Analytic Solution of Higher Order Fractional Abstract Cauchy Problem

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Abstract

In this paper, we utilize the concept of point-wise independent set of closed operators that enabled us to find atomic solutions of the non-homogeneous $\alpha-$fractional abstract Cauchy problem of order n. The proposed fractional abstract Cauchy problem is

$$A_n u^{(n\alpha)}(t) + A_{n-1}u^{((n-1)\alpha)}(t) + \cdots + A_1 u^{(\alpha)}(t) + A_\circ u(t) = f(t)$$

where the involved operators $A_n, A_{n-1}, \cdots, A_\circ$ are closed and linear on a given Banach space and the unknown function $u(t)$ is assumed to be n-times $\alpha-$differentiable. Beyond the deterministic setting, we indicate how the atomic-solution framework extends naturally when coefficients, data, or initial states are modeled as neutrosophic (single-valued) quantities, thereby accommodating uncertainty and indeterminacy at the operator or forcing level.

Keywords: α-fractional Abstract Cauchy problem of order n; Pointwise independent operators; Atomic solution; Conformable fractional derivative

1 Introduction

It is known that, constructing ordinary or partial differential equations by means of Hilbert spaces or Banach spaces leads to the so called abstract differential equations. In such cases, the values of the unknown function can be determined in these abstract spaces. However, modeling real world problems by differential equations in Banach spaces has grasped the attention of researchers in order to investigate the solutions of such problems.

Let X be a Banach space, $A : D(A) \subseteq X \to X$ be a linear operator of an appropriate type such that $x \in X$ is given, and $u : [0, \infty) \to X$ is an unknown function. Then the following initial value problem

$$\begin{aligned} &\frac{du}{dt} = Au(t), \quad t \geq 0, \\ &u(0) = x, \end{aligned} \tag{1}$$

is known as the classical abstract Cauchy problem (ACP) which has various forms and can be employed in different fields of applied sciences as well as engineering.

Though, the abstract ODEs specially the (ACP) have a long rich research history, but no comprehensive theory that rules the solutions of such problems has been developed as of yet. In fact, we are not able to obtain straightforward solutions to the abstract differential equations. Early researches for solving such problems were conducted by using both Laplace transform methods,[1],[23] and the semigroup of operators.[4]

Not long ago, in 2010, a novel approach has been proposed by Khalil[6] who introduced a new approach to handle some classes of (ACPs) for both ordinary and fractional orders. The new technique is based on the theory of tensor product of Banach spaces, and it can be utilized to obtain the so-called atomic solutions of such problems. It should be noted that, many forms of both ordinary and fractional (ACPs) can be solved and handled utilizing this technique.

Let X be a Banach space and $A_n, \cdots, A_1, A_\circ$ are closed linear operators defined on X such that for $0 \leq i \leq n$ we have

$$A_i : Dom(A_i) \subseteq X \to X.$$

Also, for $\alpha \in (0,1)$,assume $u : [0,\infty) \to X$ to be unknown n-times $\alpha-$fractional differentiable function such that

$$Rang(u) \cup Rang(u^{(\alpha)}) \cup \cdots \cup Rang(u^{(n\alpha)}) \subseteq \bigcap_{i=0}^{n} Dom(A_i).$$

In this paper, the goal is to discuss the atomic solution of the following non-homogeneous $\alpha-$fractional abstract Cauchy problem of order n

$$A_n u^{(n\alpha)}(t) + A_{n-1} u^{((n-1)\alpha)}(t) + \cdots + A_1 u^{(\alpha)}(t) + A_\circ u(t) = f(t).$$

where $f : [0,\infty) \to X$ is a given function. To do so, we utilize previous result[?] at which the ordinary case of equation

$$A_n u^{(n)}(t) + A_{n-1} u^{(n-1)}(t) + \cdots + A_1 u'(t) + A_\circ u(t) = f(t),$$

was solved by implementing the concept of point-wise independent set of closed operators together with the use of Hahn–Banach Theorem.

