1,*], **Florentin Smarandache**[2]
[1]Department of Mathematics, SRM TRP Engineering College, Tiruchirappalli-621105, Tamil Nadu, India
[2]University of New Mexico, Gallup Campus, NM 87301, USA
Emails: dkumarcnc@gmail.com; fsmarandache@gmail.com

Abstract

This paper introduces the novel concept of a Neutrosophic Signed Domination Function (NSDF) of graphs, generalizing classical domination by assigning each vertex a triple-valued influences (truth, indeterminacy, falsity) from $\{-1, 0, 1\}$. We define the Neutrosophic Signed Domination Number $\gamma_{ns}(G)$ as the optimal weighted sum under neighborhood constraints ensuring net positive influence. Fundamental properties and sharp bounds for general graphs are established. Exact values for $\gamma_{ns}(G)$ are determined for paths and cycles. This work bridges neutrosophic logic with domination theory, enabling sophisticated modeling of complex networks with uncertainty.

Keywords: Graph domination; Neutrosophic graphs; Neutrosophic domination; Signed domination; Neutrosophic signed domination

1 Introduction

Let G be a graph with m edges and n vertices. The open neighborhood of a vertex consists of all adjacent vertices, while its closed neighborhood contains both the vertex itself and its adjacent vertices. Basic notation and results can be found in.[7]

The signed domination concept was first introduced by Dunbar et al.[10] in 1995. For a graph $G = (V, E)$, a *signed dominating function* is defined as a vertex labeling $f : V \to \{-1, +1\}$ satisfying the condition that for every vertex $v \in V$, the sum of labels in its closed neighborhood satisfies:

$$f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1$$

The *signed domination number* of G, denoted $\gamma_s(G)$, represents the minimum possible weight $\sum_{v \in V} f(v)$ among all such valid functions. This fundamental concept has been extensively studied by various researchers, including,[8],[9],[11],[12],[14],[15],[16] and,[25] who have contributed significant results and extensions to the theory of signed domination.

Models of classical graph theory frequently function in a binary environment, where a vertex is either dominated (1) or not dominated (0). This was enhanced by signed domination, which is more suited for networks with competing forces (such as allies and adversaries) by adding positive (+1) and negative (-1) impacts. However, these models still fall short in handling the pervasive uncertainty and indeterminacy of real-world systems. This is where the motivation for a neutrosophic model comes from:

As a generalization of fuzzy logic, neutrosophic logic recognizes that there are three separate parts to any proposition: falsity (F), indeterminacy (I), and truth (T). This perspective of reality is more sophisticated.

In 1999, the foundational concepts were established by Florentin Smarandache introduced neutrosophic logic.[20] In 2018, Mohammad Akram studied neutrosophic graphs.[5] Numerous writers have examined the neutrosophic concept, including.[1,2,6,17,19]

Neutrosophic graph theory represents an advanced framework that generalizes both fuzzy graphs and intuitionistic fuzzy graphs. This model was developed to capture three fundamental aspects of network relationships:

- Truth membership (T) - representing confirmed or certain connections
- Indeterminacy membership (I) - capturing uncertain or ambiguous relations
- Falsity membership (F) - accounting for confirmed non-connections or contradictions

Given an underlying graph $G = (V, E)$, we define its *neutrosophic extension* $\mathcal{G} = (A, B)$ as follows:

- The vertex set is equipped with a neutrosophic structure $A = (T_A, I_A, F_A)$, where:
  - $T_A : V \to [0, 1]$ quantifies the degree of certainty for each vertex
  - $I_A : V \to [0, 1]$ measures the indeterminacy associated with each vertex
  - $F_A : V \to [0, 1]$ indicates the degree of falsity for each vertex
- The edge relations are characterized by $B = (T_B, I_B, F_B)$ with:
  - $T_B : E \to [0, 1]$ constrained by $T_B(uv) \leq \min(T_A(u), T_A(v))$
  - $I_B : E \to [0, 1]$ bounded by $I_B(uv) \leq \min(I_A(u), I_A(v))$
  - $F_B : E \to [0, 1]$ limited by $F_B(uv) \geq \max(F_A(u), F_A(v))$

Recent years have seen significant growth in neutrosophic graph theory across several important fields.[22] developed the basic theory by presenting and examining regularity in plithogenic graphs, which are an extension of neutrosophic graphs. Contributing to spectral theory,[3] extended the notion to directed networks by defining the energy of a neutrosophic digraph. A significant emphasis has been on practical applications, with[21] creating complex neutrosophic graphs for multi-attribute decision-making problems like supplier selection and[4] using single-valued neutrosophic graphs for influential node detection in social network analysis. In order to tackle com

In NSDF, the degrees of truth, indeterminacy, and falsity are independent. A node can have a high truth value and a high indeterminacy value simultaneously (e.g., a highly positive but very unpredictable actor). This is a more powerful representation than probabilistic or fuzzy models where membership degrees are often dependent.

