A Note on Multi-Neutro-Topological Space

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Abstract

Multisets have been the subject of extensive research, and their usefulness has been recognized in various areas such as computation, database management, and more. This study aims to explore certain properties of neutro-topological spaces by introducing a multi-neutro-topological space. Many fundamental features of interior, the exterior, the closure, and the boundary in a neutro-topological space are found to be preserved in a multi-neutro-topological space with the incorporation of multisets.

Keywords: Multi-Neutro-Topological Space; Multi-Neutro-Interior; Multi-Neutro-Exterior; Multi-Neutro-Closure; Multi-Neutro Boundary

1. Introduction

Neutrosophication and Antisophication led to the evolution of a neutro-topology and an anti-topology [**1**]. In order to overcome the uncertainty and vagueness faced by the classical set in defining certain classes of objects, ideas, etc. a different kind of set called a fuzzy set was developed by Zadeh [2]. In a fuzzy set, each member has a range of membership from 0 to 1 and is not defined in terms of absolute membership, say 1 or absolute exclusion, say 0, as in the case of a Cantorian set. K. T. Atanassov [3] incorporated a range of non-inclusion for every element to the fuzzy set and his category of fuzzy set is called intuitionistic fuzzy set (IFS). Further, F. Smarandache [4] added a degree of non-determinacy to every element of a fuzzy set and the new fuzzy set originated the neutrosophic set. Thus, each element in a neutrosophic set will have three parts namely: the extent of inclusion, the extent of in-determinacy, and the extent of non-inclusion. The components of a neutrosophic set are also expressed as <Truth, Indeterminacy, Falsity> or <T, I, F> or <Truth, Neutro, Anti>. Studies related to "T" are the Classical Structures, studies related to "I" gave rise to the NeutroStructures such as the Neutro-Group [5], Neutro-Metric Space [6], Neutro-Vector Space [7], Neutro-Topological Space [1], and studies related to "F" led to the AntiStructures such as the Anti-Topological Space [1]. Basumatary *et al.* [8] found that most of the characteristics of the three aspects of interior, closure, and boundary that are true for a classical topological space are also true in a neutro-topological space.

This article extends the study made by Basumatary *et al.* [8] by adding the definition of the exterior in a neutro-topological space, and providing some of its properties. The article extends the studies in the characteristics of the four aspects, namely interior, exterior, closure, and boundary, in a neutro-topological space by defining a multi-neutro-topological space, multi-neutro-interior (exterior, closure, and boundary) with the use of multisets. Multiset is a collection of members which can occur more than once unlike in the classical set where every member is unique and well defined. The origin of the study of multiset or at least their mention can be traced back centuries, although in different terms but meaning the occurrence of an object more than once in a collection. At different times, different scholars to mean the repetitions of objects in a collection had used different terms like list, bunch, bag, heap etc. Blizzard [9] formally used the term multiset and did away with the other terms used previously for the collection of objects where repetitions were allowed. According to him, a classical set is a multiset where the multiplicity of each member is unity. Girish *et al.* [10] formalised the definition of a multiset by suggesting certain notations and symbols and, introducing a multiset topological space. They also analyzed the features of interior, closure and limit points in a multiset topological space. El-Sheikh *et al.* [11] studied some axioms of separation in

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a multiset topological space. Das *et al.* [12] added the study on exterior and boundary in a multiset topological space. Ray *et al.* [13] studied some axioms of separation in a fuzzy multiset topological space.

2. Related Work

Definition 2.1: Neutro-Topological Space [1]

For non-null set *X* and a class *T* of subclasses of *X*, whenever at least any of the three points below is true, then (X, T) is a neutro-topological space (N-TS) and *T,* a neutro-topology on *X*.

(i) The null set and the universal set do not simultaneously belong to the class *T* or, it is not clear whether they belong to the class *T*.
(ii) For some members in *T*, their finite intersection belongs to *T* while for some members in *T* the finite intersection does not belong to *T* or, it is not clear whether finite intersection of some other members in *T* belong to *T*.
(iii) For some member in *T*, their random union belongs to *T* while random union of some members in *T* does not belong to *T* or, it is not clear whether random union of some other members in *T* belong to *T*.

Remark 2.2 [1]

For every topology T on the set X, $T \setminus \emptyset$ and $T \setminus X$ are both neutro topology on X. That is, a neutro-topology is deducible from any topology on the set X.

