A Conceptual Approach for Algebraic Structure of Multi-Neutrosophic BCI/BCK Algebras

Omaima Al-Shanqiti[1], **Santhakumar S.**[2,*], **Sumathi I. R.**[2]
[1]Umm Al-Qura University, Department of Mathematics, Mekkah, Saudia Arabia
[2]Department of Mathematics, Amrita School of Physical Sciences, Coimbatore,
Amrita Vishwa Vidyapeetham, India
Emails: omshanqiti@uqu.edu.sa; s_santhakumar@cb.amrita.edu; ir_sumathi@cb.amrita.edu

Abstract

A multi-neutrosophic set is a collection in which each element has a vector of truth indeterminacy, and falsity membership degree, rather than a Neutrosophic set. These vectors may correspond to multiple criteria, perspectives, or layers of information. Multi-neutrosophic sets are a more adaptive strategy for handling ambiguity in complex systems because they broaden neutrosophic sets and allow for better modeling of uncertain information. In this study, we have proposed the fundamental structure of multi-neutrosophic BCI/BCK Algebra and extended it to the category of multi-neutrosophic BCI(BCK) algebras. Theoretical results are presented along with examples. This study advances algebraic structure to multi-neutrosophic set and provides novel directions for future research in non-classical logic.

Keywords: Neutrosophic set; multi-neutrosophic set; multi-neutrosophic BCI/BCK algebra; Category of multi-neutrosophic BCI/BCK algebra

1 Introduction

Neutrosophic Set[3] is a mathematical tool to handle ambiguous, inconsistent, and incomplete data which is a generalization of Fuzzy Set (FS)[1] and Intuitionistic Fuzzy Set (IFS).[2] Many theoretical and practical approaches to Neutrosophic logic have emerged recently, gaining more attention among researchers. K. Iseki proposed the concept of BCK-algebras in 1996, and BCI-algebras are algebraic representations of set differences with characteristics based on set theory and implicational functors.[4,5] The notion of this BCI algebra was extended from classical set to fuzzy set in.[6,7,9] In,[8,10] intuitionistic fuzzification of subalgebras and ideals in BCK-algebras were discussed along with their properties. In recent decades,

DOI: https://doi.org/10.54216/IJNS.270232

researchers have focused on the algebraic properties of neutrosophic logic. The notion of BCI/BCK algebra was expanded in the neutrosophic environment.The articles[11–20] explores Neutrosophic BCI/BCK algebra, NeutroHyperGroups, MBJ-neutrosophic hyper BCK-ideal, Category of SuperHyper BCI-Algebra and BMBJ-Neutrosophic Hyper-BCK-Ideals.[21–23] describes Smarandache's SuperHyper Algebras and SuperHyperGraph, which inspired the research of hyperstructures. In 2023,[24] Smarandache coined the concept of multi-neutrosophic which is an analytical tool for tackling complicated phenomena characterized by contradictions and multiple layers of ambiguity. In,[26] the expansions of the MultiNeutrosophic Set and the Refined Neutrosophic Set through the use of HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets were investigated with comprehensive analysis and illustrative examples.

In,[27] the concept of partner sets for generalizing multi-neutrosophic sets is presented, which describes the method of determining the priority of the membership function, as well as the requirements and attributes that might be selected. multi-neutrosophic sets that describe the method of determining the priority of the membership function, as well as the requirements and attributes that might be selected. The multi-neutrosophic ARAS is applied to detect inconsistencies and uncertainty in sensory perceptions, with a focus on responses that are opposed. It incorporates neutrosophy with sensory marketing, providing practical principles for balanced multisensory experiences and customization that increase customer involvement in competitive marketplaces.[28] Additionally, it enhances organizational resilience in volatile environments by integrating predictive financial analysis, risk management, innovation, and adaptability.[29] As only a few studies have been published in the multi-neutrosophic set, we have proposed a novel algebraic structure called Multi-Neutrosophic BCI/BCK Algebras. Also, we have discussed some of its properties along with examples. The paper's structure is as follows: The key concepts are presented in the preliminaries section 2. In section3, we have developed multi-neutrosophic BCI/BCK algebra, and its characteristics are established. In Section 4, the category of multi-neutrosophic BCI/BCK algebra and its features were studied.An application of multi-neutrosophic set and BCI alebra in decision making problem is given in 5. Finally in section6, we have highlighted our substantial results and proposed prospects for further research.

2 Preliminary Concepts

This section contains definitions of fundamental concepts pertinent to the research.

Definition 2.1. [3]

For a given universal set S, a neutrosophic set A on S is collection of quadruples which is given as $A = \{(x, T_A(x), I_A(x), F_A(x)) | x \in S\}$, where $T_A, I_A, F_A : S \to [0,1]$, $T(x)$ reflects the degree of truth membership, $I(x)$ indicates the degree of indeterminacy, and $F(x)$ represents the degree of falsity and $0 \leq T_A(x), I_A(x), F_A(x) \leq 3$.

Definition 2.2. [24]

For any set S, a multi-neutrosophic set A on S is given by $A = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}),$ $I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}),) | \tilde{\varrho} \in S\}$, where $T_i : S \to [0,1]$ are represents truth membership functions, $I_j : S \to [0,1]$ are represents indeterminacy membership functions and $F_k : S \to [0,1]$ are represents falsity membership functions respectively, $p, q, r \geq 0$, $p + q + r = n$, at least one p, q, r is ≥ 2, a $0 \leq \sum_{i=1}^{p} inf(T_i(\tilde{\varrho})) + \sum_{j=1}^{q} inf(I_A(\tilde{\varrho})) + \sum_{k=1}^{r} inf(F_A(\tilde{\varrho})) \leq \sum_{i=1}^{p} sup(T_i(\tilde{\varrho})) + \sum_{j=1}^{q} sup(I_A(\tilde{\varrho})) + \sum_{k=1}^{r} sup(F_A(\tilde{\varrho})) \leq n.$

Definition 2.3. [4,5]
A BCI-Algebra is a non-empty set B with the binary operation ’$\boxdot$’ and the constant 0, if it satisfies the following identities.

1. $((b_1 \boxdot b_2) \boxdot (b_1 \boxdot b_3)) \boxdot (b_3 \boxdot b_2) = 0$
2. $(b_1 \boxdot (b_1 \boxdot b_2)) \boxdot b_2 = 0$
3. $b_1 \boxdot b_1 = 0$
4. $b_1 \boxdot b_2 = 0$ and $b_2 \boxdot b_1 = 0 \implies b_1 = b_2$

$\forall b_1, b_2, b_3 \in B$.