It should be noted that many systems are driven by imprecise information either in parameters, operators, or inputs. Interpreting $A_\circ, A_1, \cdots, A_n$, the right-hand side f, or initial data as neutrosophic-valued (single-valued neutrosophic sets) allows the present atomic solution to serve as the deterministic backbone within which truth, indeterminacy, and falsity memberships propagate through the same linear mechanism. In this way, the analytic solution we derive becomes the certain core of a neutrosophic model, to which membership evaluations can be attached at each step.[10]

2 Preliminaries

2.1 Atoms Operators

In this section, we introduce some results in functional analysis related to the main result of this paper.

Definition 2.1. Let V and W be any two Banach spaces and V^* is the dual space of V. For $v \in V$ and $w \in W$, the operator $A : V^* \to W$, defined by $A(v^*) = v^*(v)w = \langle v, v^* \rangle\, w$, is a bounded one rank linear operator since the range of A is the span of w. We write $v \otimes w$ for A and such operators are called atoms.

To give an example on the atom operators, let us consider V and W to be two Banach spaces such that $V = W = C\,[0,1]$, with the norm $\|h\| = \sup\limits_{t\in[0,1]} |h(t)|$ for any $h \in C\,[0,1]$. Define V^* to be the space of all regular Borel measures on $[0,1]$. So, if $f \in V^*$, then $f(h) = \int\limits_0^1 h\, df$. Now, let $h_1 \in V$, $h_2 \in W$, and $f \in V^*$, then $(h_1 \otimes h_2)\, f = f(h_1)h_2 = h_2 \int\limits_0^1 h_1(t)\, df(t)$. Further, $h_1 \otimes h_2$ is a bounded linear operator. Also, indeed, $\|(h_1 \otimes h_2)\,(f)\| = |f(h_1)|\, \|h_2\| \leq \|f\|\, \|h_1\|\, \|h_2\|$ for all $f \in V^*$.

Atoms operators are assumed to be basic components in the theory of tensor product of Banach spaces.

The Hahn-Banach theorem is known as one of the most important results in functional analysis. This theorem enables us to extend bounded linear functionals that is defined on a subspace of a vector space to the whole space. Consequently, the dual space X^* is a non-trivial space for X.

Theorem 2.2. [5] *(The Hahn-Banach Theorem) Let M be a linear subspace of a normed linear space N and let $f \in M^*$. Then f can be extended to a bounded linear functional $F \in N^*$ (defined on the whole linear space N) such that $F|_M = f$ and $\|f\|_{M^*} = \|F\|_{N^*}$.*

Corollary 2.3. [5] *Let N be a normed linear space and let*

$$S(N) = \{x \in N : \|x\| = 1\}.$$

Then there exists $F \in N^*$ such that $\|F\|_{N^*} = 1$ where $F(x_\circ) = 1$ for all $x_\circ \in S(N)$.

Next, for an inner product space V, the orthogonal complement of a subset of V is defined as follows:

Definition 2.4. [5] *Let Y be an inner product space such that Z is a subset of Y.* Then $Z^\perp$ is the orthogonal complement of Z in Y such that

$$Z^\perp = \{y \in Y : \langle y, z\rangle = 0 \text{ for every } z \in Z\}.$$

Theorem 2.5. [5] *Let Y be an inner product space such that D is a finite dimensional subspace of Y. Then $Y = D \oplus D^\perp$.*

It is known that $\dim D^\perp = \dim Y - \dim D$. Moreover, $D = \left(D^\perp\right)^\perp$.

Now, we derive the following definition of the so called point-wise independent set of closed operators:

Definition 2.6. Let $\{T_\circ, T_1, \cdots, T_n\}$ *be a set of closed operators* on a Banach space V such that, *for any* $v \in V/\{0\}$, the set $\{T_\circ v, T_1 v, \cdots, T_n v\}$ *is an independent set in V.* Then the set $\{T_\circ, T_1, \cdots, T_n\}$ *is called point-wise independent.*

2.2 Fractional Derivative

In,[7] an alternative definition, for both Caputo and Riemann Liouville fractional derivatives, was introduced. This new definition is called the α-conformable fractional derivative.