Theorem 2.3 [8]

For a non-void universe *X* and *T*, a class of subclasses of *X*, called neutro-open sets, (*X, T*) will be a N-TS if at least one of the points below is true. Also, *T* will be a neutro-topology on *X*.

(i) The void set or the universe is not in *T*.
(ii) For some members in *T,* their union is not a member of *T*.
(iii) For some members in *T,* their intersection is not a member of *T*.

If $Q \in T$, then cQ is a neutro-closed set.

Definition 2.4: Neutro-Interior [8]

If (*X, T*) is a N-TS with *Q* being a subset of *X*, the neutro-interior of *Q* is an union of every subset of *Q* that belongs to *T* and expressed as *Nu-Int(Q)*. Thus, *Nu-Int(Q)* = $\cup \{O_i$: each $O_i \in T$ and $O_i \subseteq Q\}$.

Theorem 2.5 [8]

If (*X, T*) is a N-TS with *P*, *Q* ⊆ *X*, the results below holds true:

(i) *Nu-Int(P)* ⊆ *P*. If *P* is neutro-open then *Nu-Int(P)* = *P*, however the converse is not always true.
(ii) *Nu-Int(X)* ⊆ *X*; *Nu-Int*(∅) = ∅
(iii) *Nu-Int(Nu-Int(P)) = Nu-Int(P)*
(iv) *Nu-Int(P)* ⊆ *Nu-Int(Q) whenever P* ⊆ *Q.*
(v) *Nu-Int(P* ∩ *Q)* ⊆ *Nu-Int(P)* ∩ *Nu-Int(Q)*

Definition 2.6: Neutro-Closure [8]

Let (*X, T*) be a N-TS with *Q* ⊆ *X*. The neutro-closure of *Q*, represented by *Nu-Cl(Q),* is defined as intersection of every neutro-closed supersets of *Q*. Thus, *Nu-Cl(Q)* = $\cap \{C_i$: each $cC_i \in T$ and $Q \subseteq C_i\}$.

Theorem 2.7 [8]

Suppose (*X, T*) be a N-TS with *P*, *Q* ⊆ *X*, the results below holds true:

(i) *Nu-Cl(P)* ⊇ *P*. And, *Nu-Cl(P)* = *P* if *P* is neutro-closed, however the converse may not be true.
(ii) *Nu-Cl(Nu-Cl(P))= Nu-Cl(P)*
(iii) *Nu-Cl(P)* ⊆ *Nu-Cl(Q) whenever P* ⊆ *Q.*
(iv) *Nu-Cl(P* ∪ *Q)* ⊇ *Nu-Cl(P)* ∪ *Nu-Cl(Q)*

Definition 2.8: Neutro-Boundary [8]

Let (*X, T*) be a N-TS with *Q* ⊆ *X*. The neutro-boundary of *Q*, represented by *Nu-Bd(Q)*, is characterized as an intersection of the neutro-closure of *Q* and the neutro-closure of *cQ*. Thus, *Nu-Bd(Q)* = *Nu-Cl(Q)* ∩ *Nu-Cl(cQ)*.

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Definition 2.9: Multiset [10]

A multiset M, written in short as m-set, formed from the members of a set X is characterized as $C_M: X \to N$; here N stands for natural numbers. If $X = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ then $C_M(\alpha)$ stands for the multiples of $\alpha \in X$ appearing in M, and written as: $M = \{\frac{n_1}{\alpha_1}, \frac{n_2}{\alpha_2}, \frac{m_3}{\alpha_3}, \ldots, \frac{n_n}{\alpha_n}\}$ where n_i stand for how many times α_i occur in M. Members that do not appear in the m-set M will have count zero.

Operations on two multisets P and Q:

(i) Two multisets P and Q are equal provided $\mathcal{C}_P(\beta) = \mathcal{C}_Q(\beta)$ for every β in X.
(ii) $R = P \cup Q$ provided $\mathcal{C}_R(\beta) =$ maximum$\{\mathcal{C}_P(\beta), \mathcal{C}_Q(\beta)\}$ for every β in X.
(iii) $S = P \cap Q$ provided $\mathcal{C}_S(\beta) =$minimum$\{\mathcal{C}_P(\beta), \mathcal{C}_Q(\beta)\}$ for every β in X.