Application in Social Network Analysis and Opinion Dynamics

Modeling the spread of opinions or influence in a social network (e.g., regarding a political issue, a new product, or a public health policy). Vertices (V): Individuals or groups in the network. Edges (E): Social connections (friendship, fol $f_T(v)$ Degree of positive support/agreement (+1 = strong supporter, 0 = neutral, -1 = active opposition). $f_I(v)$ Degree of uncertainty or indecision. $f_F(v)$ Degree of negative stance/disagreement.

Application in Cybersecurity and Trust Networks

Securing a distributed network (e.g., sensor network, peer-to-peer network) where nodes can be trustworthy, malicious, or compromised (uncertain). Vertices (V): Devices or agents in the network. Edges (E) : Communication links. $f_T(v)$: Trustworthiness score (+1 = verified secure, -1 = known malicious). $f_I(v)$: Uncertainty about the node's state. $f_F(v)$: Level of malicious activity or distrust

The domination conditions can be seen as a security protocol. For every node, the collective trustworthiness of its neighbors must meet a minimum threshold ($\sum f_T \geq 1$), and the collective negative influence must be bounded ($\sum f_F \leq 1$). This ensures that no node is isolated in a cluster of malicious or uncertain actors.

Application in Economic and Financial Networks

Analyzing systemic risk in a financial network where institutions (banks, funds) are interconnected by liabilities. Vertices (V): Financial institutions. Edges (E): Financial exposure (e.g., loans, derivatives). $f_T(v)$: Financial health/solvency (+1 = very healthy, -1 = insolvent).
$f_I(v)$: Uncertainty or opacity of the institution's true state. $f_F(v)$: Level of financial distress.

The condition $\sum_{u \in N[v]} f_T(u) \geq 1$ can be interpreted as a requirement for **local stability**: the combined health of an institution and its direct partners must be above a critical threshold to prevent a local failure from cascading.

This work presents a novel Neutrosophic Signed Domination Function(NSDF) for graphs, defined as a triple mapping $f = (f_T, f_I, f_F) : V \to [-1, 0, 1]^3$ where each vertex $v \in V$ is assigned.

- A truth component $f_T(v) \in [-1, 0, 1]$ representing positive influence
- An indeterminacy component $f_I(v) \in [-1, 0, 1]$ capturing neutral or uncertain influence
- A falsity component $f_F(v) \in [-1, 0, 1]$ indicating negative influence

The function must satisfy the following neighborhood conditions for every vertex $v \in V$

$$\sum_{u \in N[v]} f_T(u) \geq 1 \quad \text{(Truth domination)}$$
$$\sum_{u \in N[v]} f_I(u) \geq 0 \quad \text{(Indeterminacy threshold)}$$
$$\sum_{u \in N[v]} f_F(u) \leq 1 \quad \text{(Falsity constraint)}$$

The Neutrosophic Signed Domination Number(NSDN) $\gamma_{ns}(G)$ is defined as the minimum weighted sum is

$$\gamma_{ns}(G) = \min_f \left(w_T \sum_{v \in V} f_T(v) + w_I \sum_{v \in V} f_I(v) + w_F \sum_{v \in V} f_F(v) \right)$$

where $w_T, w_I, w_F \geq 0$ are importance weights for each influence type.

NSDF extend classical domination by incorporating truth (T), indeterminacy (I), and falsity (F) states to better model real-world uncertainty. They overcome limitations of binary domination in handling partial information and conflicting influences. NSDF provides a flexible framework for analyzing networks where nodes exhibit neutral or ambiguous behaviors. This generalization enables more robust applications in social networks, cybersecurity, and decision-making systems with incomplete information.

In section 2 included structural characterizations, sharp bounds for general graphs, and basic features of NSDF of graphs. We determine the NSDN for pathways and cycles in section 3. The work opens new research directions in neutrosophic graph theory and its applications to complex network analysis.

2 NSDF of graphs

The basic properties of NSDF of graphs were discussed in this section, and we discovered structural characterizations and sharp bounds for general graphs.

Lemma 2.1. *If G is a graph containing a vertex of degree k. Then $\gamma_{ns}(G) > (2 + k - n)(w_T + w_I - w_F)$*

Proof. Consider a vertex $v \in V$ with $\deg(v) = k$. Let $f = (f_T, f_I, f_F)$ be a NSDF on G.For the closed neighborhood $N[v]$, we must have $\sum_{u\in N[v]} f_T(u) \geq 1, \quad \sum_{u\in N[v]} f_I(u) \geq 0, \quad \sum_{u\in N[v]} f_F(u) \leq 1$. The minimal weight for f will now achieved if $f_T(u) = -1$, $f_I(u) = -1$, $f_F(u) = 1$ for $u \in N[v]$ and $f_T(u) = 1$, $f_I(u) = 1$, $f_F(u) = -1$ for $u \notin N[v]$. Thus

$$\begin{aligned}\gamma_{ns}(G) &= \sum_{u\in V}[w_T f_T(u) + w_I f_I(u) + w_F f_F(u)] \\ &= (k+1)(-w_T - w_I + w_F) + (n-k-1)(w_T + w_I - w_F) \\ &= (2+k-n)(w_T + w_I - w_F)\end{aligned}$$

Since any valid NSDF must satisfy stricter constraints than this extremal case, we obtain the strict inequality. □

