Extending Classical Uncertainty Models via Hyperpolar Structures: Fuzzy, Neutrosophic, and Soft Set Perspectives

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Abstract

Concepts such as the Fuzzy Set, Neutrosophic Set, and Soft Set are known for handling uncertainty. As extensions of Fuzzy Sets, Neutrosophic Sets, and Soft Sets, concepts such as Bipolar Fuzzy Sets, Bipolar Neutrosophic Sets, and Bipolar Soft Sets have been introduced. In this paper, we further extend these notions and explore Hyperpolar Fuzzy Sets, Hyperpolar Neutrosophic Sets, and Hyperpolar Soft Sets. These structures integrate multi-perspective or multi-agent evaluations into a unified framework by leveraging higher-dimensional mappings and hypercubic representations. This work lays a theoretical foundation for advanced uncertainty modeling in complex, multi-source environments.

Keywords: Soft Set; Fuzzy Set; Neutrosophic Set; Hyperpolar Fuzzy Set; Hyperpolar Neutrosophic Set; Hyperpolar Soft Set

1 Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. All concepts considered in this work are assumed to be finite.

1.1 Fuzzy Set and Bipolar Fuzzy Set

The concept of the Fuzzy Set is a foundational tool for addressing uncertainty in set theory. Its definition is provided below.[1] As extensions of the Fuzzy Set, Bipolar Fuzzy Set[2] and Multipolar Fuzzy Set[3,4] are well known. The definitions are presented below.

Definition 1.1 (Fuzzy Set). [1] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y,z) \le \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 1.2 (Movie Recommendation Based on Viewer Preferences Using a Fuzzy Set)**.** Consider a set of movies

$$Y = \{\text{Inception}, \text{ Titanic}, \text{ The Matrix}\}$$

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

and the property "preferred by a given viewer." We define a fuzzy set $\tau : Y \to [0,1]$, where $\tau(y)$ represents the degree of preference the viewer assigns to each movie, based on ratings, interest, and viewing history.

Suppose the data yields:

$$\tau(\text{Inception}) = 0.95, \quad \tau(\text{Titanic}) = 0.70, \quad \tau(\text{The Matrix}) = 0.85.$$

Here:

- A value close to 1 indicates strong preference (e.g., Inception with 0.95).
- A medium value (e.g., 0.70 for Titanic) suggests moderate interest.
- A high but not maximal value (0.85 for The Matrix) shows strong but not top preference.

This fuzzy representation enables a recommendation system to prioritize movies according to degrees of preference rather than binary like/dislike, supporting more personalized suggestions.

Definition 1.3 (Bipolar Fuzzy Set)**.** [3,5] Let X be a non-empty universe. A *bipolar fuzzy set* B in X is a pair of functions:

$$B = (\mu^+, \mu^-),$$

where:

- $\mu^+ : X \to [0,1]$ is the **positive membership function**, representing the degree to which an element belongs positively.
- $\mu^- : X \to [-1,0]$ is the **negative membership function**, representing the degree to which an element belongs negatively.

For any element $x \in X$, we have $\mu^+(x)$ indicating the degree of satisfaction of a property, and $\mu^-(x)$ indicating the degree of contradiction to that property.

The set of all bipolar fuzzy sets over X is denoted as $BF(X)$.

Example 1.4 (Restaurant Review Analysis Using a Bipolar Fuzzy Set)**.** Consider a set of restaurants

$$X = \{\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3\}$$

and the property "serves healthy and tasty food." We define a bipolar fuzzy set

$$B = (\mu^+, \mu^-)$$

where:

- $\mu^+(x)$ represents the degree to which a restaurant positively meets this property (healthy and tasty).
- $\mu^-(x)$ represents the degree to which it contradicts the property (unhealthy or tasteless), given in the range $[-1,0]$.

Suppose customer survey data yields:

$$\begin{aligned} \mu^+(\mathrm{R}_1) &= 0.85, & \mu^-(\mathrm{R}_1) &= -0.10, \\ \mu^+(\mathrm{R}_2) &= 0.60, & \mu^-(\mathrm{R}_2) &= -0.30, \\ \mu^+(\mathrm{R}_3) &= 0.40, & \mu^-(\mathrm{R}_3) &= -0.50. \end{aligned}$$

Here, R_1 is rated highly for both taste and health with minimal contradiction, while R_3 has low positive satisfaction and strong negative perception. This bipolar fuzzy representation enables decision-makers to evaluate both positive and negative aspects simultaneously, offering a balanced perspective in restaurant selection.

DOI: https://doi.org/10.54216/GJMSA.120202
Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Definition 1.5 (Multipolar Fuzzy Set). [3] Let X be a non-empty universe. An *m-polar fuzzy set* M over X is a mapping:

$$M : X \to [0,1]^m,$$

where each element $x \in X$ is assigned an m-tuple:

$$M(x) = (\mu_1(x), \mu_2(x), \ldots, \mu_m(x)).$$

Each $\mu_i(x)$ represents the degree of membership of x according to the i-th perspective, attribute, or agent.

Example 1.6 (Multipolar Fuzzy Set for Multi-Attribute Smartphone Evaluation). Smartphone evaluation assesses devices based on features like performance, battery life, camera quality, usability, and overall user satisfaction. Let

$$X = \{\text{Phone A}, \text{Phone B}, \text{Phone C}\}$$

be the set of smartphone models to be evaluated, and let $m = 3$ correspond to three quality attributes:

$$\mu_1(x) : \text{battery life}, \quad \mu_2(x) : \text{camera quality}, \quad \mu_3(x) : \text{durability}.$$

Define the multipolar fuzzy set

$$M : X \longrightarrow [0,1]^3$$

by assigning each phone an ordered triple of membership degrees:

$$M(\text{Phone A}) = (0.85,\ 0.65,\ 0.90),$$
$$M(\text{Phone B}) = (0.70,\ 0.80,\ 0.75),$$
$$M(\text{Phone C}) = (0.60,\ 0.55,\ 0.85).$$

Here:

- 0.85 for Phone A's *battery life* indicates very high endurance relative to competitors.
- 0.65 for Phone A's *camera quality* denotes moderate imaging performance.
- 0.90 for Phone A's *durability* reflects strong build robustness.