Definition 2.7. Let $\alpha \in (0, 1)$, and $u : I \subseteq (0, \infty) \to \mathbb{R}$. For $\varkappa \in I$, let

$$D^\alpha u(\varkappa) = \lim_{\epsilon \to 0} \frac{u(\varkappa + \epsilon \varkappa^{1-\alpha}) - u(\varkappa)}{\epsilon}. \tag{2}$$

If the limit exists, then it is called the α-conformable fractional derivative of u at $\varkappa$. Moreover, u is said to be α-differentiable on $(0, r)$ for some $r > 0$, if $\lim\limits_{\epsilon \to 0^+} D^\alpha u(\varkappa)$ exists then we write

$$D^\alpha u(0) = \lim_{\varkappa \to 0} D^\alpha u(\varkappa). \tag{3}$$

For $\alpha \in (0, 1]$ and u, v *are* α-differentiable at a point $\varkappa$, one can easily see that the conformable derivative satisfies

$$\begin{aligned} &(i)\ D^\alpha(c_1 u + c_2 v) = c_1 D^\alpha(u) + c_2 D^\alpha(v), \text{ for all } c_1, c_2 \in \mathbb{R},\\ &(ii)\ D^\alpha(k) = 0, \text{ for all constant functions } f(\varkappa) = k,\\ &(iii)\ D^\alpha(uv) = uD^\alpha(v) + vD^\alpha(u),\\ &(iv)\ D^\alpha(\tfrac{u}{v}) = \tfrac{vD^\alpha(u) - uD^\alpha(v)}{v^2}, v(\varkappa) \neq 0. \end{aligned} \tag{4}$$

In the following, we provide the α-conformable fractional derivatives of some basic functions,

$$\begin{aligned} &(i)\ D^\alpha(\varkappa^p) = p\varkappa^{p-\alpha},\\ &(ii)\ D^\alpha(\sin(\tfrac{1}{\alpha}\varkappa^\alpha)) = \cos(\tfrac{1}{\alpha}\varkappa^\alpha),\\ &(iii)\ D^\alpha(\cos(\tfrac{1}{\alpha}\varkappa^\alpha)) = -\sin(\tfrac{1}{\alpha}\varkappa^\alpha),\\ &(iv)\ D^\alpha(e^{\frac{1}{\alpha}\varkappa^\alpha}) = e^{\frac{1}{\alpha}\varkappa^\alpha}. \end{aligned} \tag{5}$$

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives. For more on fractional calculus and its applications we refer to[8] and.[9] Many real world problems can be modeled utilizing fractional differential equations.

Let us write $D_s^{\alpha}u$ and $D_t^{\alpha}u$ to denote the partial α-conformable fractional derivative with respect to s and t respectively. Moreove we write $D_s^{2\alpha}u$ to denote $D_s^{\alpha}D_s^{\alpha}u$ and similarly, for $D_t^{2\alpha}u$.

Definition 2.8. [7] The α-fractional integral of a function f starting from $\mathbf{a} \geq 0$ is denoted by $I_{\alpha}^{\mathbf{a}}(u)(\varkappa)$ such that

$$I_{\alpha}^{\mathbf{a}}(u)(\varkappa) = I_1^{\mathbf{a}}\left(\varkappa^{\alpha-1}u\right) = \int_{\mathbf{a}}^{\varkappa} \frac{u(t)}{t^{1-\alpha}}\, dt, , \tag{6}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.

3 Main Result

In this section, we aim to find the atomic solution of the following non-homogeneous $\alpha-$fractional ACP of order n

$$A_n u^{(n\alpha)}(t) + A_{n-1}u^{((n-1)\alpha)}(t) + \cdots + A_1 u^{(\alpha)}(t) + A_{\circ}u(t) = f(t), \tag{7}$$

where $A_n, A_{n-1}, \cdots, A_{\circ}$ are closed linear operators defined on a Banach space X such that $A_i : Dom(A_i) \subseteq X \to X$ (for $0 \leq i \leq n$), the unknown function $u : [0,\infty) \to X$ is n-times $\alpha-$fractional differentiable for $\alpha \in (0,1)$ with