Theorem 2.2. *If f denote any NSDF of the graph $G = (V, E)$ with $f = (f_T, f_I, f_F)$ and let $P_T = \{v \in V \mid f_T(v) > 0\}$, $M_T = \{v \in V \mid f_T(v) \leq 0\}$, $p = |P_T|$, $n = |V|$ and $o(G)$ = number of odd-degree vertices in G. Then $\sum_{v\in P_T} d_G(v) \geq \sum_{v\in M_T} d_G(v) + 2(n-p) + o(G)$, $\sum_{v\in P_T} d_{G[P_T]}(v) \geq \sum_{v\in P_T} \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ where $d_G(v)$ is the degree of v in G and $G[P_T]$ is the subgraph induced by P_T.*

Proof. For any NSDF $f = (f_T, f_I, f_F)$, consider the truth component f_T as a NSDF. For $v \in P_T$.

$$\begin{aligned}\sum_{u\in N[v]} f_T(u) \geq 1 &= f_T(v) + \sum_{u\in N(v)} f_T(u) \geq 1 \\ &= 1 + \sum_{u\in N(v)} f_T(u) \geq 1 \quad \text{(since } f_T(v) > 0) \\ &= \sum_{u\in N(v)} f_T(u) \geq 0\end{aligned}$$

For $v \in M_T$:

$$\begin{aligned}\sum_{u\in N[v]} f_T(u) \geq 1 &= f_T(v) + \sum_{u\in N(v)} f_T(u) \geq 1 \\ &= 0 + \sum_{u\in N(v)} f_T(u) \geq 1 \quad \text{(since } f_T(v) \leq 0) \\ &= w \sum_{u\in N(v)} f_T(u) \geq 1\end{aligned}$$

Following the same edge-counting argument as in given bellow and we obtain

$$\sum_{v\in P_T} d_G(v) \geq \sum_{v\in M_T} d_G(v) + 2(n-p) + o(G)$$

For $v \in P_T$, the condition $\sum_{u\in N[v]} f_T(u) \geq 1$ implies that v must have sufficient positive influence in its neighborhood. Considering only the subgraph $G[P_T]$:

$$d_{G[P_T]}(v) = |\{u \in P_T \mid uv \in E\}| \geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor$$

Summing over all $v \in P_T$ gives the second inequality. □

Theorem 2.3. *Let G be a graph of order n, size m, maximum degree Δ, and minimum degree δ. Let $\gamma_{ns}(G)$ be the NSDFN of G with weights $w_T, w_I, w_F \geq 0$. Then, the following bounds hold:*

1. $\gamma_{ns}(G) \leq \frac{(\Delta^*+2-\delta)n+2o(G)}{\Delta^*+2+\delta}$

2. $\gamma_{ns}(G) \leq \frac{2m+2n+o(G)}{\delta+1} - n$

3. $\gamma_{ns}(G) \leq n - \frac{2m-o(G)}{\Delta^*+1}$

4. $\gamma_{ns}(G) \leq 2\left\lfloor \frac{-\Delta^*+\sqrt{\Delta^{*2}+8(\Delta^*+2)n+8o(G)}}{4} \right\rfloor - n$

5. $\gamma_{ns}(G) \leq 2\left\lfloor \frac{1+\sqrt{1+8(m+n)+4o(G)}}{4} \right\rfloor - n$. *Where* $\Delta^* = \max_{v\in V} \sum_{u\in N[v]} |f_T(u)|$ $o(G) = o$.

Proof. Let $f = (f_T, f_I, f_F)$ be an optimal NSDF for G with $\gamma_{ns}(G) = \sum_{v \in V} (w_T f_T(v) + w_I f_I(v) + w_F f_F(v))$. Let $P = \{v \in V \mid f_T(v) > 0\}$ (truth-dominated vertices), $M = \{v \in V \mid f_F(v) < 0\}$ (falsity-dominated vertices), $p = |P|$, $q = |M|$.

Proof of Bound (1): Degree-Based Bound
From the *truth constraint* $\sum_{u \in N[v]} f_T(u) \geq 1$, summing over all $v \in V$:

$$\sum_{v \in V} \sum_{u \in N[v]} f_T(u) \geq n.$$

By double-counting:

$$\sum_{v \in P} d_G(v) + 2p \geq \sum_{v \in M} d_G(v) + 2(n-p) + o(G).$$

Using $d_G(v) \leq \Delta^*$ for $v \in M$ and $d_G(v) \geq \delta$:

$$p\Delta^* + 2p \geq (n-p)\delta + 2(n-p) + o(G).$$

Solving for p:

$$p \geq \frac{(\delta+2)n + o(G)}{\Delta^* + \delta + 2}.$$

Since $\gamma_{ns}(G) \leq 2p - n$, we obtain:

$$\gamma_{ns}(G) \leq \frac{(\Delta^* + 2 - \delta)n + 2o(G)}{\Delta^* + 2 + \delta}.$$

.