Definition 2.10: Support Set [10]

M^* stands for the root set or support set of M, where M^* is a subclass of X and it is characterized by $M^* = \{\beta \in X: C_M(\beta) > 0\}$.

Definition 2.11: Empty multiset [10]

A null m-set is classified as $C_M(\beta) = 0$, for any β in X and represented by $\emptyset$.

Definition 2.12: Multiset Space [10]

An m-set space $[X]^n$ contains all m-sets having members from X in such a way that all members appear at most n times.

Definition 2.13: Multiset-subset [10]

An m-set B is termed a subm-set of A if $C_B(\beta) \leq C_A(\beta)$ for every β in X.

Definition 2.14: Power-multiset [10]

Let $Q \in [X]^n$ be an m-set, then we have the following definitions:

(i) $P(Q)$ stands for the power m-set of Q, and all subm-sets of A belongs to $P(Q)$.
(ii) $P^*(Q)$ stands for the power m-set for the root set of $P(Q)$.

Definition 2.15: Multiset-complement [10]

For a subm-set N of an m-set M, the m-complement is expressed as cN and is classified as $cN = M \ominus N$, where $M \ominus N =$ maximum$\{\mathcal{C}_M(x) - \mathcal{C}_N(x), 0\}$.

3. Main results

3.1 Neutro-Topological Spaces

Definition 3.1.1: Neutro-Exterior

If (*X, T*) is a N-TS with *Q* being a subset of *X*, neutro-exterior of *Q* is characterized as a union of every neutro-open subset of the complement of *Q* and is represented by *Nu-Ext(Q)*. Thus, *Nu-Ext(Q)* = $\cup \{O_i$: each $O_i \in T$ and $O_i \subseteq cQ\}$, where *cQ* denotes the complement of *Q*. That is, *Nu-Ext*(*Q*) = *Nu-Int*(*cQ*).

Theorem 3.1.2

If (*X, T*) is a N-TS with *P*, *Q* being subsets of *X*, the following holds:

(i) *Nu-Ext(P)* ⊆ *cP*. If *A* is neutro-closed then *Nu-Ext(P)=cP.* The converse is not true.
(ii) *Nu-Ext(P)* ∩ *Nu-Int(P)*= ∅
(iii) *Nu-Ext(P)= Nu-Int(cP)*
(iv) *Nu-Ext(c(Nu-Ext(P))= Nu-Ext(P)*
(v) *Nu-Ext(cP)= Nu-Int(P)*
(vi) *Nu-Ext(P)* ⊇ *Nu-Ext(Q)* whenever *P* ⊆ *Q*.
(vii) *Nu-Int(P)* ⊆ *Nu-Ext(Nu-Ext(P))*
(viii) *Nu-Ext(P* ∪ *Q)* ⊆ *Nu-Ext(P)* ∩ *Nu-Ext(Q)*
(ix) *Nu-Ext(P)* ∪ *Nu-Ext(Q)* ⊆ *Nu-Ext(P* ∩ *Q)*

Proof: All the proofs are simple and follow from the definition and properties of the neutro-interior given in **theorem 2.12**.

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3.2 Multi-neutro-topological space

Definition 3.2.1: Multi-neutro-topological space

If $M \in [X]^n$ and $T \subseteq P^*(M)$, we define a multi-neutro-topology (*MN-T*) on *M* if one or more of the following are true for *T*:

(i) *M* or ∅ is not in *T*.

(ii) There exist members in *T* whose union is in *T* and there exists members in *T* whose union may not be in *T*.

(iii) There exist members in *T* whose intersection is in *T* and there exists members in *T* whose intersection may not be in *T*.

The space *(M, T)* will be called a multi-neutro-topological space or *MN*-topological space (*MN-TS*) on *M* and the members of *T* will be multi-neutro-open (*MN-Open*) m-sets and the m-complements of such m-sets will be called multi-neutro-closed (*MN-Closed*) m-sets.

Example 3.2.2

Let us assume that *X* = {*x,y,z,w*}, *n* = *4* and *M* $=\left\{\frac{3}{x},\frac{2}{y},\frac{3}{z},\frac{4}{w}\right\}$ is a m-set formed on *X*. Let T = $\left\{\emptyset,\left\{\frac{2}{x},\frac{1}{y}\right\},\left\{\frac{3}{x},\frac{1}{y}\right\},\left\{\frac{1}{x},\frac{2}{y}\right\},\left\{\frac{2}{x},\frac{1}{z}\right\},\left\{\frac{1}{x},\frac{2}{y},\frac{1}{z}\right\},\left\{\frac{2}{y},\frac{2}{z},\frac{2}{w}\right\}\right\}$. Then (*M*, *T*) is a *MN-TS* on *M*.