A BCI-Algebra B satifying the additional identity $0 \boxdot b_1 = 0$ then B is called as BCK -Algebra

3 Multi-neutrosophic BCI/BCK Algebra

In this part, we present the Multi-neutrosophic BCI/BCK Algebra and deduce some theorems.

Definition 3.1.
Let A be a multi-neutrosophic set on a BCI(BCK) algebra B, of the form $A = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$. Then A is said to Multi-neutrosophic BCI(BCK) sub-algebra if $T_i(\tilde{\varrho} \boxdot \tilde{\nu}) \geq T_i(\tilde{\varrho}) \wedge T_i(\tilde{\nu})$ for all $i = 1, 2, 3, ...p$, $I_j(\tilde{\varrho} \boxdot \tilde{\nu}) \geq I_j(\tilde{\varrho}) \wedge I_j(\tilde{\nu})$ for all $j = 1, 2, 3, ...q$ and $F_k(\tilde{\varrho} \boxdot \tilde{\nu}) \leq F_k(\tilde{\varrho}) \vee F_k(\tilde{\nu})$ for all $k = 1, 2, 3, ...r$.

Note: Every Multi-neutrosophic BCK sub-algebra on a BCK-algebra is Multi-neutrosophic BCI sub-algebra on a BCI-algebra. The following example establishes the existence of Multi-neutrosophic BCK sub-algebra

Example 3.2.
Let $B = \{0, 1\}$ be a BCI(BCK)-algebra with binary operation $\oslash : B \times B \to B$ defined as in the table below:

$\oslash$	0	1
0	0	0
1	1	0

Let A be a multi-neutrosophic set defined by

A	$T_1(\tilde{\varrho})$	$T_2(\tilde{\varrho})$	$T_3(\tilde{\varrho})$	$I_1(\tilde{\varrho})$	$I_2(\tilde{\varrho})$	$F_1(\tilde{\varrho})$	$F_2(\tilde{\varrho})$
0	0.8	0.3	0.7	0.4	0.7	0.5	0.3
1	0.6	0.1	0.5	0.3	0.5	0.6	0.6

Then we can easily verify that, A is Multi-neutrosophic BCI(BCK) sub-algebra of B.

Example 3.3.
Consider the BCI (BCK) algebra $B = \{0, \varrho_1, \varrho_2, \varrho_3\}$ with the Cayley's table

$\boxdot$	0	ϱ_1	ϱ_2	ϱ_3
0	0	0	0	0
ϱ_1	ϱ_1	0	0	ϱ_1
ϱ_2	ϱ_2	ϱ_1	0	ϱ_2
ϱ_3	ϱ_3	ϱ_3	ϱ_3	0

Also consider the multi-neutrosophic set A_1 on B defined by

A_1	$T_1(\tilde{\varrho})$	$T_2(\tilde{\varrho})$	$T_3(\tilde{\varrho})$	$I_1(\tilde{\varrho})$	$I_2(\tilde{\varrho})$	$I_3(\tilde{\varrho})$	$F_1(\tilde{\varrho})$	$F_2(\tilde{\varrho})$	$F_3(\tilde{\varrho})$
0	0.8	0.9	0.5	0.7	0.9	0.5	0.2	0.1	0.1
ϱ_1	0.7	0.5	0.4	0.5	0.9	0.3	0.4	0.3	0.5
ϱ_2	0.6	0.4	0.2	0.5	0.8	0.2	0.5	0.6	0.6
ϱ_3	0.5	0.1	0.1	0.2	0.8	0.1	0.6	0.8	0.9

Then we can easily verify that, A_1 is Multi-neutrosophic BCI(BCK) sub-algebra of B.

Now consider the multi-neutrosophic set A_2 on B defined by

A_2	$T_1(\tilde{\varrho})$	$T_2(\tilde{\varrho})$	$T_3(\tilde{\varrho})$	$I_1(\tilde{\varrho})$	$I_2(\tilde{\varrho})$	$I_3(\tilde{\varrho})$	$F_1(\tilde{\varrho})$	$F_2(\tilde{\varrho})$	$F_3(\tilde{\varrho})$
0	0.8	0.9	0.5	0.7	0.9	0.5	0.5	0.1	0.6
ϱ_1	0.7	0.5	0.4	0.5	0.2	0.3	0.4	0.3	0.5
ϱ_2	0.6	0.4	0.2	0.5	0.4	0.2	0.5	0.6	0.6
ϱ_3	0.5	0.1	0.1	0.2	0.8	0.1	0.6	0.8	0.9

Consider $F_1(\varrho_1 \boxdot \varrho_1) = F_1(0) = 0.5$ and $F_1(\varrho_1) \vee F_1(\varrho_1) = 0.4$. Clearly, $F_1(\varrho_1 \boxdot \varrho_1) \nleq F_1(\varrho_1) \vee F_1(\varrho_1)$. Hence, A_2 is not Multi-neutrosophic BCI(BCK) sub-algebra of B.

Theorem 3.4.
Let $A_1 = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho})) | \tilde{\varrho} \in B\}$ *be a multi-neutrosophic BCI(BCK) sub-algebras of a BCI(BCK) algebra* B*. Then,* $T_i(0) \geq T_i(\tilde{\varrho})$*,* $I_j(0) \leq I_j(\tilde{\varrho})$ *and* $F_k(0) \leq F_k(\tilde{\varrho})$ *for all* $\tilde{\varrho}$*,* $1 \leq i \leq p$*,* $1 \leq j \leq q$ *and* $1 \leq k \leq r$ *.*

Proof. Since $\tilde{\varrho} \boxdot \tilde{\varrho} = 0$ for all $\tilde{\varrho}$. Then $T_i(\tilde{\varrho}) = T_i(\tilde{\varrho}) \wedge T_i(\tilde{\varrho}) \leq T_i(\tilde{\varrho} \boxdot \tilde{\varrho}) = T_i(0)$ for all $\tilde{\varrho}$, $1 \leq i \leq p$. Similarly, we can prove the indetermincay and falsity membership. □

The following theorem shows that the intersection of two Multi-neutrosophic BCI(BCK) algebra defined in[24] is again a Multi-neutrosophic BCI(BCK) algebra

Theorem 3.5.
Let $A_1 = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$ *and* $A_2 = \{(\tilde{\varrho}, T'_1(\tilde{\varrho}), T'_2(\tilde{\varrho}), ..., T'_p(\tilde{\varrho}), I'_1(\tilde{\varrho}), I'_2(\tilde{\varrho}), ..., I'_q(\tilde{\varrho}), F'_1(\tilde{\varrho}), F'_2(\tilde{\varrho}), ..., F'_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$ *be two multi-neutrosophic BCI(BCK) sub-algebras of a BCI(BCK) algebra* B*. Then, the intersection of* A_1 *and* A_2 *is again a multi-neutrosophic BCI(BCK) sub-algebra.*