Proof of Bound (2): Size-Based Bound
From the *truth constraint* and double-counting:

$$2\sum_{v \in P} d_G(v) \leq 2m + 2(n-p) + o(G).$$

Using $\sum_{v \in P} d_G(v) \geq p\delta$:

$$2p\delta \leq 2m + 2(n-p) + o(G).$$

Solving for p:

$$p \leq \frac{2m + 2n + o(G)}{2(\delta+1)}.$$

Thus:

$$\gamma_{ns}(G) \leq \frac{2m + 2n + o(G)}{\delta + 1} - n.$$

Proof of Bound (3): Max-Degree Bound
From the *falsity constraint* $\sum_{u \in N[v]} f_F(u) \leq 1$, summing over $v \in V$:

$$\sum_{v \in V} \sum_{u \in N[v]} f_F(u) \leq n.$$

Double-counting gives:

$$\sum_{v \in M} d_G(v) + 2q \leq \sum_{v \in P} d_G(v) + 2(n-q) - o(G).$$

Using $d_G(v) \leq \Delta^*$ for $v \in M$:

$$2m \leq 2\Delta^* q + 2(n-q) - o(G).$$

Solving for q:

$$q \geq \frac{2m - 2n + o(G)}{2(\Delta^* - 1)}.$$

Since $\gamma_{ns}(G) \geq n - 2q$, we obtain:

$$\gamma_{ns}(G) \leq n - \frac{2m - o(G)}{\Delta^* + 1}.$$

Proof of Bound (4): Quadratic Truth-Dominated Bound
Consider the subgraph $G[P]$ induced by truth-dominated vertices P. From Lemma 1, we have:

$$\sum_{v \in P} d_{G[P]}(v) \leq \sum_{v \in P} \left\lfloor \frac{d_G(v)}{2} \right\rfloor.$$

Since $G[P]$ is simple, $G[P]$ has at most $\frac{p(p-1)}{2}$ edges, thus:

$$\sum_{v \in P} d_{G[P]}(v) \leq p(p-1).$$

Combining these:

$$p(p-1) \leq \sum_{v \in P} \left\lfloor \frac{d_G(v)}{2} \right\rfloor \leq \frac{1}{2} \sum_{v \in P} d_G(v).$$

From the truth constraint (as in Bound (1)):

$$\sum_{v \in P} d_G(v) \leq \sum_{v \in M} d_G(v) + 2(n-p) + o(G) \leq \Delta^*(n-p) + 2(n-p) + o(G).$$

Thus:

$$2p(p-1) \leq \Delta^*(n-p) + 2(n-p) + o(G).$$

Rearranging gives the quadratic inequality:

$$2p^2 + (\Delta^* + 2)p - (\Delta^* + 2)n - o(G) \leq 0.$$

Solving for p using the quadratic formula:

$$p \leq \left\lfloor \frac{-(\Delta^* + 2) + \sqrt{(\Delta^* + 2)^2 + 8(\Delta^* + 2)n + 8o(G)}}{4} \right\rfloor.$$

Substituting $\gamma_{ns}(G) = 2p - n$ yields the bound:

$$\gamma_{ns}(G) \leq 2 \left\lfloor \frac{-\Delta^* + \sqrt{\Delta^{*2} + 8(\Delta^* + 2)n + 8o(G)}}{4} \right\rfloor - n.$$

Proof of Bound (5): General Quadratic Bound
From Bound (2), we have:

$$\sum_{v \in P} d_G(v) \leq m + (n-p) + \frac{o(G)}{2}.$$

From the edge-counting argument in Bound (4):

$$2p(p-1) \leq \sum_{v \in P} d_G(v).$$

Combining these:

$$2p(p-1) \leq m + (n-p) + \frac{o(G)}{2}.$$

Rearranging gives:

$$2p^2 - p - m - n - \frac{o(G)}{2} \leq 0.$$

Solving the quadratic inequality for p:

$$p \leq \left\lfloor \frac{1 + \sqrt{1 + 8(m+n) + 4o(G)}}{4} \right\rfloor.$$

Thus:

$$\gamma_{ns}(G) \leq 2 \left\lfloor \frac{1 + \sqrt{1 + 8(m+n) + 4o(G)}}{4} \right\rfloor - n.$$

Hence complete the proof.

□

Theorem 2.4. *For any graph G of order n, the NSDN is $\gamma_{ns}(G) \geq 2 \left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor - n$ and this bound is sharp.*

Proof. Let $f = (f_T, f_I, f_F)$ be an optimal NSDF for G, $A = \{v \in V(G) \mid f_T(v) > 0\}$ (truth-dominated vertices), and $B = V(G) \setminus A$ (remaining vertices). Let $|A| = t$, then $|B| = n - t$ and $\gamma_{ns}(G) = \sum_{v \in A} f_T(v) - \sum_{v \in B} |f_F(v)| \geq 2t - n$ (since $f_T(v) \leq 1$, $f_F(v) \geq -1$).