Definition 3.2.3: Multi-Neutro-Interior

Let (*M*, *T*) be a *MN-TS* and *Q* be a subm-set of *M*. The *MN*-interior of *Q* is a union of every *MN*-open subm-set of *Q* and is represented by *MN-Int(Q)*.

Remark 3.2.4

The *MN*-interior of a subm-set *Q* of a *MN-TS* (*M*, *T*) is not the largest *MN*-open subm-set of *Q*. The following example justifies this statement.

Let *X* = {*x, y, z, w*}, *n* = *4* and *M* = $\left\{\frac{4}{x},\frac{4}{y},\frac{3}{z},\frac{4}{w}\right\}$. Let *T* = $\left\{\emptyset,\left\{\frac{1}{x},\frac{2}{y}\right\},\left\{\frac{2}{x},\frac{2}{y}\right\},\left\{\frac{1}{x},\frac{2}{y},\frac{3}{z}\right\},\left\{\frac{2}{x},\frac{2}{z}\right\},\left\{\frac{2}{y},\frac{3}{z},\frac{2}{w}\right\}\right\}$ be a *MN*-topology on *M* and let *Q* = $\left\{\frac{2}{x},\frac{2}{y},\frac{2}{z}\right\}$ then *MN-Int(Q)* = $\left\{\frac{1}{x},\frac{2}{y}\right\} \cup \left\{\frac{2}{x},\frac{2}{y}\right\} \cup \left\{\frac{2}{x},\frac{2}{z}\right\} = \left\{\frac{2}{x},\frac{2}{y},\frac{2}{z}\right\}$ = *Q*. However, it may be observed that the m-set *Q* is not *MN*-open in *T*.

Theorem 3.2.5

Assume *Q* to be a subm-set of a *MN-TS* (*M*, *T*). *MN-Int(Q) = Q* whenever *Q* ∈ *T*. The converse is necessarily not true and it can be seen from the example provided in **remark 3.2.4**.

Theorem 3.2.6

If (*M, T*) is a *MN-TS* with *P* and *Q* being subm-sets of *M*, then the results below are true:

(i) *MN-Int(M)* ⊆ *M*

(ii) *MN-Int(MN-Int(P)) = MN-Int(P)*

(iii) *MN-Int(P)* ⊆ *MN-Int(Q)* whenever *P* ⊆ *Q*.

(iv) *MN-Int(P ∩ Q)*⊆ *MN-Int(P)* ∩ *MN-Int(Q)*

Remark 3.2.7

Equality will not always be true in (i) of **theorem 3.2.6** because the m-set *M* need not necessarily be *MN*-open in a *MN*-topology and union of *MN*-open sets need not necessarily be equal to *M*. Equality will not hold in (iv) as can be seen from the example that follows: Let *X* = {*x,y,z,w*}, *n* = *4* and *M* = $\left\{\frac{3}{x},\frac{3}{y},\frac{3}{z},\frac{3}{w}\right\}$. Let *T* = $\left\{\emptyset,\left\{\frac{1}{x},\frac{2}{y}\right\},\left\{\frac{2}{x},\frac{2}{z}\right\},\left\{\frac{1}{y},\frac{2}{z}\right\},\left\{\frac{1}{x},\frac{2}{y},\frac{3}{z}\right\},\left\{\frac{2}{y},\frac{3}{z},\frac{2}{w}\right\}\right\}$ be a *MN*-topology on *M*. Now, if we consider the m-sets *P* = $\left\{\frac{1}{x},\frac{2}{y},\frac{3}{z}\right\}$ and *Q* = $\left\{\frac{2}{y},\frac{3}{z},\frac{3}{w}\right\}$ which are subm-sets of *M* and it can be verified that *MN-Int(P ∩ Q)* = ∅ and *MN-Int(P)* ∩ *MN-Int(Q)* = $\left\{\frac{2}{y},\frac{3}{z}\right\}$.