Proof. Let $\tilde{\varrho}$ & $\tilde{\nu} \in B$,
1) For truth values:

$$\begin{aligned}(T_i \cap T'_i)(\tilde{\varrho} \boxdot \tilde{\nu}) &= T_i(\tilde{\varrho} \boxdot \tilde{\nu}) \wedge T'_i(\tilde{\varrho} \boxdot \tilde{\nu}) \\ &\geq T_i(\tilde{\varrho}) \wedge T_i(\tilde{\nu}) \wedge T'_i(\tilde{\varrho}) \wedge T'_i(\tilde{\nu}) \\ &= T_i(\tilde{\varrho}) \wedge T'_i(\tilde{\varrho}) \wedge T_i(\tilde{\nu}) \wedge T'_i(\tilde{\nu}) \\ &= (T_i \cap T'_i)(\tilde{\varrho}) \wedge (T_i \cap T'_i)(\tilde{\nu})\end{aligned}$$

2) For indeterminancy values:

$$(I_i \cap I'_i)(\tilde{\varrho} \boxdot \tilde{\nu}) = I_i(\tilde{\varrho} \boxdot \tilde{\nu}) \vee I'_i(\tilde{\varrho} \boxdot \tilde{\nu})$$
$$(I_i \cap I'_i)(\tilde{\varrho}) \wedge (I_i \cap I'_i)(\tilde{\nu}) = [I_i(\tilde{\varrho}) \vee I'_i(\tilde{\varrho})] \wedge [I_i(\tilde{\nu}) \vee I'_i(\tilde{\nu})]$$

Since $I_i(\tilde{\varrho} \boxdot \tilde{\nu}) \geq I_i(\tilde{\varrho}) \wedge I_i(\tilde{\nu})$, $I'_i(\tilde{\varrho} \boxdot \tilde{\nu}) \geq I'_i(\tilde{\varrho}) \wedge I'_i(\tilde{\nu})$, and

$$\begin{aligned}(I_i \cap I'_i)(\tilde{\varrho}) \wedge (I_i \cap I'_i)(\tilde{\nu})) &= [I_i(\tilde{\varrho}) \wedge I_i(\tilde{\nu})] \vee [I_i(\tilde{\varrho}) \wedge I'_i(\tilde{\nu})] \vee [I'_i(\tilde{\varrho}) \wedge I_i(\tilde{\nu})] \vee [I'_i(\tilde{\varrho}) \wedge I'_i(\tilde{\nu})] \\ &\leq [I_i(\tilde{\varrho}) \wedge I_i(\tilde{\nu})] \vee [I'_i(\tilde{\varrho}) \wedge I_i(\tilde{\nu})] \\ &\leq I_i(\tilde{\varrho} \boxdot \tilde{\nu}) \vee I'_i(\tilde{\varrho} \boxdot \tilde{\nu}) \\ &= (I_i \cap I'_i)(\tilde{\varrho} \boxdot \tilde{\nu})\end{aligned}$$

Hence $(I_i \cap I'_i)(\tilde{\varrho} \boxdot \tilde{\nu}) \leq (I_i \cap I'_i)(\tilde{\varrho}) \wedge (I_i \cap I'_i)(\tilde{\nu})$.
3) Similarly we can prove for false values. Hence the theorem. □

Here, we define a new notion of intersection between two Multi-neutrosophic sets namely $I-intersetion$, which differs by the indeterminacy from the classical one introduced in.[24]

Definition 3.6.
Let A_1 and A_2 be two multi-neutrosophic set on a set S where $A_1 = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}),)|\tilde{\varrho} \in B\}$ and $A_2 = \{(\tilde{\varrho}, T'_1(\tilde{\varrho}), T'_2(\tilde{\varrho}), ..., T'_p(\tilde{\varrho}), I'_1(\tilde{\varrho}), I'_2(\tilde{\varrho}), ..., I'_q(\tilde{\varrho}), F'_1(\tilde{\varrho}), F'_2(\tilde{\varrho}), ..., F'_r(\tilde{\varrho}),)|\tilde{\varrho} \in B\}$. Then, the I-intersection of A_1 and A_2, defined by $A_1 \cap_I A_2 = \{(\tilde{\varrho}, T_1(\tilde{\varrho}) \wedge T'_1(\tilde{\varrho}), T_2(\tilde{\varrho}) \wedge T'_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}) \wedge T'_p(\tilde{\varrho}), I_1(\tilde{\varrho}) \wedge I'_1(\tilde{\varrho}), I_2(\tilde{\varrho}) \wedge I'_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}) \wedge I'_q(\tilde{\varrho}), F_1(\tilde{\varrho}) \vee F'_1(\tilde{\varrho}), F_2(\tilde{\varrho}) \vee F'_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}) \vee F'_r(\tilde{\varrho}),)|\tilde{\varrho} \in B\}$.

Theorem 3.7.
Let $A_1 = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$ *and* $A_2 = \{(\tilde{\varrho}, T'_1(\tilde{\varrho}), T'_2(\tilde{\varrho}), ..., T'_p(\tilde{\varrho}), I'_1(\tilde{\varrho}), I'_2(\tilde{\varrho}), ..., I'_q(\tilde{\varrho}), F'_1(\tilde{\varrho}), F'_2(\tilde{\varrho}), ..., F'_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$ *be two multi-neutrosophic BCI(BCK) sub-algebras of a BCI(BCK) algebra* B*. Then, the* I*-intersection of* A_1 *and* A_2 *is again a multi-neutrosophic BCI(BCK) sub-algebra.*

Proof. The truth and False value cases are same as the proof of Theorem 3.5. We need to prove the indeterminacy case only. Let $\tilde{\varrho}$ & $\tilde{\nu} \in B$,

$$(I_i \cap_I I'_i)(\tilde{\varrho}) \wedge (I_i \cap_I I'_i)(\tilde{\nu}) = [I_i(\tilde{\varrho}) \wedge I'_i(\tilde{\varrho})] \wedge [I_i(\tilde{\nu}) \wedge I'_i(\tilde{\nu})]$$