Step 1: Neighborhood Constraints For every vertex $u \in B$, the truth constraint requires:

$$\sum_{w \in N[u]} f_T(w) \geq 1 \implies |N[u] \cap A| \geq 2.$$

Thus, the number of edges between A and B satisfies:

$$|E(A, B)| \geq 2(n - t).$$

This implies at least one vertex $v \in A$ has degree:

$$d_B(v) \geq \left\lceil \frac{2(n-t)}{t} \right\rceil.$$

Consequently, its closed neighborhood satisfies:

$$|N[v] \cap A| \geq 1 + \left\lceil \frac{2(n-t)}{t} \right\rceil.$$

Step 2: Solving the Inequality Since $|A| = t \geq |N[v] \cap A|$, we derive:

$$t \geq 1 + \frac{2(n-t)}{t}.$$

Rearranging yields the quadratic inequality:

$$t^2 + t - 2n \geq 0.$$

Solving for t:

$$t \geq \frac{-1 + \sqrt{1 + 8n}}{2}.$$

As t must be an integer:

$$t \geq \left\lfloor \frac{-1 + \sqrt{1 + 8n}}{2} \right\rfloor.$$

Substituting into $\gamma_{ns}(G) \geq 2t - n$ gives the bound.

Sharpness Construction: For any $n \geq 1$, we exhibit a graph achieving equality:

- **Case** $n = 1, 2$: Take $G = K_n$ with $f_T(v) = 1$ for all v.

- **Case** $n \geq 3$: Let $t = \left\lfloor \frac{-1+\sqrt{1+8n}}{2} \right\rfloor$. Construct G by:
  1. Starting with $K_t \cup K_{n-t}$,
  2. Connecting each vertex in K_{n-t} to exactly 2 vertices in K_t,
  3. Ensuring no pair in K_t shares more than one common neighbor from K_{n-t}.

  Define f by:

$$f_T(v) = \begin{cases} 1 & \text{if } v \in V(K_t), \\ -1 & \text{if } v \in V(K_{n-t}). \end{cases}$$

  Then f is an NSDF with $\gamma_{ns}(G) = 2t - n$, matching the bound.

□

Theorem 2.5. *For any graph G of order n and size m, the NSDN satisfies $\gamma_{ns}(G) \geq n - \frac{2}{3}m$.*

Proof. Let $f = (f_T, f_I, f_F)$ be an optimal NSDF for G, $A = \{v \in V \mid f_T(v) > 0\}$ (truth-dominated vertices), and $B = V \setminus A$ (remaining vertices). Let $|A| = t$, then $|B| = n - t$ and $\gamma_{ns}(G) \geq 2t - n$ (since $f_T(v) \leq 1$ and $f_F(v) \geq -1$).

Step 1: Edge Counting For every vertex $u \in B$, the truth constraint requires:

$$\sum_{w \in N[u]} f_T(w) \geq 1 \implies |N[u] \cap A| \geq 2.$$

Thus, the number of edges between A and B satisfies:

$$|E(A, B)| \geq 2(n - t).$$

Step 2: Degree Analysis in A For any $v \in A$, the domination constraint implies:

$$\sum_{u \in N[v]} f_T(u) \geq 1 \implies d_A(v) \geq d_B(v),$$

where $d_A(v)$ and $d_B(v)$ denote the number of neighbors of v in A and B respectively. Therefore:

$$2|E(G[A])| = \sum_{v \in A} d_A(v) \geq \sum_{v \in A} d_B(v) = |E(A, B)| \geq 2(n - t).$$

Step 3: Total Edge Bound The total number of edges satisfies:

$$m \geq |E(G[A])| + |E(A, B)| \geq (n - t) + 2(n - t) = 3(n - t).$$

Solving for t:

$$t \geq n - \frac{m}{3}.$$

Substituting into $\gamma_{ns}(G) \geq 2t - n$ yields:

$$\gamma_{ns}(G) \geq 2\left(n - \frac{m}{3}\right) - n = n - \frac{2}{3}m.$$

□

Theorem 2.6. *For any graph G of order n with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$, the NSDFN satisfies:*

$$\gamma_{ns}(G) \geq \frac{\delta(G) + 2 - \Delta(G)}{\delta(G) + 2 + \Delta(G)} n.$$

Proof. Let $f = (f_T, f_I, f_F)$ be an optimal NSDF for G. Define $A = \{v \in V(G) \mid f_T(v) > 0\}$ (truth-dominated vertices), $B = V(G) \setminus A$ (remaining vertices). Let $|A| = t$, then $|B| = n - t$ and $\gamma_{ns}(G) = 2t - n$.

Step 1: Neighborhood Constraints For any vertex $v \in V(G)$, the NSDF constraints require:

$$\sum_{u \in N[v]} f_T(u) \geq 1 \implies |N[v] \cap A| \geq \left\lceil \frac{\delta(G)}{2} \right\rceil + 1.$$

This follows because:

$$\begin{aligned} &|N[v] \cap A| - |N[v] \cap B| \geq 1, \\ &|N[v] \cap A| + |N[v] \cap B| = d(v) + 1 \geq \delta(G) + 1. \end{aligned}$$

Step 2: Edge Counting For vertices in B, we have:

$$\sum_{v \in B} |N[v] \cap A| = |E(A, B)| + |B| \geq \left(\left\lfloor \frac{\delta(G)}{2} \right\rfloor + 1 \right) (n - t).$$

Thus, there exists $u \in A$ with at least $\left\lceil \frac{(\lfloor \delta(G)/2 \rfloor + 1)(n-t)}{t} \right\rceil$ neighbors in B.