Definition 3.3.1: Multi-Neutro-Exterior

Let (*M*, *T*) be a *MN-TS* and *Q* be a subm-set of *M*. The *MN*-exterior of *Q* is the union of every *MN*-open subm-set of the m-complement *Q* and is represented by *MN-Ext(Q)*. Thus, *MN-Ext*(*Q*) = *MN-Int*(*cQ*).

Theorem 3.3.2

If (*M, T*) be a *MN-TS* with *P* and *Q* as subm-sets of *M*, then the following holds:

(i) *MN-Ext(P)* ⊆ *cP*. If *P* is *MN*-closed then *MN-Ext(P)=cP.* The converse is not true.
(ii) *MN-Ext(P)* ∩ *MN-Int(P)* = ∅
(iii) *MN-Ext(P) = MN-Int(cP)*
(iv) *MN-Ext(c(MN-Ext(P)) = MN-Ext(P)*
(v) *MN-Ext(cP) = MN-Int(P)*
(vi) *MN-Ext(P)* ⊇ *MN-Ext(Q)* whenever *P* ⊆ *Q*.
(vii) *MN-Int(P)* ⊆ *MN-Ext(MN-Ext(P))*
(viii) *MN-Ext(P* ∪ *Q)* ⊆ *MN-Ext(P)* ∩ *MN-Ext(Q)*
(ix) *MN-Ext(P)* ∪ *MN-Ext(Q)* ⊆ *MN-Ext(P* ∩ *Q)*

Proof: (i) The first part is true from the definition. For the second part, if *P* is *MN*-closed the *cP* is *MN*-Open and by **theorem 3.2.5** *MN-Int*(*cP*) = *cP* which by definition gives *MN-Ext*(*P*) = *cP*. For the third part pertaining to the converse part of the second part not being true, let us assume the following example: Let *X* = {*x,y,z,w*}, *n = 4* and *M* = $\left\{\frac{3}{x},\frac{3}{y},\frac{3}{z},\frac{3}{w}\right\}$. Let *T* = {∅, $\left\{\frac{1}{x},\frac{2}{y}\right\}$, $\left\{\frac{2}{x},\frac{2}{y}\right\}$, $\left\{\frac{1}{y},\frac{2}{dw}\right\}$, $\left\{\frac{1}{x},\frac{2}{y},\frac{3}{z}\right\}$, $\left\{\frac{2}{y},\frac{3}{z},\frac{2}{w}\right\}$} be a *MN*-topology on *M*. Let *P* = $\left\{\frac{1}{x},\frac{1}{y},\frac{1}{z},\frac{1}{w}\right\}$ then *cP* = $\left\{\frac{2}{x},\frac{2}{y},\frac{2}{z},\frac{2}{w}\right\}$ and *MN-Int*(*cP*) = $\left\{\frac{1}{x},\frac{2}{y}\right\} \cup \left\{\frac{2}{x},\frac{2}{z}\right\} \cup \left\{\frac{1}{y},\frac{2}{w}\right\} = \left\{\frac{2}{x},\frac{2}{y},\frac{2}{z},\frac{2}{w}\right\} = cP$. But by **definition 3.3.1** *MN-Int*(*cP*) = *MN-Ext*(*P*). Thus *MN-Ext*(*P*) = *cP* but *P* is not *MN*-closed as *cP* is not *MN*-open.

Remark 3.3.3

The other results can be similarly proved using the definition and properties of the *MN*-interior.

Definition 3.4.1: Multi-Neutro-Closure

Let (*M*, *T*) be a *MN-TS* and *Q* be a subm-set of *M*. The *MN*-closure of *Q* is an intersection of every *MN*-closed m-set that contains *Q* and is written as *MN-Cl(Q).*

Theorem 3.4.2

Let (*M*, *T*) be a *MN-TS*. If *Q* is a *MN*-closed subm-set of *M*, then *MN-Cl(Q) = Q*.