Since $I_i(\tilde{\varrho} \boxdot \tilde{\nu}) \geq I_i(\tilde{\varrho}) \wedge I_i(\tilde{\nu})$, $I'_i(\tilde{\varrho} \boxdot \tilde{\nu}) \geq I'_i(\tilde{\varrho}) \wedge I'_i(\tilde{\nu})$, then

$$\begin{aligned}(I_i \cap_I I'_i)(\tilde{\varrho} \boxdot \tilde{\nu}) &= I_i(\tilde{\varrho} \boxdot \tilde{\nu}) \wedge I'_i(\tilde{\varrho} \boxdot \tilde{\nu}) \\ &\geq [I_i(\tilde{\varrho}) \wedge I_i(\tilde{\nu})] \wedge [I'_i(\tilde{\varrho}) \wedge I'_i(\tilde{\nu})]\end{aligned}$$

Hence $(I_i \cap I'_i)(\tilde{\varrho} \boxdot \tilde{\nu}) \leq (I_i \cap I'_i)(\tilde{\varrho}) \wedge (I_i \cap I'_i)(\tilde{\nu})$. Hence the theorem. □

Definition 3.8.
Let $A = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$ be a multi-neutrosophic sub-set of B. Then, the $(t_1, t_2, ..., t_p, i_1, i_2, ...i_q, f_1, f_2, ..., f_r)$-level set of A is a subset B' of B, $B' = \{\tilde{\varrho}|T_i(\tilde{\varrho}) \geq l_i, I_j(\tilde{\varrho}) \geq m_j \; and \; F_k(\tilde{\varrho}) \leq n_k\}$.

Note: Let B' be a $(t_1, t_2, ..., t_p, i_1, i_2, ...i_q, f_1, f_2, ..., f_r)$-level set of a multi-neutrosophic sub-set $A = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$ of B. Then the notation $A|_{B'}$ denotes the multi-neutrosophic sub-set with

$$T_i^{(B')}(\tilde{\varrho}) = \begin{cases} T_i & \tilde{\varrho} \in B' \\ 0 & \tilde{\varrho} \in (B - B') \end{cases}$$

$$I_j^{(B')}(\tilde{\varrho}) = \begin{cases} I_j & \tilde{\varrho} \in B' \\ 0 & \tilde{\varrho} \in (B - B') \end{cases}$$

$$F_k^{(B')}(\tilde{\varrho}) = \begin{cases} F_k & \tilde{\varrho} \in B' \\ 1 & \tilde{\varrho} \in (B - B') \end{cases}$$

of B. That means, the elements of $B - B'$ have all the truth and indeterminate memberships are zero and the false memberships are one. Also, the elements of B' hold the membership as there in A.

Theorem 3.9.
Let B' be a $(t_1, t_2, ..., t_p, i_1, i_2, ...i_q, f_1, f_2, ..., f_r)$-level set of multi-neutrosophic BCI(BCK) sub-algebra $A = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$ of a BCI(BCK) algebra B.. Then, $A|_{B'}$ multi-neutrosophic BCI(BCK) sub-algebras of B.

Proof. Let $\tilde{\varrho}, \tilde{\nu} \in B'$, then

$$\begin{aligned}T_i^{(B')}(\tilde{\varrho} \boxdot \tilde{\nu}) &= T_i(\tilde{\varrho} \boxdot \tilde{\nu}) \\ &\geq T_i(\tilde{\varrho}) \wedge T_i(\tilde{\nu}) \\ &= T_i^{(B')}(\tilde{\varrho}) \wedge T_i^{(B')}(\tilde{\nu}).\end{aligned}$$

Let $\tilde{\varrho} \in B - B'$ and $\tilde{\nu} \in B'$, then

$$\begin{aligned} T_i^{(B')}(\tilde{\varrho} \boxdot \tilde{\nu}) &\geq 0 \\ &= 0 \wedge T_i^{(B')}(\tilde{\nu}) \\ &= T_i^{(B')}(\tilde{\varrho}) \wedge T_i^{(B')}(\tilde{\nu}). \end{aligned}$$

Similar approach can be applied for I_j and F_k. Hence, $A|_{B'}$ multi-neutrosophic BCI(BCK) sub-algebras of B. □

4 Category of Multi-neutrosophic BCI/BCK Algebra

This section introduces the category of multi-neutrosophic BCI/BCK algebra and few results are derived.

4.1 Morphisms for Category of Multi-neutrosophic BCI/BCK Algebra

In this subsection we introduce the multi-neutrosophic homomorphism and multi-neutrosophic anti-homomorphism between two multi-neutrosophic BCI/BCK algebras.

Definition 4.1.
Let $A_1 = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B_1\}$ and $A_2 = \{(\tilde{\varrho}, T'_1(\tilde{\varrho}), T'_2(\tilde{\varrho}), ..., T'_p(\tilde{\varrho}), I'_1(\tilde{\varrho}), I'_2(\tilde{\varrho}), ..., I'_q(\tilde{\varrho}), F'_1(\tilde{\varrho}), F'_2(\tilde{\varrho}), ..., F'_r(\tilde{\varrho}))|\tilde{\varrho} \in B_2\}$ be two multi-neutrosophic BCI(BCK) sub-algebras of a BCI(BCK) algebras B_1 and B_2, respectively. Then a homomorphism $f : B_1 \to B_2$ is said to be a multi-neutrosophic homomorphism if

$$\begin{aligned} T_i(\tilde{\varrho} \boxdot \tilde{\nu}) &\leq T'_i(f(\tilde{\varrho}) \boxdot f(\tilde{\nu})),\ 1 \leq i \leq p \\ I_j(\tilde{\varrho} \boxdot \tilde{\nu}) &\leq I'_j(f(\tilde{\varrho}) \boxdot f(\tilde{\nu})),\ 1 \leq j \leq q \\ F_k(\tilde{\varrho} \boxdot \tilde{\nu}) &\geq F'_k(f(\tilde{\varrho}) \boxdot f(\tilde{\nu})),\ 1 \leq k \leq r. \end{aligned}$$

A homomorphism $f : B_1 \to B_2$ is said to be a multi-neutrosophic anti-homomorphism if

$$\begin{aligned} T_i(\tilde{\varrho} \boxdot \tilde{\nu}) &\geq T'_i(f(\tilde{\varrho}) \boxdot f(\tilde{\nu})),\ 1 \leq i \leq p \\ I_j(\tilde{\varrho} \boxdot \tilde{\nu}) &\geq I'_j(f(\tilde{\varrho}) \boxdot f(\tilde{\nu})),\ 1 \leq j \leq q \\ F_k(\tilde{\varrho} \boxdot \tilde{\nu}) &\leq F'_k(f(\tilde{\varrho}) \boxdot f(\tilde{\nu})),\ 1 \leq k \leq r \end{aligned}$$

Example 4.2.
Let $A_1 = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), T_2(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), I_2(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), F_2(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B\}$ be a multi-neutrosophic BCI(BCK) sub-algebra of a BCI(BCK) algebra B. Then the identity $i : B \to B$ is a multi-neutrosophic homomorphism and anti-homomorphism.