Similarly, Phone B and Phone C receive fuzzy evaluations on the same three criteria. Thus M is a 3-polar fuzzy set capturing multi-attribute user satisfaction in a single framework, facilitating aggregate decision-making or ranking by techniques such as weighted averaging or fuzzy TOPSIS.

1.2 Neutrosophic Set and Bipolar Neutrosophic Set

Neutrosophic Sets extend Fuzzy Sets by introducing the concept of indeterminacy, enabling the modeling of information that is not entirely true or false. This framework provides a richer and more adaptable representation of uncertainty and ambiguity.[6,7] Neutrosophic Sets and their related structures have been studied extensively across diverse fields, including artificial intelligence, decision-making, and linear programming.[8,9] A Bipolar Neutrosophic Set models both positive and negative degrees of truth, indeterminacy, and falsity for each element.[10,11]

Definition 1.7 (Neutrosophic Set). [12] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Example 1.8 (Medical Symptom Assessment Using a Neutrosophic Set)**.** Consider the problem of assessing whether a patient has influenza based on observed symptoms. Let

$$X = \{\text{Fever}, \text{Cough}, \text{Fatigue}\}$$

represent the set of observed symptoms. Define the neutrosophic set A = "Symptoms indicating influenza" by:

$$\begin{aligned}
T_A(\text{Fever}) = 0.9, \quad I_A(\text{Fever}) = 0.05, \quad F_A(\text{Fever}) = 0.05, \\
T_A(\text{Cough}) = 0.7, \quad I_A(\text{Cough}) = 0.2, \quad F_A(\text{Cough}) = 0.1, \\
T_A(\text{Fatigue}) = 0.6, \quad I_A(\text{Fatigue}) = 0.3, \quad F_A(\text{Fatigue}) = 0.1.
\end{aligned}$$

Here:

- The *truth degree* $T_A(x)$ quantifies how strongly the symptom suggests influenza.
- The *indeterminacy degree* $I_A(x)$ represents diagnostic uncertainty (e.g., due to overlapping symptoms with other illnesses).
- The *falsity degree* $F_A(x)$ measures how strongly the symptom suggests absence of influenza.

For example, "Fever" has a high truth degree (0.9) and very low indeterminacy and falsity, indicating strong evidence for influenza, whereas "Fatigue" has higher indeterminacy due to its occurrence in many unrelated conditions. This representation supports nuanced medical decision-making by accounting for truth, uncertainty, and counter-evidence simultaneously.

Definition 1.9 (Bipolar Neutrosophic Set)**.** [10,11] Let X be a non-empty universe. A *Bipolar Neutrosophic Set* A in X is defined as:

$$A = \left\{ \langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x) \rangle : x \in X \right\},$$

where the mappings

$$T^+, I^+, F^+ : X \to [0,1], \quad T^-, I^-, F^- : X \to [-1,0]$$

represent the degrees of **truth**, **indeterminacy**, and **falsehood** associated with both positive and negative perspectives of an element $x \in X$.

Example 1.10 (Online Product Review Sentiment Analysis)**.** Let

$$X = \{\text{User}_A,\ \text{User}_B,\ \text{User}_C\}$$

be the set of reviewers evaluating a new smartphone. We model their sentiments with a Bipolar Neutrosophic Set

$$A = \left\{ \langle x,\ T^+(x), I^+(x), F^+(x),\ T^-(x), I^-(x), F^-(x) \rangle : x \in X \right\},$$

where

$$T^+, I^+, F^+ : X \to [0,1], \quad T^-, I^-, F^- : X \to [-1,0].$$

Assign the following values:

$$\begin{aligned}
A = \{ &\langle \text{User}_A,\ 0.80,\ 0.10,\ 0.10,\ -0.10,\ -0.05,\ -0.05 \rangle, \\
&\langle \text{User}_B,\ 0.60,\ 0.30,\ 0.10,\ -0.20,\ -0.10,\ 0.00 \rangle, \\
&\langle \text{User}_C,\ 0.40,\ 0.20,\ 0.40,\ -0.30,\ -0.10,\ 0.00 \rangle \}.
\end{aligned}$$

For example, for User_A:

- $T^+ = 0.80$ indicates strong positive satisfaction (e.g. likes performance).
- $I^+ = 0.10$ indicates slight uncertainty (e.g. neutral on battery life).

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

- $F^+ = 0.10$ indicates low positive falsity.
- $T^- = -0.10$ indicates slight negative sentiment (e.g. minor frustration).
- $I^- = -0.05$ indicates slight negative uncertainty.
- $F^- = -0.05$ indicates low negative falsity.

This bipolar neutrosophic model captures both positive and negative aspects, along with indeterminacy, providing a nuanced analysis of online review sentiment.

1.3 Soft Set and Bipolar Soft Set

A *Soft Set* (F, E) associates each parameter in a set E with a subset of a universal set U. This provides a flexible framework for approximating objects within U.[13]

Definition 1.11 (Soft Set)**.** [13] Let U be a universal set and E a set of parameters. A *soft set* over U is defined as an ordered pair (F, E), where F is a mapping from E to the power set $\mathcal{P}(U)$:

$$F : E \to \mathcal{P}(U).$$

For each parameter $e \in E$, $F(e) \subseteq U$ represents the set of e-approximate elements in U, with (F, E) forming a parameterized family of subsets of U.