Step 3: Degree Bound The vertex u must satisfy:

$$\Delta(G) \geq d(u) \geq 2 \left\lceil \frac{(\delta(G) + 2)(n - t)}{2t} \right\rceil \geq \frac{(\delta(G) + 2)(n - t)}{t}.$$

Solving for t:

$$t \geq \frac{(\delta(G) + 2)n}{\delta(G) + 2 + \Delta(G)}.$$

Substituting into $\gamma_{ns}(G) = 2t - n$:

$$\gamma_{ns}(G) \geq \frac{(\delta(G) + 2 - \Delta(G))n}{\delta(G) + 2 + \Delta(G)}.$$

Hence complete the proof. □

Theorem 2.7. *A graph G has $\gamma_{ns}(G) = n(w_T - w_F)$ if and only if every vertex $v \in V$ is either isolated, an endvertex or adjacent to an end vertex.*

Proof. Suppose $\gamma_{ns}(G) = n(w_T - w_F)$. We show that every vertex in G must satisfy one of the three conditions.If a vertex v has degree ≥ 2 and none of its neighbors are endvertices, then closed neighborhood $N[v]$ contains v and at least two other vertices. To satisfy $\sum_{u \in N[v]} f_T(u) \geq 1$, at least one vertex in $N[v]$ must have $f_T(u) > -1$. If we assign $f_T(v) = -1$ and $f_T(u) = 1$ for all $u \neq v$, the sum condition holds, but the total contribution to $\gamma_{ns}(G)$ would be less than $n(w_T - w_F)$, contradicting minimality. Hence, no such vertex can exist, meaning every non-endvertex must be adjacent to an endvertex.

Assume every vertex in G is either isolated, an endvertex, or adjacent to an endvertex. We construct an NSDF $f = (f_T, f_I, f_F)$ as follows $f_T(v) = 1, \quad f_I(v) = 0, \quad f_F(v) = -1$ for all $v \in V$. From the above labeling any $\sum_{u \in N[v]} f_T(u) \geq 1$, for all $v \in V$. holds since $f_T(u) = 1$ for all u $\sum_{u \in N[v]} f_I(u) = 0 \geq 0$ and $\sum_{u \in N[v]} f_F(u) = -|N[v]| \leq 1$ (since $|N[v]| \geq 1$). Thus

$$\gamma_{ns}(G) = \sum_{v \in V} (w_T f_T(v) + w_I f_I(v) + w_F f_F(v)) = n(w_T - w_F).$$

Hence completing the proof. □

3 NSDN for cycles and paths

This section contains the NSDN for cycles and paths

Theorem 3.1. *For a cycle graph C_n with $n \geq 3$ vertices, the NSDN is given by*

$$\gamma_{ns}(C_n) = \begin{cases} w_T \left\lceil \frac{n}{3} \right\rceil - nw_F & \text{if } w_I \geq 0, \\ w_T \left\lceil \frac{n}{3} \right\rceil + w_I \left\lfloor \frac{n}{3} \right\rfloor - nw_F & \text{if } w_I < 0. \end{cases}$$

Proof. Let $f = (f_T, f_I, f_F)$ be a NSDF.

Truth Membership f_T

The truth constraint requires:

$$\sum_{u \in N[v]} f_T(u) \geq 1 \quad \forall v \in V.$$

Since C_n is 2-regular, $|N[v]| = 3$. The minimal assignment occurs when:

- For $n \equiv 0 \mod 3$: Set $f_T(v) = \frac{1}{3}$ for all v, giving $\sum f_T = \frac{n}{3}$.
- For $n \not\equiv 0 \mod 3$: Assign $f_T(v) = 1$ to $\lceil \frac{n}{3} \rceil$ vertices (a minimal dominating set) and 0 elsewhere, giving $\sum f_T = \lceil \frac{n}{3} \rceil$.

Thus, the minimal $\sum f_T = \lceil \frac{n}{3} \rceil$.

Indeterminacy Membership f_I

The indeterminacy constraint requires:

$$\sum_{u \in N[v]} f_I(u) \geq 0 \quad \forall v \in V.$$

- If $w_I \geq 0$, set $f_I(v) = 0$ for all v, giving $\sum f_I = 0$.
- If $w_I < 0$, assign $f_I(v) = 1$ to $\lfloor \frac{n}{3} \rfloor$ vertices and 0 elsewhere, ensuring constraints are satisfied, giving $\sum f_I = \lfloor \frac{n}{3} \rfloor$.

Falsity Membership f_F

The falsity constraint requires:

$$\sum_{u \in N[v]} f_F(u) \leq 1 \quad \forall v \in V.$$

Set $f_F(v) = -1$ for all v. Then $\sum_{u \in N[v]} f_F(u) = -3 \leq 1$, satisfying the constraint, and $\sum f_F = -n$.