Remark 3.4.3

The converse of **theorem 3.4.2** is not always true. It may be seen from the example that follows: Let *X* = $\{x, y, z, w\}$, *n* = *3*, *M* = $\left\{\frac{3}{x},\frac{3}{y},\frac{2}{z},\frac{3}{w}\right\}$ and let *T* =$\left\{\emptyset, \left\{\frac{1}{x},\frac{2}{y}\right\}, \left\{\frac{2}{x},\frac{2}{y}\right\}, \left\{\frac{2}{x},\frac{2}{z}\right\}, \left\{\frac{1}{x},\frac{2}{y},\frac{2}{z}\right\}, \left\{\frac{2}{y},\frac{2}{z},\frac{2}{w}\right\}\right\}$ be a MN-topology on *M*. Here the *MN*-closed m-sets are: *M*, $\left\{\frac{2}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$, $\left\{\frac{1}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$, $\left\{\frac{1}{x},\frac{3}{y},\frac{1}{z},\frac{3}{w}\right\}$, $\left\{\frac{2}{x},\frac{1}{y},\frac{1}{z},\frac{3}{w}\right\}$, and $\left\{\frac{3}{x},\frac{1}{y},\frac{1}{z},\frac{1}{w}\right\}$. Assume that *Q* = $\left\{\frac{1}{x},\frac{1}{y},\frac{2}{z},\frac{3}{w}\right\}$ then *MN-Cl(Q)* = $\left\{\frac{2}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\} \cap \left\{\frac{1}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\} = \left\{\frac{1}{x},\frac{1}{y},\frac{2}{z},\frac{3}{w}\right\} = Q$. But *Q* is not *MN*-closed.

Theorem 3.4.4

If (*M*, *T*) is a *MN-TS* with *P*, *Q* being subm-sets of *M*, the results below are true:

(i) *P* ⊆ *MN-Cl(P)*
(ii) *MN-Cl(MN-Cl(P)) = MN-Cl(P)*
(iii) *MN-Cl(P)* ⊆ *MN-Cl(Q)* whenever *P* ⊆ *Q*.
(iv) *MN-Cl(P* ∪ *Q)* ⊇ *MN-Cl(P)* ∪ *MN-Cl(Q)*

Remark 3.4.5

Equality will not always be true in (iv) of **theorem 3.4.4** and the following example justifies it. Let X = {*x, y, z, w*}, *n* = *3*, *M* = $\left\{\frac{3}{x},\frac{3}{y},\frac{3}{z},\frac{3}{w}\right\}$ and let *T* =$\left\{\emptyset, \left\{\frac{1}{x},\frac{2}{y}\right\}, \left\{\frac{2}{x},\frac{2}{y}\right\}, \left\{\frac{2}{x},\frac{2}{z}\right\}, \left\{\frac{1}{x},\frac{2}{y},\frac{2}{z}\right\}, \left\{\frac{2}{y},\frac{2}{z},\frac{2}{w}\right\}\right\}$ be a *MN*-topology on *M*. Here the *MN*-closed m-sets are: *M*, $\left\{\frac{2}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$, $\left\{\frac{1}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$, $\left\{\frac{1}{x},\frac{3}{y},\frac{1}{z},\frac{3}{w}\right\}$, $\left\{\frac{2}{x},\frac{1}{y},\frac{1}{z},\frac{3}{w}\right\}$, and $\left\{\frac{3}{x},\frac{1}{y},\frac{1}{z},\frac{1}{w}\right\}$. Assume that *P* = $\left\{\frac{1}{x},\frac{3}{y},\frac{1}{z},\frac{1}{w}\right\}$, *Q* = $\left\{\frac{2}{x},\frac{1}{y},\frac{2}{z},\frac{3}{w}\right\}$ then *MN-Cl(P)* = $\left\{\frac{1}{x},\frac{3}{y},\frac{1}{z},\frac{3}{w}\right\}$ and *MN-Cl(Q)* = $\left\{\frac{2}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$ and *MN-Cl(P)* ∪ *MN-Cl(Q)*= $\left\{\frac{2}{x},\frac{3}{y},\frac{3}{z},\frac{3}{w}\right\}$. However, *MN-Cl(P* ∪ *Q) = M* and hence the result.

Theorem 3.4.6

Let (*M, T*) be a *MN-TS* with *P* being a subm-set of *M*, then we have the following relations between the *MN-Int(P)* and *MN-Cl(P)*:

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(i) *c(MN-Int(P)) = MN-Cl*(*cP*)
(ii) *MN-Int(cP) = c(MN-Cl(P))*
(iii) *MN-Int(P) = c(MN-Cl(cP))*
(iv) *MN-Cl(P) = c(MN-Int(cP)*

Proof: (i) By definition: *MN-Int(P)* = $\cup\, Q_i$ so that each Q_i is *MN*-open and $Q_i \subseteq P$. Thus, *c(MN-Int(P))* = $c(\cup\, Q_i)$ so that $c(Q_i) \supseteq cP$. This leads to *c(MN-Int(P))* = $\cap\,(cQ_i)$ so that each $c(Q_i)$ is MN-closed and $cP \subseteq c(Q_i)$. This in turn gives: *c(MN-Int(P))* = $\cap\,(R_i)$so that each R_i is *MN*-closed and $cP \subseteq R_i$. Thus, *c(MN-Int(P)) = MN-Cl(cP)* by the definition of *MN*-closure.