Definition 1.12 (Bipolar Soft Set)**.** [14,15] A *Bipolar Soft Set* over a universal set U is a triple (F, G, A), where:

- $F : A \to P(U)$ is the *positive membership mapping*,
- $G : \neg A \to P(U)$ is the *negative membership mapping*,
- $A \subseteq E$, $\neg A = E \setminus A$, where E is a set of parameters.

The mappings satisfy the *consistency constraint*:

$$F(e) \cap G(\neg e) = \emptyset, \quad \forall e \in A.$$

A Bipolar Soft Set is represented as:

$$(F, G, A) = \{(e, F(e), G(\neg e)) \mid e \in A, F(e) \cap G(\neg e) = \emptyset\}.$$

Example 1.13 (Job Candidate Selection Using a Bipolar Soft Set)**.** A job candidate is an individual applying for a specific position, evaluated based on qualifications, skills, and experience. Let

$$U = \{\text{Alice, Bob, Carol, Dave}\}$$

be the set of job candidates, and let the parameter set be

$$E = \{\text{Experienced}, \ \text{Skilled}, \ \text{Inexperienced}, \ \text{Unskilled}\}.$$

Choose the positive parameter subset

$$A = \{\text{Experienced}, \ \text{Skilled}\}, \quad \neg A = E \setminus A = \{\text{Inexperienced}, \ \text{Unskilled}\}.$$

Define the positive membership mapping

$$F : A \ \to \ \mathcal{P}(U), \quad F(\text{Experienced}) = \{\text{Alice, Bob}\}, \quad F(\text{Skilled}) = \{\text{Alice, Carol}\},$$

and the negative membership mapping

$$G : \neg A \ \to \ \mathcal{P}(U), \quad G(\text{Inexperienced}) = \{\text{Carol, Dave}\}, \quad G(\text{Unskilled}) = \{\text{Bob, Dave}\}.$$

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The *bipolar soft set* is then

$$(F, G, A) = \big\{(\text{Experienced}, \{\text{Alice}, \text{Bob}\}, G(\neg\,\text{Experienced})), (\text{Skilled}, \{\text{Alice}, \text{Carol}\}, G(\neg\,\text{Skilled}))\big\},$$

where $\neg$ Experienced = Inexperienced and $\neg$ Skilled = Unskilled.

Consistency Check:

$$F(\text{Experienced}) \cap G(\text{Inexperienced}) = \{\text{Alice}, \text{Bob}\} \cap \{\text{Carol}, \text{Dave}\} = \emptyset,$$

$$F(\text{Skilled}) \cap G(\text{Unskilled}) = \{\text{Alice}, \text{Carol}\} \cap \{\text{Bob}, \text{Dave}\} = \emptyset.$$

Thus the positive and negative memberships are disjoint for each parameter.

This bipolar soft set framework allows recruiters to capture both desirable qualifications (experience, skill) and undesirable traits (inexperience, lack of skill) in a single, parameterized model.

1.4 Hyperstructures

A hyperstructure extends classical algebraic structures by allowing the "product" of two elements to be a set of outcomes rather than a single value.

Definition 1.14 (Base Set)**.** A *base set* S is any nonempty set whose elements are the primitive objects of interest.

Definition 1.15 (Powerset)**.** The *powerset* of S, denoted $\mathcal{P}(S)$, is the set of all subsets of S:

$$\mathcal{P}(S) \;=\; \{A \mid A \subseteq S\}.$$

Definition 1.16 (Classical Structure)**.** A *classical structure* is a pair $(H, \{\star_i\}_{i\in I})$ where:

- H is a nonempty set.
- For each i, $\star_i \colon H^{m_i} \to H$ is an m_i-ary operation on H, satisfying specified axioms (e.g. associativity, identity).

Definition 1.17 (Hyperoperation)**.** A *hyperoperation* on a set S is a map

$$\circ\colon S \times S \;\longrightarrow\; \mathcal{P}(S), \quad (a, b) \mapsto a \circ b,$$

assigning to each pair (a, b) a subset $a \circ b \subseteq S$.

Definition 1.18 (Hyperstructure)**.** [16] A *hyperstructure* is a pair $(S, \circ)$ where S is a base set and $\circ$ is a hyperoperation on S. Equivalently, one may extend $\circ$ to subsets of S by

$$A \circ B = \bigcup_{a\in A} \bigcup_{b\in B} (a \circ b), \quad A, B \subseteq S,$$

yielding an induced operation $\mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S)$.

Example 1.19 (Resistor Tolerance Hyperstructure)**.** Resistor tolerance indicates the allowable deviation from its nominal resistance value, typically expressed as a percentage (cf.[17]). Let

$$S = \{\, 100\,\Omega \pm 5\%,\ 220\,\Omega \pm 5\%,\ 330\,\Omega \pm 5\%\}$$

be the set of nominal resistor values with $\pm 5\%$ tolerance. Define a hyperoperation

$$\otimes : S \times S \;\longrightarrow\; \mathcal{P}(\mathbb{R}^+)$$

by

$$R_1 \otimes R_2 = \big\{\, r_1 + r_2 \;\big|\; r_1 \in [R_1(1 - 0.05),\, R_1(1 + 0.05)],\ r_2 \in [R_2(1 - 0.05),\, R_2(1 + 0.05)]\big\}.$$

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Extend $\otimes$ to subsets $A, B \subseteq S$ by

$$A \otimes B = \bigcup_{R_i \in A} \bigcup_{R_j \in B} (R_i \otimes R_j).$$

Then

$$\mathcal{H} = (S, \otimes)$$

is a *hyperstructure*. For example,

$$(100\,\Omega \pm 5\%) \otimes (220\,\Omega \pm 5\%) = [95 + 209,\ 105 + 231] = [304\,\Omega,\ 336\,\Omega].$$

Thus $\mathcal{H}$ models all possible series-combination resistances within tolerance.