The NSDN is:

$$\gamma_{ns}(C_n) = w_T \sum f_T + w_I \sum f_I + w_F \sum f_F.$$

Substituting the minimal sums:

$$\gamma_{ns}(C_n) = \begin{cases} w_T \lceil \frac{n}{3} \rceil - nw_F & \text{if } w_I \geq 0, \\ w_T \lceil \frac{n}{3} \rceil + w_I \lfloor \frac{n}{3} \rfloor - nw_F & \text{if } w_I < 0. \end{cases}$$

Hence complete the proof. □

Theorem 3.2. *For a path graph P_n with $n \geq 3$ vertices, the NSDN is given by:*

$$\gamma_{ns}(P_n) = \begin{cases} w_T \left\lceil \frac{n}{3} \right\rceil - nw_F & \text{if } w_I \geq 0, \\ w_T \left\lceil \frac{n}{3} \right\rceil + w_I \left\lfloor \frac{n}{3} \right\rfloor - nw_F & \text{if } w_I < 0. \end{cases}$$

Proof. We prove this by constructing an optimal NSDF $f = (f_T, f_I, f_F)$.

The truth constraint requires:

$$\sum_{u \in N[v]} f_T(u) \geq 1 \quad \forall v \in V.$$

Since P_n is a path, the closed neighborhood $|N[v]|$ has size 3 for internal vertices and 2 for endpoints. The minimal assignment occurs by placing positive truth values on a dominating set of P_n, which has size $\lceil n/3 \rceil$. Specifically:

- Assign $f_T(v) = 1$ to the vertices in a minimum dominating set (e.g., every second vertex).
- Assign $f_T(v) = 0$ to all other vertices.

This gives $\sum f_T = \lceil n/3 \rceil$, which is minimal.

Indeterminacy Membership f_I

The indeterminacy constraint requires:

$$\sum_{u \in N[v]} f_I(u) \geq 0 \quad \forall v \in V.$$

- If $w_I \geq 0$, set $f_I(v) = 0$ for all v, giving $\sum f_I = 0$.
- If $w_I < 0$, assign $f_I(v) = 1$ to $\lfloor n/3 \rfloor$ vertices (spaced appropriately to avoid violating constraints) and 0 elsewhere. This ensures each closed neighborhood has at most one positive indeterminacy value, so $\sum_{u \in N[v]} f_I(u) \geq 0$. Then $\sum f_I = \lfloor n/3 \rfloor$.

Falsity Membership f_F

The falsity constraint requires:

$$\sum_{u \in N[v]} f_F(u) \leq 1 \quad \forall v \in V.$$

Set $f_F(v) = -1$ for all v. Then $\sum_{u \in N[v]} f_F(u) = -|N[v]| \leq -2 \leq 1$, so the constraint is satisfied. Then $\sum f_F = -n$.

The NSDN is:

$$\gamma_{ns}(P_n) = w_T \sum f_T + w_I \sum f_I + w_F \sum f_F.$$

Substituting the minimal sums:

$$\gamma_{ns}(P_n) = \begin{cases} w_T \lceil n/3 \rceil - nw_F & \text{if } w_I \geq 0, \\ w_T \lceil n/3 \rceil + w_I \lfloor n/3 \rfloor - nw_F & \text{if } w_I < 0. \end{cases}$$

Hence complete the proof. □

Illustration of NSDF on P_3 is presented in Fig 1.

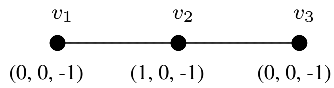

Fig. 1. NSDF on P_3

Let $V(P_3) = \{v_1, v_2, v_3\}$ with edges v_1v_2 and v_2v_3. Since $N[v_1] = \{v_1, v_2\}, N[v_2] = \{v_1, v_2, v_3\}, N[v_3] = \{v_2, v_3\}$; Define $f = (f_T, f_I, f_F) : V(P_3)$ to $\{-1, 0, 1\}$ as follows:

$$f_T(v_1) = 0, \quad f_T(v_2) = 1, \quad f_T(v_3) = 0$$
$$f_I(v_1) = 0, \quad f_I(v_2) = 0, \quad f_I(v_3) = 0$$
$$f_F(v_1) = -1, \quad f_F(v_2) = -1, \quad f_F(v_3) = -1$$

Now we consider the vertex v_1. From the above labeling, we get
$f_T(v_1) + f_T(v_2) = 0 + 1 = 1, f_I(v_1) + f_I(v_2) = 0 + 0 = 0, f_F(v_1) + f_F(v_2) = -1 + (-1) = -2.$
Next, we consider the vertex v_2. By the above labeling, we get
$f_T(v_1) + f_T(v_2) + f_T(v_3) = 0 + 1 + 0 = 1, f_I(v_1) + f_I(v_2) + f_I(v_3) = 0 + 0 + 0 = 0, f_F(v_1) + f_F(v_2) + f_F(v_3) = -1 + (-1) + (-1) = -3.$
Now, we consider the vertex v_3. According to the above labeling, we get
$f_T(v_2) + f_T(v_3) = 1 + 0 = 1, f_I(v_2) + f_I(v_3) = 0 + 0 = 0, f_F(v_2) + f_F(v_3) = -1 + (-1) + = -2.$
Thus $\gamma_{ns}(P_3) = 1 + 0 + (-3) = -2.$

4 Conclusion

This paper establishes the novel framework of Neutrosophic Signed Domination, integrating three-valued logic into graph theory. We derive sharp bounds for $\gamma_{ns}(G)$ and characterize extremal graphs achieving these bounds. Exact closed-form formulas are obtained for paths and cycles, demonstrating computability. Our work extends classical domination theory by incorporating uncertainty and influence gradations. This provides a powerful tool for analyzing complex networks where binary classifications are insufficient.