Remark 3.4.7

The results (ii), (iii) and (iv) of **theorem 3.4.6** can be proved similarly.

Definition 3.5.1: Multi-Neutro-Boundary

Let (*M, T*) be a *MN-TS* and *Q* be a subm-set of *M*. The *MN*-boundary of *Q* is defined as an intersection of *MN*-closure of *Q* and the *MN*-closure of m-complement of *Q* and is denoted by *MN-Bd(Q).* Thus, *MN-Bd(Q) = MN-Cl(Q)* ∩ *MN-Cl(cQ).*

Example 3.5.2

If *X* = {*x, y, z, w*}, *n = 3*, *M* = $\left\{\frac{3}{x},\frac{3}{y},\frac{3}{z},\frac{3}{w}\right\}$ and let *T* =$\left\{\emptyset,\left\{\frac{1}{x},\frac{2}{y}\right\},\left\{\frac{2}{x},\frac{2}{y}\right\},\left\{\frac{2}{x},\frac{2}{z}\right\},\left\{\frac{1}{x},\frac{2}{y},\frac{2}{z}\right\},\left\{\frac{2}{y},\frac{2}{z},\frac{2}{w}\right\}\right\}$ be a *MN*-topology on *M*. Here the *MN*-closed m-sets are: *M*, $\left\{\frac{2}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$, $\left\{\frac{1}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$, $\left\{\frac{1}{x},\frac{3}{y},\frac{1}{z},\frac{3}{w}\right\}$, $\left\{\frac{2}{x},\frac{1}{y},\frac{1}{z},\frac{3}{w}\right\}$, and $\left\{\frac{3}{x},\frac{1}{y},\frac{1}{z},\frac{1}{w}\right\}$. Assume that *P* = $\left\{\frac{1}{x},\frac{2}{y},\frac{1}{z},\frac{2}{w}\right\}$ then *cP* = $\left\{\frac{2}{x},\frac{1}{y},\frac{2}{z},\frac{1}{w}\right\}$. *MN-Cl(P)* = *M* ∩ $\left\{\frac{1}{x},\frac{3}{y},\frac{1}{z},\frac{3}{w}\right\}$ = $\left\{\frac{1}{x},\frac{3}{y},\frac{1}{z},\frac{3}{w}\right\}$ and *MN-Cl(cA)* = *M* ∩ $\left\{\frac{2}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$ = $\left\{\frac{2}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$. Thus, by definition, we have: *MN-Bd*(*P*) = $\left\{\frac{1}{x},\frac{3}{y},\frac{1}{z},\frac{3}{w}\right\}$ ∩ $\left\{\frac{2}{x},\frac{1}{y},\frac{3}{z},\frac{3}{w}\right\}$ = $\left\{\frac{1}{x},\frac{1}{y},\frac{1}{z},\frac{3}{w}\right\}$.

4. Conclusion

In the article, we have used multisets to study the characteristics of neutro-interior, neutro-exterior, neutro-closure and neutro-boundary by defining a multi-neutro-topological space. The basic characteristics of these aspects that had been validated in the case of a neutro-topology have also been found to be true with the multi-neutro-topology. In this article, we have added the notion of a neutro-exterior and correspondingly studied its characteristics in a multi-neutro-topology. Since a neutro-topology deviates from a classical topology in the cases of union and intersection of its constituent members, certain results of the basic aspects deviate from those in a classical topology. We have provided necessary counter examples whenever necessary and sample proofs have been provided for some of the results we have proposed in this analysis. The notion of multisets can also be applied to analyse the characteristics of the aspects of anti-interior, anti-exterior, anti-closure, and anti-boundary in an anti-topological space. It may be mentioned that in an anti-topological space, all the three axioms of a classical topology are negated. Further, the concept of multisets can also be applied to the study the characteristics of these basic aspects in a neutro-bitopological space, which is a space with two neutro-topologies on a universe. Further, separation axioms can be studied in multi-neutro-topological spaces.

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