Theorem 4.3.
Let $A_1 = \{(\tilde{\varrho}, T_1^1(\tilde{\varrho}), T_2^1(\tilde{\varrho}), ..., T_p^1(\tilde{\varrho}), I_1^1(\tilde{\varrho}), I_2^1(\tilde{\varrho}), ..., I_q^1(\tilde{\varrho}), F_1^1(\tilde{\varrho}), F_2^1(\tilde{\varrho}), ..., F_r^1(\tilde{\varrho}))|\tilde{\varrho} \in B_1\}$,
$A_2 = \{(\tilde{\varrho}, T_1^2(\tilde{\varrho}), T_2^2(\tilde{\varrho}), ..., T_p^2(\tilde{\varrho}), I_1^2(\tilde{\varrho}), I_2^2(\tilde{\varrho}), ..., I_q^2(\tilde{\varrho}), F_1^2(\tilde{\varrho}), F_2^2(\tilde{\varrho}), ..., F_r^2(\tilde{\varrho}))|\tilde{\varrho} \in B_2\}$ *and*

$A_3 = \{(\tilde{\varrho}, T_1^3(\tilde{\varrho}), T_2^3(\tilde{\varrho}), ..., T_p^3(\tilde{\varrho}),\ I_1^3(\tilde{\varrho}), I_2^3(\tilde{\varrho}), ..., I_q^3(\tilde{\varrho}), F_1^3(\tilde{\varrho}), F_2^3(\tilde{\varrho}), ..., F_r^3(\tilde{\varrho}))|\tilde{\varrho} \in B_3\}$ *be three multi-neutrosophic BCI(BCK) sub-algebras of a BCI(BCK) algebras* B_1*,* B_2*,* B_3*, respectively. Also, let* $f_1 : B_1 \to B_2$ *and* $f_1 : B_2 \to B_3$ *be two multi-neutrosophic (anti-)homomorphism. Then the composition* $f_2 \circ f_1$ *is again a multi-neutrosophic (anti-)homomorphism.*

Proof. For any $\tilde{\varrho}, \tilde{\nu} \in B_1$,

$$\begin{aligned} T_i^1(\tilde{\varrho} \boxdot \tilde{\nu}) &\leq T_i^2(f_1(\tilde{\varrho}) \boxdot f_1(\tilde{\nu})) \\ &\leq T_i^3((f_2 \circ f_1)(\tilde{\varrho}) \boxdot (f_2 \circ f_1)f(\tilde{\nu})) \end{aligned}$$

Similarly, $I_j^1(\tilde{\varrho} \boxdot \tilde{\nu}) \leq I_j^3((f_2 \circ f_1)(\tilde{\varrho}) \boxdot (f_2 \circ f_1)f(\tilde{\nu}))$ and $F_k^1(\tilde{\varrho} \boxdot \tilde{\nu}) \leq F_k^3((f_2 \circ f_1)(\tilde{\varrho}) \boxdot (f_2 \circ f_1)f(\tilde{\nu}))$. Hence, the composition $f_2 \circ f_1$ is again a multi-neutrosophic homomorphism.

Dual proof can be employed for multi-neutrosophic anti-homomorphism. □

4.2 Different Categories by Multi-neutrosophic BCI/BCK Algebras

In this subsection we discuss different variety of categories by the concept of multi-neutrosophic BCI/BCK algebras.

Theorem 4.4.
Collection of all multi-neutrosophic BCI(BCK) sub-algebras of a BCI(BCK) algebra B *along with multi-neutrosophic homomorphism form a category.*

Proof. By Theorem 4.3 and Example 4.2 the proof follows. □

Theorem 4.5.
Collection of all multi-neutrosophic BCI(BCK) sub-algebras of a BCI(BCK) algebra B *along with multi-neutrosophic anti-homomorphism form a category.*

Proof. Applying Theorem 4.3 and Example 4.2 the result can be proved. □

Theorem 4.6.
Collection of all multi-neutrosophic BCI(BCK) sub-algebras of all BCI(BCK) algebras along with multi-neutrosophic homomorphism form a category.

Proof. Using Theorem 4.3 and Example 4.2 the result can be proved. □

Theorem 4.7.
Collection of all multi-neutrosophic BCI(BCK) sub-algebras of all BCI(BCK) algebras along with multi-neutrosophic anti-homomorphism form a category.

Proof. By Theorem 4.3 and Example 4.2 the proof follows. □

Note: In all the above categories, the multi-neutrosophic BCI(BCK) sub-algebras with same number of truth, Indeterminacy and False membership functions, only can have morphism between them. Two multi-neutrosophic BCI(BCK) sub-algebras with different number of truth or Indeterminacy or False membership functions can't have a morphism between them.

4.3 Functors Between Categories of Multi-neutrosophic BCI/BCK Algebras by Onto Homomorphism

In this subsection we explored some functors between categories of multi-neutrosophic BCI/BCK algebras.

Suppose B_1 and B_2 are two BCI(BCK) algebras and $f : B_1 \to B_2$ is an onto homomorphism from B_1 to B_2. Then f can be extend to a functor from the category of multi-neutrosophic BCI(BCK) sub-algebras of B_1 to the category of multi-neutrosophic BCI(BCK) sub-algebras of B_2 by the following construction.

For any $A_1 = \{(\tilde{\varrho}, T_1(\tilde{\varrho}), ..., T_p(\tilde{\varrho}), I_1(\tilde{\varrho}), ..., I_q(\tilde{\varrho}), F_1(\tilde{\varrho}), ..., F_r(\tilde{\varrho}))|\tilde{\varrho} \in B_1\}$ multi-neutrosophic BCI (BCK) sub-algebra of B_1, we will define a multi-neutrosophic BCI(BCK) sub-algebra A_2 of B_2 as follows.

For any element $\varrho \in B_2$, define

$$T'_i(\varrho) = \wedge_{\tilde{\varrho}\in f^{-1}(\varrho)} T_i(\tilde{\varrho}), I'_j(\varrho) = \wedge_{\tilde{\varrho}\in f^{-1}(\varrho)} I_j(\tilde{\varrho}) \text{ and } F'_k(\varrho) = \vee_{\tilde{\varrho}\in f^{-1}(\varrho)} F_k(\tilde{\varrho})$$

. Then, $A_2 = \{(\varrho, T'_1(\varrho), T'_2(\varrho), ..., T'_p(\varrho), I'_1(\varrho), I'_2(\varrho), ..., I'_q(\varrho), F'_1(\varrho), F'_2(\varrho), ..., F'_r(\varrho))|\varrho \in B_2\}$ is a multi-neutrosophic set on B_2.