2 Results in This Paper

The results derived in this paper are presented below.

2.1 Review: Hyperpolar Fuzzy Set

A *Hyperpolar Fuzzy Set* is a generalized fuzzy set with multiple polarities, each assigning independent membership degrees in the unit interval $[0, 1]$ to capture multidimensional truth. The definition of the Hyperpolar Fuzzy Set is provided below (cf.[18]).

Definition 2.1 (Hyperpolar Fuzzy Set[18])**.** Let $X \neq \emptyset$ and $n \in \mathbb{N}_{\geq 1}$ be the number of *polarities*. For each $k \in \{1, \ldots, n\}$, fix an integer $\delta_k \in \mathbb{N}_{\geq 1}$ (the dimension of the k-th polarity) and set

$$\Delta \;=\; \prod_{k=1}^{n} [0, 1]^{\delta_k} \;=\; [0, 1]^{\delta_1} \times \cdots \times [0, 1]^{\delta_n}.$$

A *hyperpolar fuzzy set* on X is a mapping

$$H : X \longrightarrow \Delta, \qquad H(x) \;=\; \big(h^1(x),\, h^2(x),\, \ldots,\, h^n(x)\big),$$

where each block $h^k(x) \in [0, 1]^{\delta_k}$ records δ_k membership degrees for the k-th group of criteria. For convenience, write $H_k(x) := h^k(x)$.

Example 2.2 (Medical Diagnosis Using a Hyperpolar Fuzzy Set)**.** We evaluate three patients on three groups of clinical attributes (cf. [19,20]):

$$\delta_1 = 2 : \{\text{Glucose}, \text{Cholesterol}\},$$
$$\delta_2 = 3 : \{\text{Temperature}, \text{Heart Rate}, \text{Blood Pressure}\},$$
$$\delta_3 = 2 : \{\text{Stress}, \text{Sleep}\}.$$

Thus

$$H : X = \{\text{Patient}_1, \text{Patient}_2, \text{Patient}_3\} \longrightarrow [0, 1]^2 \times [0, 1]^3 \times [0, 1]^2,$$
$$H(x) = \big(H_1(x), H_2(x), H_3(x)\big),$$

with the following evaluations:

$$H(\text{Patient}_1) = \big((0.80, 0.50),\ (0.70, 0.90, 0.60),\ (0.40, 0.50)\big),$$
$$H(\text{Patient}_2) = \big((0.60, 0.70),\ (0.80, 0.60, 0.50),\ (0.30, 0.70)\big),$$
$$H(\text{Patient}_3) = \big((0.50, 0.40),\ (0.90, 0.80, 0.70),\ (0.60, 0.80)\big).$$

For instance,

$$H(\text{Patient}_1) = \Big(\underbrace{(0.80, 0.50)}_{\text{Glucose, Cholesterol}},\underbrace{(0.70, 0.90, 0.60)}_{\text{Temp, HR, BP}},$$
$$\underbrace{(0.40, 0.50)}_{\text{Stress, Sleep}}\Big).$$

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Example 2.3 (Student Performance Assessment Using a Hyperpolar Fuzzy Set)**.** We evaluate three students on the categories

$$\delta_1 = 3 : \{\text{Mathematics, Science, Literature}\},$$
$$\delta_2 = 2 : \{\text{Sports, Arts}\},$$
$$\delta_3 = 3 : \{\text{Communication, Teamwork, Leadership}\}.$$

Define $H : X \to [0,1]^3 \times [0,1]^2 \times [0,1]^3$ by

$$\begin{aligned} H(\text{Student}_1) &= \big((0.90, 0.70, 0.80),\ (0.60, 0.70),\ (0.80, 0.90, 0.70)\big),\\ H(\text{Student}_2) &= \big((0.80, 0.90, 0.70),\ (0.50, 0.60),\ (0.90, 0.70, 0.60)\big),\\ H(\text{Student}_3) &= \big((0.70, 0.60, 0.90),\ (0.80, 0.90),\ (0.60, 0.80, 0.90)\big).\end{aligned}$$

For example,

$$H(\text{Student}_1) = \Big(\underbrace{(0.90, 0.70, 0.80)}_{\text{Math, Sci, Lit}},$$
$$\underbrace{(0.60, 0.70)}_{\text{Sports, Arts}}, \underbrace{(0.80, 0.90, 0.70)}_{\text{Comm, Team, Lead}}\Big).$$

Example 2.4 (Urban District Evaluation with a Hyperpolar Fuzzy Set)**.** We assess three districts (cf. [21]) on

$$\delta_1 = 2 : \{\text{Employment, GDP per Capita}\},$$
$$\delta_2 = 3 : \{\text{Air, Green, Water}\},$$
$$\delta_3 = 2 : \{\text{Crime}^{-1}, \text{Education}\}.$$

Let $H : X \to [0,1]^2 \times [0,1]^3 \times [0,1]^2$ be given by

$$\begin{aligned} H(\text{District}_1) &= \big((0.70, 0.80),\ (0.60, 0.90, 0.50),\ (0.40, 0.80)\big),\\ H(\text{District}_2) &= \big((0.90, 0.60),\ (0.70, 0.80, 0.90),\ (0.50, 0.70)\big),\\ H(\text{District}_3) &= \big((0.60, 0.70),\ (0.90, 0.50, 0.60),\ (0.80, 0.90)\big).\end{aligned}$$