Mathematics Subject Classification (2020): 05C69; 03E72.

References

[1] D. Ajay, P. Chellamani, Pythagorean neutrosophic fuzzy graphs, *International Journal of Neutrosophic Science*, 11 (2020), pp. 108-114

[2] A. B. Smith and C. D. Johnson, "Neutrosophic Graph Theory: New Applications and Developments," *Journal of Applied Mathematics and Computing*, vol. 36, no. 1, pp. 101-115.

[3] R. K. Gupta and S. P. Singh, "Fuzzy Graphs and Their Applications in Decision Making," *International Journal of Fuzzy Systems*, vol. 23, no. 3, pp. 789-798.

[4] Al-Hamido, R. K., Al-Tahan, M and Elewe, A. M. Neutrosophic Graph for Social Network Analysis. *Journal of Intelligent and Fuzzy Systems*, 44(5), 2023, 7187-7198.

[5] Akram, M., & Shahzadi, S. Operations on Neutrosophic Graphs, *Journal of Fuzzy Mathematics*, 26(3), 2018, 707-725.

[6] M. Akram, S. Shahzadi, A. B. Saeid, Single-valued neutrosophic hypergraphs, *TWMS Journal of Applied and Engineering Mathematics*, 8, (2018), pp. 122-135

[7] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Elsevier, North Holland, New York, (1986).

[8] G.J. Chang, S.-C. Liaw, H.-G. Yeh, k-Subdomination in graphs, *Discrete Appl. Math.*, 120, (2002) 44-60.

[9] E.J. Cockayne and C.M. Mynhardt, On a generalization of signed dominating functions of graphs, *Ars Combin.*, 43, (1996) 235-245.

[10] J.E. Dunbar, S.T. Hedetniemi, M. A. Henning and P. J. Slater, Signed domination in graphs. In: Graph Theory, Combinatorics and Applications. Proc. 7th Internat. conf. Combinatorics, Graph Theory, Applications, (Y. Alavi, A. J. Schwenk, eds.). John Wiley & Sons, Inc., 1 (1995) 311-322.

[11] O. Favaron, Signed domination in regular graphs, *Discrete Math.*, 158, (1996) 287-293.

[12] Z. Füredi and D. Mubayi, Signed domination in regular graphs and set-systems, *J. Combin. Theory Series B*, 76, (1999) 223-239.

[13] Hafiz, M. A and Al-Masarwah, A. Domination in Bipolar Neutrosophic Graphs and Its Application. *AIMS Mathematics*, 9(4), 2024, 8234-8255.

[14] K Iyappan, V Nirmala, D. Kumar, Signed domination function of graphical image of trisomy ultrascan, *Math. Eng. Sci. Aerosp*, **16**(1), (2025), 269–276.

[15] N. Jayalakshmi, P. Thangaraj, D. Kumar, Non-negative isolated signed dominating function of graphs, *Advances and Applications in Mathematical Sciences* **22**(12) (2023), 2239–2280.

[16] D. Kumar, P. Thangaraj, N. Jayalakshmi, Non-negative unique isolated signed dominating function of graphs, *Math. Eng. Sci. Aerosp*, **13**(4) (2022), 959–964.

[17] Kumari RS, Kalayathankal S, George M, Smarandache F. N-cylindrical fuzzy neutrosophic sets. *Int J Neutrosophic Sci*, 18(4), 2022.

[18] Meenakshi A, Mythreyi O, Bramila M, Kannan A, Senbagamalar J. Application of neutrosophic optimal network using operations. *J Intell Fuzzy Syst*, 45(1), 2023, 421–33.

[19] Şahin R, Liu P. Possibility-induced simplified neutrosophic aggregation operators and their application to multi-criteria group decision-making, *J Exp Theor Artif Intell* , 29(4), 2017, 769–85.

[20] Smarandache, F. (1999). *A Unifying Field in Logics: Neutrosophic Logic*. American Research Press.

[21] Ullah, K., Mahmood, T and Garg, H. Complex Neutrosophic Graph with Application. *Computational and Applied Mathematics*, 42(5), 2023, 218.

[22] Vasantha, W. B., Kandasamy, I and Smarandache, F. (2022). On Regularity in Plithogenic Graphs. *Neutrosophic Sets and Systems*, 50, 2022, 266-279.

[23] Ye, J. (2014). *Aggregation Operators of Neutrosophic Sets*. Journal of Intelligent Systems, 23(2), 213-226.

[24] Wen-Sheng Li, Hua-Ming Xing, and Moo Young Sohn, *On the signed total domination number of generalized petersen graphs* $P(n, 2)$, Bull. Korean Math. Soc. 50 (2013), No. 6, pp. 2021-2026.

[25] Z. Zhang, B. Xu, Y. Li and L. Liu, A note on the lower bounds of signed domination number of a graph, /it Discrete Math., 195 (1999), 295-298.