Theorem 4.8.
For any multi-neutrosophic BCI(BCK) sub-algebra A_1 of BCI(BCK) algebra B_1, the multi-neutrosophic set A_2 defined as above is a multi-neutrosophic BCI(BCK) sub-algebra of B_2.

Proof. For any two element ϱ & $\nu \in B_2$, $T'_i(\varrho \boxdot_2 \nu) = \wedge_{\tilde{\mu}\in f^{-1}(\varrho\boxdot_2\nu)} T_i(\tilde{\mu})$ and $T'_i(\varrho) \wedge T'_i(\nu) = \left[\wedge_{\tilde{\mu}\in f^{-1}(\varrho)} T_i(\tilde{\mu})\right] \wedge \left[\wedge_{\tilde{\mu}\in f^{-1}(\nu)} T_i(\tilde{\mu})\right]$. For any $\tilde{\varrho} \in f^{-1}(\varrho)$ & $\tilde{\nu} \in f^{-1}(\nu)$ we have that $f(\tilde{\varrho} \boxdot_1 \tilde{\nu}) = f(\tilde{\varrho}) \boxdot_2 f(\tilde{\nu}) = \varrho \boxdot_2 \nu$. i.e., $f(\tilde{\varrho} \boxdot_1 \tilde{\nu}) = \varrho \boxdot_2 \nu$. Hence, $T'_i(\varrho \boxdot_2 \nu) \le T_i(\tilde{\varrho} \boxdot_1 \tilde{\nu}) \le T_i(\tilde{\varrho}) \wedge T_i(\tilde{\nu})$ for all $\tilde{\varrho} \in f^{-1}(\varrho)$ & $\tilde{\nu} \in f^{-1}(\nu)$. Therefore, $T'_i(\varrho \boxdot_2 \nu) \le T'_i(\varrho) \wedge T'_i(\nu)$. Similarly, we can prove in the case of Indeterminacy and false values. □

From this argument, we can define a functor $\mathcal{F}$ between the category of multi-neutrosophic BCI(BCK) sub-algebras of B_1 to the category of multi-neutrosophic BCI(BCK) sub-algebras of B_2. For any multi-neutrosophic BCI(BCK) sub-algebra A_1 of BCI(BCK) algebra B_1, the image under $\mathcal{F}$ is A_2 of B_2 defined above.

Theorem 4.9.
The function $\mathcal{F}$ defined above is a functor between the category of multi-neutrosophic BCI(BCK) sub-algebras of B_1 to the category of multi-neutrosophic BCI(BCK) sub-algebras of B_2.

Proof. It is direct from the above discussion. □

4.4 Comparing Categories of Multi-neutrosophic BCI/BCK Algebras with other existing Categories

When we compare the categories of Multi-neutrosophic BCI/BCK sub algebras with other we will get the following results.

Theorem 4.10.
Category of BCI/BCK Algebra is the subcategory of category of all multi-neutrosophic BCI(BCK) sub-algebras of all BCI(BCK) algebras.

Proof. For any BCI/BCK algebra to every element assign truth values are 1, Indeterminacy and False value are 0. Then It is a multi-neutrosophic BCI/BCK Algebra. By this we will get the theorem. ☐

Theorem 4.11.
Category of neutrosophic BCI(BCK) algebra is the subcategory of category of all multi-neutrosophic BCI(BCK) sub-algebras of all BCI(BCK) algebras.

Proof. For any neutrosophic BCI/BCK algebra is a special case of multi-neutrosophic BCI/BCK Algebra. By this we will get the theorem. . ☐

5 Application in Decsion Making:

In this section, we have discussed a decision-making problem using multi-neutrosophic set with BCI algebra.
Algorithm :

- Step 1: Consider a set of alternatives $A = \{A_1, \ldots, A_n\}$ and for each alternative A_i, the criterion $C_j = \{C_1, \ldots, C_m\}$ is given with a multineutrosophic value.
- Step 2 : Convert a multineutronosophic set to a single-valued neutrosophic set by applying the classical average to truth, indeterminacy, and falsehood.
- Step 3: Construct Cayley table for BCI algebra $X = \{0, 1, 2, 3\}$.
- Step 4: Calculate the score function using the definition[24] and map the score to the element in X by defining the threshold.
- Step 5: Aggregate the Criterion using cayley table by (((C1 * C2) * C3)* C4).
- Step 6: Choose the best alternative.

5.1 Illustrative example:

Step 1: Consider Let $MN = \{A1, A2, A3, A4\}$ represent a collection of candidates. Several panels (information sources and experts) assess their interview performance. The selection criteria include technical proficiency (C1), strategic business acumen (C2), problem-solving and critical thinking (C3), and communication and collaboration (C4). Panel members come from the following departments. T-{Tech Experts- E1, E2, E2}, and H-{HR Professionals, H1, H2}. S-{Senior Managers (S1, S2, S3)} P- Product Manager, P1.

Assume that four panels, E1, E2, E3, and P1, evaluate the degrees of positive knowledge (truth) of the candidate's technical knowledge, and Panel E1 assigns the value T1 to all candidates; Panel E2 assigns the value T2 to all candidates; Panel E3 assigns the value T3, and P1 assigns the value T4 to all candidates, as follows:

$$A1(T1 = 0.8, T2 = 0.8, T3 = 0.7, T4 = 0.7); A2(T1 = 0.6, T2 = 0.55, T3 = 0.6, T4 = 0.65)$$
$$A3(T1 = 0.65, T2 = 0.6, T3 = 0.6, T4 = 0.55); A4(T1 = 0.5, T2 = 0.3, T3 = 0.4, T4 = 0.4)$$

However, two panels, H1 and H2, are unsure about the applicants' technical performance and provide ambiguous degrees (I1 and I2, respectively) to the students.

$$A1(I1 = 0.3, I2 = 0.1); A2(I1 = 0.2, I2 = 0.4);$$
$$A3(I1 = 0.4, I2 = 0.4); A4(I1 = 0.5, I2 = 0.7)$$

Furthermore, three S1, S2, and S3, who are displeased with the applicants' technical performance, give negative evaluations (falsehood degrees), F1, F2, and F3, respectively:

$$A1(F1 = 0.2, F2 = 0.2, F3 = 0.2); A2(F1 = 0.3, F2 = 0.4, F3 = 0.2);$$
$$A3(F1 = 0.4, F2 = 0.5, F3 = 0.3); A4(F1 = 0.6, F2 = 0.65, F3 = 0.7).$$