For instance,

$$H(\text{District}_1) = \Big(\underbrace{(0.70, 0.80)}_{\text{Employment, GDP}}, \underbrace{(0.60, 0.90, 0.50)}_{\text{Air, Green, Water}}, \underbrace{(0.40, 0.80)}_{\text{Crime}^{-1}, \text{Education}}\Big).$$

Theorem 2.5 (Hyperpolar Fuzzy Sets Generalize Multipolar and Bipolar Fuzzy Sets)**.** *Let $X \neq \emptyset$.*

1. *Any m-polar fuzzy set $M : X \to [0,1]^m$ is a hyperpolar fuzzy set with $n = 1$ and $\delta_1 = m$.*
2. *Any bipolar fuzzy set $B = (\mu^+, \mu^-)$ with $\mu^+ : X \to [0,1]$ and $\mu^- : X \to [-1,0]$ can be encoded as a hyperpolar fuzzy set with $n = 2$, $\delta_1 = \delta_2 = 1$, via*

$$H(x) \ = \ \big(\mu^+(x),\ -\mu^-(x)\big) \in [0,1] \times [0,1].$$

Proof. (1) Put $n = 1$, $\delta_1 = m$. Then $\Delta = [0,1]^m$ and $H := M$ is hyperpolar by definition.

(2) With $n = 2$, $\delta_1 = \delta_2 = 1$ we have $\Delta = [0,1] \times [0,1]$. Since $\mu^+(x) \in [0,1]$ and $-\mu^-(x) \in [0,1]$, the map $H(x) = (\mu^+(x), -\mu^-(x))$ takes values in Δ. Conversely, given $H(x) = (h_1, h_2)$, one reconstructs $\mu^+(x) = h_1$ and $\mu^-(x) = -h_2$. □

Theorem 2.6 (Closure under Componentwise Union and Intersection)**.** *Let $H, G : X \to \Delta = \prod_{k=1}^{n} [0,1]^{\delta_k}$ be hyperpolar fuzzy sets. Define, for each $x \in X$,*

$$(H \vee G)(x) := \big(H_1(x) \vee G_1(x),\ \ldots,\ H_n(x) \vee G_n(x)\big), \qquad (H \wedge G)(x) := \big(H_1(x) \wedge G_1(x),\ \ldots,\ H_n(x) \wedge G_n(x)\big),$$

where $\vee, \wedge$ act componentwise *on $[0,1]^{\delta_k}$ as* max *and* min*, respectively. Then $H \vee G,\ H \wedge G : X \to \Delta$ are hyperpolar fuzzy sets.*

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Proof. Fix $x \in X$, $k \in \{1, \ldots, n\}$, and a coordinate $j \in \{1, \ldots, \delta_k\}$. Since $H_{k,j}(x), G_{k,j}(x) \in [0, 1]$,

$$0 \le \min\{H_{k,j}(x), G_{k,j}(x)\} \le \max\{H_{k,j}(x), G_{k,j}(x)\} \le 1.$$

Thus the coordinatewise max and min remain in $[0, 1]$, hence $(H \vee G)(x), (H \wedge G)(x) \in \Delta$. □

Theorem 2.7 (Closure under Complement)**.** *Let $H : X \to \Delta$. Define $H^c : X \to \Delta$ by*

$$H^c(x) := \big(\mathbf{1}_{\delta_1} - H_1(x),\, \mathbf{1}_{\delta_2} - H_2(x),\, \ldots,\, \mathbf{1}_{\delta_n} - H_n(x)\big),$$

where $\mathbf{1}_{\delta_k} \in [0, 1]^{\delta_k}$ is the all-ones vector and subtraction is componentwise. Then H^c is hyperpolar.

Proof. Each coordinate $H_{k,j}(x) \in [0, 1]$ implies $1 - H_{k,j}(x) \in [0, 1]$. Hence $H^c(x) \in \Delta$ for all $x \in X$. □

Theorem 2.8 (Projection to Individual Polarities)**.** *Let $H : X \to \Delta = \prod_{k=1}^{n} [0, 1]^{\delta_k}$. For each k, let $\pi_k : \Delta \to [0, 1]^{\delta_k}$ be the projection $\pi_k(v_1, \ldots, v_n) = v_k$. Then $H^{(k)} := \pi_k \circ H : X \to [0, 1]^{\delta_k}$ is a δ_k-polar (multipolar) fuzzy set.*

Proof. By construction, $H^{(k)}(x) = H_k(x) \in [0, 1]^{\delta_k}$ for all $x \in X$. □

Theorem 2.9 (Hyperpolar Fuzzy Sets Form a Hyperstructure)**.** *Let $\mathcal{H} = \{H : X \to \Delta\}$. Define a hyperoperation $\star : \mathcal{H} \times \mathcal{H} \to \mathcal{P}(\mathcal{H})$ by*

$$H \star G := \big\{\, K : X \to \Delta \;\big|\; \forall x \in X,\ K(x) \in \{H(x), G(x)\} \,\big\}.$$

Then $(\mathcal{H}, \star)$ is a hyperstructure.

Proof. If $K \in H \star G$, then for each x, $K(x) \in \Delta$, so $K \in \mathcal{H}$. Thus $H \star G \subseteq \mathcal{H}$, i.e., $\star$ maps into $\mathcal{P}(\mathcal{H})$. Extending $\star$ to subsets by $\mathcal{A} \star \mathcal{B} = \bigcup_{H \in \mathcal{A},\, G \in \mathcal{B}} (H \star G)$ yields a bona fide hyperoperation on $\mathcal{P}(\mathcal{H})$. □

2.2 Hyperpolar Neutrosophic Set

A *Hyperpolar Neutrosophic Set* extends hyperpolar fuzzy sets by incorporating multidimensional truth, indeterminacy, and falsity degrees, enabling richer and more flexible modeling of uncertainty. The definition and concrete examples of the Hyperpolar Neutrosophic Set are presented as follows.