$$A_i u^{(i\alpha)}(t) = A_i \underbrace{D^{\alpha}D^{\alpha}\cdots D^{\alpha}}_{i\text{- times}} u(t) \text{ , for } 0 \leq i \leq n,$$

and

$$Rang(u) \cup Rang(u^{(\alpha)}) \cup \cdots \cup Rang(u^{(n\alpha)}) \subseteq \bigcap_{i=0}^{n} Dom(A_i),$$

and the right hand side of (7) that is $f : [0,\infty) \to X$ is a given function.

In the following, our goal is to determine an atomic solution of (7). In order to do that, we state the following theorem.

Theorem 3.1. *Let X be a Banach space and let $\{A_{\circ},\ A_1, \cdots,\ A_n\}$ be a set of point-wise independent closed operator on X such that*

$$Rang(u) \cup Rang(u^{(\alpha)}) \cdots \cup Rang(u^{(n\alpha)}) \subseteq \bigcap_{i=0}^{n} Dom(A_i).$$

Then, for $\alpha \in (0,1)$, the following nonhomogeneous $\alpha-$fractional ACP of order n

$$A_n u^{(n\alpha)}(t) + A_{n-1}u^{((n-1)\alpha)}(t) + \cdots + A_1 u^{(\alpha)}(t) + A_{\circ}u(t) = f(t), \tag{8}$$

has an atomic solution.

Proof. Consider that

$$u(t) = \upsilon(t)x = \upsilon \otimes x \text{ and } f(t) = g(t)y = g \otimes y, \tag{9}$$

where both $\upsilon, g : [0,\infty) \to \mathbb{R}$ are scalar valued functions and $x, y \in X \setminus \{0\}$. Also, we assume that $g(t)$ and y are known. Now, by (9), the main $\alpha-$fractional *ACP* (8), can be transformed into the following tensor product form

$$\upsilon^{(n\alpha)}(t) \otimes A_n x + \upsilon^{((n-1)\alpha)}(t) \otimes A_{n-1}x + \cdots + \upsilon(t) \otimes A_{\circ}x = g(t) \otimes y. \tag{10}$$

Since $\{A_{\circ},\ A_1, \cdots,\ A_n\}$ is point-wise independent, then $\{A_{\circ}x,\ A_1x, \cdots,\ A_nx\}$ is an independent set in X.

Now, Define $x^* \in X^*$ on $Y = span\{A_{\circ}x,\ A_1x, \cdots,\ A_nx\}$ such that

$$x^*(A_{\circ}x) = x^*(A_1x) = \cdots = x^*(A_nx) = 1.$$

Now, since $Y = span\{A_\circ x,\ A_1 x, \cdots,\ A_2 x\}$ is, clearly, a finite dimensional subspace in X. Consequently, Y is closed subspace of X. Also, since every linear functional on a finite dimensional normed space is continuous, then it is bounded on Y. Thus, by the Hahn-Banach Theorem, x^* can be extended as a bounded linear functional on the whole space X. Therefore, (10) becomes

$$\upsilon^{(n\alpha)}(t) + \upsilon^{((n-1)\alpha)}(t) + \cdots + \upsilon(t) = g(t)\langle y, x^*\rangle . \tag{11}$$

But, $\langle y, x^*\rangle$ is a known real number because, we assume that y is given. Thus, we get

$$\upsilon^{(n\alpha)}(t) + \upsilon^{((n-1)\alpha)}(t) + \cdots + \upsilon(t) = \lambda(t), \tag{12}$$

where $\lambda(t) = g(t)\langle y, x^*\rangle$. Clearly, (12) is a non-homogeneous linear $\alpha-$fractional differential equation with constant coefficients of order n and can be solved by variation of parameters.