The aforementioned assignment has been done for Criteria 1. The following multi-neutosophic set, where each element has the form based on the criteria 1 is provided by

$$\begin{aligned} MN1 = \{ & A1(\{0.8, 0.8, 0.7, 0.7\}, \{0.3, 0.1\}\{0.2, 0.2, 0.2\}), \\ & A2(\{0.7, 0.6, 0.65, 0.65\}, \{0.3, 0.3\}\{0.3, 0.25, 0.35\}), \\ & A3(\{0.2, 0.25, 0.25, 0.1\}, \{0.77, 0.83\}\{0.7, 0.85, 0.85\}), \\ & A4(\{0.8, 0.8, 0.7, 0.5\}, \{0.3, 0.5\}\{0.2, 0.2, 0.5\})\} \end{aligned}$$

Similarly, we can assign the other criterion 2 based on expertise. Assume four panels (P1, S1, S2, and S3) evaluate the candidate's Strategic Business Acumen in terms of positive knowledge (truthfulness). H1 and H2 assign an indeterminacy degree, whereas E1, E2, and E3 assign a falsity degree. The multineutrophic set, based on criterion 2, is provided by

$$\begin{aligned} MN2 = \{ & A1(\{0.6, 0.55, 0.6, 0.65\}, \{0.3, 0.1\}\{0.3, 0.4, 0.2\}); \\ & A2(\{0.7, 0.65, 0.7, 0.75\}, \{0.2, 0.2\}\{0.15, 0.1, 0.05\}), \\ & A3(\{0.5, 0.45, 0.5, 0.55\}, \{0.5, 0.5\}\{0.5, 0.4, 0.6\}), \\ & A4(\{0.6, 0.55, 0.60.25\}, \{0.55, 0.65\}\{0.3, 0.4, 0.5\})\} \end{aligned}$$

Likewise the foru panels S1, S2 , S3 and P1 evaluate the degrees of positive knowledge (truth) of the candidate's Problem-Solving and Critical Thinking indeterminacy by H1 and H2 and falsity by E1, E2, and E3. The multi neutrosophic set, based on the criteria 3 is given by

$$\begin{aligned} MN3 =\{&A1(\{0.65, 0.6, 0.6, 0.55\}, \{0.4, 0.4\}\{0.4, 0.5, 0.3\}),\\ &A2(\{0.55, 0.55, 0.45, 0.45\}, \{0.55, 0.65\}\{0.75, 0.55, 0.8\}),\\ &A3(\{0.8, 0.8, 0.75, 0.85\}, \{0.3, 0.1\}\{0.25, 0.35, 0.3\}),\\ &A4(\{0.65, 0.6, 0.6, 0.55\}, \{0.6, 0.6\}\{0.6, 0.6, 0.6\})\} \end{aligned}$$

For Criterion 4 the three panels H1, H2, and P1 evaluate the degrees of positive knowledge (truth) of the candidate's communication and collaboration, indeterminacy by E1, E2, and E3, and falsity by S1, S2, and S3. The multi-neutrosophic set is given by

$$\begin{aligned} MN4 =\{&A1(\{0.5, 0.3, 0.4\}, \{0.5, 0.7, 0.6\}, \{0.6, 0.65, 0.7\}),\\ &A2(\{0.65, 0.6, 0.55\}, \{0.45, 0.35, 0.4\}, \{0.35, 0.4, 0.45\}),\\ &A3(\{0.7, 0.65, 0.75\}, \{0.25, 0.15, 0.2\}, \{0.2, 0.15, 0.25\}),\\ &A4(\{0.6, 0.7, 0.5\}, \{0.3, 0.4, 0.4\}\{0.3, 0.4, 0.2\})\} \end{aligned}$$

Step 2: Convert multi-neutrosophic to single valued neutrosophic by calculating the average of truth, indeterminacy, and falsity.[24] The table below shows the single valued neutrosophic set with the corresponding truth, indeterminacy, and falsity values for each alternative with respect to the criterion.

Table 1

Criterion	A1 (T, I, F)	A2 (T, I, F)	A3 (T, I, F)	A4 (T, I, F)
C1	(0.75,0.2,0.2)	(0.65,0.3,0.3)	(0.2,0.8,0.8)	(0.7,0.4.0.3)
C2	(0.6,0.3,0.3)	(0.7,0.2,0.1)	(0.5,0.5,0.5)	(0.5,0.6,0.4)
C3	(0.6,0.4,0.4)	(0.5,0.6,0.7)	(0.8,0.2,0.3)	(0.6,0.6,0.6)
C4	(0.4,0.6,0.65)	(0.6,0.4,0.4)	(0.7,0.2,0.2)	(0.6,0.4,0.3)

Step 3: Consturt the cayley for BCI Algebra X = {0, 1, 2, 3}.

Table 2: BCI -Algebra (X, *, 0)

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Step 4: Calculate the score function for the table 1. The score function is assigned to four levels(L) ie., 3 - Excellent; 2- Good ; 1 - Average and 0 - Poor. The classification is if $S \geq 0.75$ then L = 3 (Excellent); $0.60 \leq S < 0.75$ then L= 2 (Good); $0.40 \leq S < 0.60$ L =1 (Average) and $0.0 \leq S < 0.40$ then L= 0 (Poor).

Table 3: Score of each criterion with level

Alternative	C1 [Level]	C2 [Level]	C3 [Level]	C4 [Level]	[Level Vector]
$A1$	0.7833 [3]	0.6667 [2]	0.6 [2]	0.3833 [0]	[3 2 2 0]
$A2$	0.6833 [2]	0.8 [3]	0.4 [1]	0.6 [2]	[2 3 1 2]
$A3$	0.3667 [0]	0.5 [1]	0.7667 [3]	0.7667 [3]	[0 1 3 3]
$A4$	0.6667 [2]	0.5 [1]	0.4667 [1]	0.6333 [2]	[2 1 1 2]

Step 5: Finally we aggregate the criterion for each alternative by applying (((C1 * C2) * C3)* C4) from table 2.

Table 4: Aggrgation of the criterion for each alternative

Alternative	Criterion level vector	(((C1 * C2) * C3)* C4)-Final Level
$A1$	[3 2 2 0]	3-Excellent
$A2$	[2 3 1 2]	2-Good
$A3$	[0 1 3 3]	1-Average
$A4$	[2 1 1 2]	0-Poor

Step 6: From the table 4 we conclude A1 is the best alternative.