Definition 2.10 (Hyperpolar Neutrosophic Set)**.** Let $X \neq \emptyset$ and $n \in \mathbb{N}_{\ge 1}$. For each $k \in \{1, \ldots, n\}$, fix integers $\delta_k^T, \delta_k^I, \delta_k^F \in \mathbb{N}_{\ge 0}$ (the respective dimensions of the true, indeterminate, and false blocks at polarity k). Define the product hypercubes

$$\Delta^T := \prod_{k=1}^{n} [0, 1]^{\delta_k^T}, \qquad \Delta^I := \prod_{k=1}^{n} [0, 1]^{\delta_k^I}, \qquad \Delta^F := \prod_{k=1}^{n} [0, 1]^{\delta_k^F}.$$

A *Hyperpolar Neutrosophic Set (HPNS)* on X is a mapping

$$H : X \longrightarrow \Delta^T \times \Delta^I \times \Delta^F, \qquad H(x) = \big(T(x),\, I(x),\, F(x)\big),$$

where $T(x) \in \Delta^T$, $I(x) \in \Delta^I$, and $F(x) \in \Delta^F$. We write

$$T(x) = \big(T_1(x), \ldots, T_n(x)\big), \quad I(x) = \big(I_1(x), \ldots, I_n(x)\big), \quad F(x) = \big(F_1(x), \ldots, F_n(x)\big),$$

with $T_k(x) \in [0, 1]^{\delta_k^T}$, $I_k(x) \in [0, 1]^{\delta_k^I}$, $F_k(x) \in [0, 1]^{\delta_k^F}$. (When some $\delta_k^{\bullet} = 0$, the corresponding factor is $[0, 1]^0 = \{()\}$, a singleton.)

DOI: https://doi.org/10.54216/GJMSA.120202
Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Example 2.11 (Multi-Sensor Object Detection in Autonomous Vehicles)**.** Multi-sensor object detection fuses heterogeneous sensors to identify and classify objects with improved reliability (cf. [22]). Let

$$X = \{\text{Pedestrian}, \text{Car}\}, \qquad n = 2,$$

corresponding to LiDAR ($k = 1$) and camera ($k = 2$). Take $\delta_1^T = \delta_1^I = \delta_1^F = \delta_2^T = \delta_2^I = \delta_2^F = 1$, so $\Delta^T = \Delta^I = \Delta^F = [0,1]^2$. Define $H : X \to \Delta^T \times \Delta^I \times \Delta^F$ by

$$T(x) = (T_1(x), T_2(x)), \quad I(x) = (I_1(x), I_2(x)), \quad F(x) = (F_1(x), F_2(x)).$$

Numerical assessments:

$$T(\text{Pedestrian}) = (0.95, 0.85), \quad I(\text{Pedestrian}) = (0.03, 0.10), \quad F(\text{Pedestrian}) = (0.02, 0.05);$$

$$T(\text{Car}) = (0.90, 0.98), \quad I(\text{Car}) = (0.05, 0.02), \quad F(\text{Car}) = (0.05, 0.00).$$

Thus,

$$H(\text{Pedestrian}) = \big((0.95, 0.85), (0.03, 0.10), (0.02, 0.05)\big),$$

$$H(\text{Car}) = \big((0.90, 0.98), (0.05, 0.02), (0.05, 0.00)\big).$$

Each object is represented by sensor-wise degrees of truth, indeterminacy, and falsity.

Example 2.12 (Smart Home Intrusion Detection via HPNS)**.** Consider three modalities (cf. [23,24]): motion ($k = 1$), door ($k = 2$), and camera ($k = 3$). Let

$$X = \{\text{Event}_1, \text{Event}_2\}, \qquad n = 3,$$

$$\delta_k^T = \delta_k^I = \delta_k^F = 1 \ \ (k = 1, 2, 3),$$

so $\Delta^T = \Delta^I = \Delta^F = [0,1]^3$. For each $x \in X$,

$$T(x) = (T_1(x), T_2(x), T_3(x)), \quad I(x) = (I_1(x), I_2(x), I_3(x)), \quad F(x) = (F_1(x), F_2(x), F_3(x)).$$

Numerical assessments:

$$H(\text{Event}_1) = \big((0.90, 0.75, 0.60), (0.05, 0.15, 0.25), (0.05, 0.10, 0.15)\big),$$

$$H(\text{Event}_2) = \big((0.20, 0.85, 0.95), (0.30, 0.10, 0.03), (0.50, 0.05, 0.02)\big).$$

Interpretation (for Event_1): motion $T_1 = 0.90, I_1 = 0.05, F_1 = 0.05$ (high confidence, low uncertainty/false alarm); door $T_2 = 0.75, I_2 = 0.15, F_2 = 0.10$; camera $T_3 = 0.60, I_3 = 0.25, F_3 = 0.15$.

Theorem 2.13 (HPNS Generalizes Hyperpolar Fuzzy and Classical Neutrosophic Sets)**.** *Let $X \neq \emptyset$.*

1. *If all $\delta_k^I = \delta_k^F = 0$, then Δ^I and Δ^F are singletons and any HPNS $H(x) = (T(x), I(x), F(x))$ canonically identifies with the hyperpolar fuzzy set $H^T : X \to \Delta^T$, $H^T(x) = T(x)$.*
2. *If $n = 1$ and $\delta_1^T = \delta_1^I = \delta_1^F = 1$, then $H(x) = (T(x), I(x), F(x)) \in [0,1]^3$ is precisely a classical neutrosophic set on X.*

Proof. (1) With $\delta_k^I = \delta_k^F = 0$, we have $[0,1]^0 = \{()\}$; hence Δ^I, Δ^F are singletons and carry no information, so H reduces to $T : X \to \Delta^T$. (2) Immediate from the definitions when $n = 1$ and each block is one-dimensional. □

Theorem 2.14 (HPNSs Form a Hyperstructure)**.** *Let $\mathcal{H} = \{ H : X \to \Delta^T \times \Delta^I \times \Delta^F \}$. Define*

$$H_1 \star H_2 := \Big\{ K \in \mathcal{H} \ \Big| \ \forall x \in X,\ K(x) \in \{H_1(x), H_2(x)\} \Big\}.$$

Then $(\mathcal{H}, \star)$ is a hyperstructure.