The associated characteristic equation is

$$r^n + r^{n-1} + \cdots + r + 1 = 0. \tag{13}$$

Hence, if all the roots $r_1, \cdots, r_{n-1}, r_n$ of (13)are distinct and real, then the homogeneous solution $\upsilon_\hbar(t)$ of (12) is

$$\upsilon_\hbar(t) = c_1 e^{r_1(t^\alpha)/\alpha} + \cdots + c_n e^{r_n(t^\alpha)/\alpha}, \tag{14}$$

where $c_1, \cdots, c_n$ are constants and

$$\upsilon_1(t) = e^{r_1(t^\alpha)/\alpha}, \upsilon_2(t) = e^{r_2(t^\alpha)/\alpha}, \cdots, \upsilon_n(t) = e^{r_n(t^\alpha)/\alpha}, \tag{15}$$

are the fundamental set of solutions to the associated homogeneous equation.

For the particular solution $\upsilon_p(t)$ of (12), we have

$$\upsilon_p(t) = p_1(t)\upsilon_1(t) + p_2(t)\upsilon_2(t) + \cdots + p_n(t)\upsilon_n(t), \tag{16}$$

such that

$$p_i(t) = \int \frac{\lambda(t) W_i^\alpha(t)}{W^\alpha(t) t^{1-\alpha}} dt,\ i = 1, \cdots, n \tag{17}$$

where $W^\alpha(t) = W^\alpha(\upsilon_1, \cdots, \upsilon_n)(t)$ and $W_i^\alpha(t)$ is the determinant obtained by replacing i^{th} column of $W^\alpha(t)$ with $(0, 0, \cdots, 0, 1)$. So, the general solution $\upsilon_g(t)$ of (12) can be given in the form

$$\begin{aligned} \upsilon_g(t) &= \upsilon_\hbar(t) + \upsilon_p(t) \\ &= \sum_{i=1}^{n} c_i \upsilon_i(t) + \sum_{i=1}^{n} p_i(t)\upsilon_i(t), \end{aligned} \tag{18}$$

where for $i = 1, \cdots, n$ both $\upsilon_i(t)$ and $p_i(t)$ are, respectively, given in (15) and (17).

Now, back to (9). In order to obtain the value of the unknown function $u(t)$ which is the solution to (8), we need to find x.

Consider (10) for $t_\circ \in [0, \infty)$ as

$$\upsilon^{(n\alpha)}(t_\circ) A_n x + \upsilon^{((n-1)\alpha)}(t_\circ) A_{n-1} x + \cdots + \upsilon(t_\circ) A_\circ x = g(t_\circ) y, \tag{19}$$

where $\upsilon(t_\circ) \neq 0$. Choose $x^* \in X^*$ such that $x^*(A_\circ x) = 1$ and $x^*(A_k x) = 0$ for all $1 \leq k \leq n$ where $\upsilon^{(i\alpha)}(t_\circ) \neq 0$ for all $1 \leq i \leq n$ and $g(t_\circ) \neq 0$. So, in this case, we can write (19) as

$$\upsilon(t_\circ)\langle A_\circ x, x^*\rangle = g(t_\circ)\langle y, x^*\rangle . \tag{20}$$

Consequently,

$$\langle x, A_\circ^*(x^*)\rangle = \frac{g(t_\circ)}{\upsilon(t_\circ)}\langle y, x^*\rangle , \tag{21}$$

by which the value of x can be obtained and then the atomic solution $u(t) = \upsilon(t)x$ to equation (8) is fully obtained by taking into account (18).

4 Conclusion

In this paper, we derive atomic solutions of a non-homogeneous $\alpha-$fractional ACP of order n. The suggested $\alpha-$fractional ACP is assumed to be non-homogeneous where the involved operators are closed and linear on a given Banach space and the unknown function is assumed to be n-times differentiable. The tensor product techniques are combined with the Hahn–Banach theorem and the conformable fractional derivative. Moreover, the point-wise independent property of closed operators is utilized. The present work is limited to the conformable fractional derivative and extensions to other fractional operators remain open. More importantly, the analytic framework developed here opens the way to neutrosophic extensions, where uncertainty and indeterminacy are explicitly taken into account. Future work may focus on adapting the atomic solution method to neutrosophic fractional differential equations and related models in applied sciences.

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