Remark 5.1. Suppose there is a tie in the alternatives; we can apply the accuracy function and certainty function as in.[24] The above example is for the criterion having equal importance. We can also assign the weight for each criterion based on its importance, and we can perform the same by using the weighted average and corresponding score function, accuracy function, and certainty function.[24]

6 Conclusion

In this article, we have presented a novel algebraic structure of the multi-neutrosophic set. Further, we have discussed some of its characteristics with illustrative examples. Also, we have put forward the concept of category in the multi-neutrosophic BCI/BCK algebra.The characteristics of category of multi-neutrosophis BCI/BCK algebra is discussed.Moreover, we can develop this concept in hyperstructure and Superhyper and may expand the notion in the soft set, rough set, etc. From a theoretical perspective, further studies may focus on other algebraic structures such as BL-algebra, MV-algebra, etc., on multineutrosophic set. In addition, representation theorems and complexity analysis offer rich avenues for advancing the foundational understanding of these BCI/BCK algebraic systems." Combining the suggested algebraic structure with machine learning techniques may result in models that balance interpretability and prediction accuracy.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8(3):338–353.

[2] Atanassov, K. T. (1986). Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems, 20(1):87–96.

[3] Hossain, M. S., Rahman, M. M. (2022). Neutrosophic set theory and its applications in decision-making problems. Journal of Applied Mathematics and Computational Mechanics, 21(1), 15-26.

[4] Iséki, K. (1966). An algebra related with a propositional calculus. Proceedings of the Japan Academy, 42(1), 26-29.

[5] Yasuyuki, I. M. A. I. (1966). On Axiom Systems of Propositional Calculi. XIV. Proceedings of the Japan Academy, 42(1), 19-22.

[6] Jun, Y. B., & Xin, X. L. (2001). Involutory and invertible fuzzy BCK-algebras. Fuzzy sets and systems, 117(3), 463-469.

[7] Jun, Y. B., & Xin, X. L. (2001). Fuzzy prime ideals and invertible fuzzy ideals in BCK-algebras. Fuzzy sets and systems, 117(3), 471-476.

[8] Jun, Y. B., & Kim, K. H. (2000). Intuitionistic fuzzy ideals of BCK-algebras. International journal of mathematics and mathematical sciences, 24(12), 839-849.

[9] Zhan, J., Hamidi, M., and Borumand Saeid, A. (2016). Extended Fuzzy BCK-Subalgebras. Iranian Journal of Fuzzy Systems, 13(4):125–144.

[10] Jana, C., Senapati, T., & Pal, M. (2016). $(\epsilon, \epsilon \vee q)$-intuitionistic fuzzy BCI-subalgebras of a BCI-algebra. Journal of Intelligent & Fuzzy Systems, 31(1), 613-621.

[11] Agboola, A. A. A., Davvaz, B., et al. (2015). Introduction to neutrosophic bci/bck-algebras. International Journal of Mathematics and Mathematical Sciences, 2015.

[12] Jun, Y. B. (2017). Neutrosophic subalgebras of several types in BCK/BCI-algebras. Annals of Fuzzy Mathematics and Informatics. Volume 14, No. 1, (2017), pp. 75–86

[13] Ozturk, M. A. and Jun, Y. B. (2018). Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points. Journal of the International Mathematical Virtual Institute Vol. 8(2018), 1-17

[14] Khademan, S., Zahedi, M., Borzooei, R., and Jun, Y. (2019). Neutrosophic Hyper BCK- ideals. Neutrosophic Sets and Systems, 27:201-217.

[15] Alharthy, K., Alshahrani, A., Khan, M. (2023). Analyzing neutrosophic structures in decision-making frameworks: A review. Journal of Computational and Theoretical Nanoscience, 20(4), 1234-1245.

[16] Bhatia, S., Kumar, R. (2021). Advanced neutrosophic algebraic structures and their applications in optimization problems. International Journal of Computational Mathematics, 99(10), 987-1004.

[17] Ibrahim, M. and Agboola, A. A. A. (2020). Introduction to NeutroHyperGroups. Neutrosophic Sets and Systems, 38:15–32.

[18] Alsubie, A., Al-Masarwah, A., Jun, Y. B., and Ahmad, A. G. (2021). An approach to BMBJ- Neutrosophic Hyper-BCK-Ideals of Hyper-BCK-Algebras. Journal of Mathematics, 1–10.

[19] Alsubie, A. and Al-Masarwah, A. (2021). MBJ-Neutrosophic Hyper BCK Ideals in Hyper BCK-Algebras. AIMS Mathematics, 6(6): 6107–6121.

[20] Santhakumar, S., Sumathi, I. R., & Mahalakshmi, J. (2023). A Novel Approach to the Algebraic Structure of Neutrosophic SuperHyper Algebra. Neutrosophic Sets and Systems, 60(1), 39.

[21] Smarandache, F. (2022). Introduction to SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra. Journal of Algebraic Hyperstructures and Logical Algebras, 3(2):17-24.

[22] Zhang, Y., Liu, X. (2024). Extensions of fuzzy algebraic structures: A comprehensive analysis. Journal of Mathematical Analysis and Applications, 510(2), 567-580.

[23] Smarandache, F. (2023a). New types of soft set "hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set": An improved version. Neutrosophic Systems with Applications, 8:35–41.

[24] Smarandache, F. (2023). Introduction to the multi-neutrosophic Set. Neutrosophic Sets and Systems, 61, 89-99.

[25] Santhakumar, S., Sumathi, I. R., & Mahalakshmi, J. (2025). A note on Category of SuperHyper BCI-Algebra. International Journal of Neutrosophic Science (IJNS), 25(2).

[26] Fujita, Takaaki & Smarandache, Florentin. (2025). Some Types of HyperNeutrosophic Set (6): multi-neutrosophic Set and Refined Neutrosophic Set.

[27] Al-Tamimi, T. A., Al-Swidi, L. A., & Al-Obaidi, A. H. (2024). Partner Sets for Generalizations of multi-neutrosophic Sets. International Journal of Neutrosophic Science (IJNS), 24(1).

[28] Zenozain, E. A. M., Delgado, R. M. C., & Rischmoller, J. C. V. (2025). Sensory Marketing Analysis: Exploring Uncertainty and Contradiction in Consumer Decision-Making. A multi-neutrosophic Analysis with ARAS. Neutrosophic Sets and Systems, 84, 175-187.

[29] Arias, E. A. O., Vargas, J. L. C., Sánchez, J. A. F., & Estrada, S. M. D. (2025). multi-neutrosophic Analysis of Financial Statements as a Basis for Strategic Decision-Making in Uncertain Environments. Neutrosophic Sets and Systems, 82, 370-380.