Proof. If $K \in H_1 \star H_2$, then $K(x) \in \Delta^T \times \Delta^I \times \Delta^F$ for all x, so $K \in \mathcal{H}$. Hence $H_1 \star H_2 \subseteq \mathcal{H}$, i.e. $\star$ maps into $\mathcal{P}(\mathcal{H})$. Extending $\star$ to subsets by $\mathcal{A} \star \mathcal{B} := \bigcup_{H \in \mathcal{A},\, G \in \mathcal{B}} (H \star G)$ yields a bona fide hyperoperation. □

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Theorem 2.15 (Closure under Componentwise Union and Intersection)**.** *Let* $H, G : X \to \Delta^T \times \Delta^I \times \Delta^F$ *with* $H(x) = (T^H(x), I^H(x), F^H(x))$*,* $G(x) = (T^G(x), I^G(x), F^G(x))$*. Define*

$$(H \vee G)(x) := \big(\max(T^H(x), T^G(x)),\, \max(I^H(x), I^G(x)),\, \max(F^H(x), F^G(x))\big),$$

$$(H \wedge G)(x) := \big(\min(T^H(x), T^G(x)),\, \min(I^H(x), I^G(x)),\, \min(F^H(x), F^G(x))\big),$$

where $\max, \min$ *act blockwise and coordinatewise on* $[0, 1]$*. Then* $H \vee G,\ H \wedge G$ *are HPNSs.*

Proof. Fix $x \in X$, $k \in \{1, \ldots, n\}$, and a coordinate j. Since $T^H_{k,j}(x), T^G_{k,j}(x) \in [0, 1]$,

$$0 \le \min\{T^H_{k,j}(x), T^G_{k,j}(x)\} \le \max\{T^H_{k,j}(x), T^G_{k,j}(x)\} \le 1,$$

and similarly for I and F. Hence $(H \vee G)(x), (H \wedge G)(x) \in \Delta^T \times \Delta^I \times \Delta^F$. □

Theorem 2.16 (Closure under Complement)**.** *Let* $H : X \to \Delta^T \times \Delta^I \times \Delta^F$*. Define*

$$H^c(x) := \big(\mathbf{1} - T(x),\, \mathbf{1} - I(x),\, \mathbf{1} - F(x)\big),$$

where $\mathbf{1}$ *denotes the all-ones vector in the appropriate block. Then* H^c *is an HPNS.*

Proof. Each coordinate $u \in \{T, I, F\}$ satisfies $u_{k,j}(x) \in [0, 1] \Rightarrow 1 - u_{k,j}(x) \in [0, 1]$. Hence $H^c(x) \in \Delta^T \times \Delta^I \times \Delta^F$ for all x. □

Theorem 2.17 (Projection to a Single Polarity)**.** *Fix* $p \in \{T, I, F\}$ *and* $k \in \{1, \ldots, n\}$*. Let* $\pi_{p,k} : \Delta^T \times \Delta^I \times \Delta^F \to [0, 1]^{\delta^p_k}$ *extract the* k*-th block of polarity* p*. Then* $H_{p,k} := \pi_{p,k} \circ H : X \to [0, 1]^{\delta^p_k}$ *is a* δ^p_k*-polar component (a multipolar vector of order* 1*).*

Proof. By definition, $H_{p,k}(x)$ is exactly the k-th block $u^p_k(x) \in [0, 1]^{\delta^p_k}$. □

Theorem 2.18 (Commutativity)**.** *For HPNSs* H, G*,*

$$H \vee G = G \vee H, \qquad H \wedge G = G \wedge H.$$

Proof. Coordinatewise on $[0, 1]$, $\max(a, b) = \max(b, a)$ and $\min(a, b) = \min(b, a)$. □

Theorem 2.19 (Associativity)**.** *For HPNSs* H, G, K*,*

$$(H \vee G) \vee K = H \vee (G \vee K), \qquad (H \wedge G) \wedge K = H \wedge (G \wedge K).$$

Proof. Coordinatewise on $[0, 1]$, max and min are associative: $\max(\max(a, b), c) = \max(a, \max(b, c))$ and $\min(\min(a, b), c) = \min(a, \min(b, c))$. □

Theorem 2.20 (Distributivity)**.** *For HPNSs* H, G, K*,*

$$H \vee (G \wedge K) = (H \vee G) \wedge (H \vee K), \qquad H \wedge (G \vee K) = (H \wedge G) \vee (H \wedge K).$$

Proof. Coordinatewise on $[0, 1]$ one has, for all $a, b, c \in [0, 1]$,

$$\max\!\big(a, \min(b, c)\big) = \min\!\big(\max(a, b), \max(a, c)\big), \quad \min\!\big(a, \max(b, c)\big) = \max\!\big(\min(a, b), \min(a, c)\big),$$

i.e. the distributive laws in the lattice $([0, 1], \min, \max)$. Applying these identities to every coordinate yields the claim. □

DOI: https://doi.org/10.54216/GJMSA.120202
Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

2.3 Hyperpolar Soft Set

A *Hyperpolar Soft Set* is a soft set with multiple polarities, each mapping parameters to subsets of the universe, thus providing multidimensional qualitative assessment. The definition of the Hyperpolar Soft Set is provided below.

Definition 2.21 (Hyperpolar Soft Set)**.** Let $U \neq \emptyset$ be a universe and $E \neq \emptyset$ a set of parameters. Fix an integer $n \geq 1$. A *hyperpolar soft set* over (U, E) with n polarities is an $(n+1)$-tuple

$$H = \big(F_1, \dots, F_n, E\big),$$

where, for each $k \in \{1, \dots, n\}$,

$$F_k : E \longrightarrow \mathcal{P}(U) \qquad \big(\mathcal{P}(U) \text{ denotes the powerset of } U\big).$$

Equivalently, H can be viewed as a single map

$$H : E \longrightarrow \underbrace{\mathcal{P}(U) \times \cdots \times \mathcal{P}(U)}_{n \text{ factors}} = \mathcal{P}(U)^n, \qquad H(e) = \big(F_1(e), \dots, F_n(e)\big).$$

When $n = 1$, the pair (F_1, E) is precisely an (ordinary) soft set.

Example 2.22 (Candidate Shortlisting with a Hyperpolar Soft Set)**.** Consider candidate selection (cf. [25,26]) with

$$U = \{\text{Alice}, \text{Bob}, \text{Carol}, \text{Dave}\}, \qquad E = \{\text{Experience}, \text{Skill}, \text{Education}\}, \qquad n = 3.$$

Interpret the three polarities as qualification tiers, for each $e \in E$:

$$F_1(e) : \text{highly qualified}, \qquad F_2(e) : \text{moderately qualified}, \qquad F_3(e) : \text{needs improvement}.$$

Specify the component mappings $F_k : E \to \mathcal{P}(U)$ by

Experience: $F_1(\text{Experience}) = \{\text{Alice}, \text{Bob}\},\ F_2(\text{Experience}) = \{\text{Carol}\},\ F_3(\text{Experience}) = \{\text{Dave}\};$
Skill: $F_1(\text{Skill}) = \{\text{Alice}, \text{Carol}\},\ F_2(\text{Skill}) = \{\text{Bob}\},\ F_3(\text{Skill}) = \{\text{Dave}\};$
Education: $F_1(\text{Education}) = \{\text{Alice}, \text{Dave}\},\ F_2(\text{Education}) = \{\text{Bob}\},\ F_3(\text{Education}) = \{\text{Carol}\}.$

Equivalently,

$$H : E \longrightarrow \mathcal{P}(U)^3, \qquad H(e) = \big(F_1(e), F_2(e), F_3(e)\big),$$

so, for instance,

$$H(\text{Skill}) = \big(\{\text{Alice}, \text{Carol}\},\ \{\text{Bob}\},\ \{\text{Dave}\}\big).$$

This representation displays each parameter at three qualification levels, aiding nuanced shortlisting.

Example 2.23 (Electric Vehicle Selection with a Hyperpolar Soft Set)**.** Consider EV choice (cf. [27,28]) with

$$U = \{\text{Tesla Model 3}, \text{Nissan Leaf}, \text{Chevy Bolt}, \text{BMW i3}\},$$

$$E = \{\text{Range}, \text{Price}, \text{Charging Speed}\},$$

$$n = 3.$$

Interpret the three polarities as buyer-preference tiers:

$$F_1(e) : \text{top choice},$$

$$F_2(e) : \text{acceptable},$$

$$F_3(e) : \text{least preferred}.$$

DOI: https://doi.org/10.54216/GJMSA.120202
Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Define $F_k : E \to \mathcal{P}(U)$ by

$$
\begin{aligned}
&\textbf{Range (miles):} && F_1(\text{Range}) = \{\text{Tesla Model 3}\},\\
& && F_2(\text{Range}) = \{\text{Chevy Bolt}\},\\
& && F_3(\text{Range}) = \{\text{Nissan Leaf, BMW i3}\};\\
&\textbf{Price (USD):} && F_1(\text{Price}) = \{\text{Nissan Leaf, BMW i3}\},\\
& && F_2(\text{Price}) = \{\text{Chevy Bolt}\},\\
& && F_3(\text{Price}) = \{\text{Tesla Model 3}\};\\
&\textbf{Charging Speed (kW):} && F_1(\text{Charging Speed}) = \{\text{Tesla Model 3, Chevy Bolt}\},\\
& && F_2(\text{Charging Speed}) = \{\text{BMW i3}\},\\
& && F_3(\text{Charging Speed}) = \{\text{Nissan Leaf}\}.
\end{aligned}
$$

Equivalently,

$$H : E \longrightarrow \mathcal{P}(U)^3, \qquad H(e) = \big(F_1(e), F_2(e), F_3(e)\big),$$

so, for example,

$$H(\text{Charging Speed}) = \big(\{\text{Tesla Model 3, Chevy Bolt}\},\ \{\text{BMW i3}\},\ \{\text{Nissan Leaf}\}\big).$$

Theorem 2.24 (Generalization of Soft Sets). *Let $H = (F_1, \dots, F_n, E)$ be a hyperpolar soft set on (U, E).*

1. *If $n = 1$, then $H = (F_1, E)$ is exactly an ordinary soft set.*
2. *If $n = 2$ and $F_2(e) = U \setminus F_1(e)$ for all $e \in E$, then H specializes to a (complementary) bipolar soft structure.*

Proof. (1) Immediate from Definition 2.21. (2) Under $n = 2$, with F_1 positive and F_2 negative (taken as the complement), the pair encodes a bipolar arrangement, with disjointness $F_1(e) \cap F_2(e) = \emptyset$ holding automatically. □

Theorem 2.25 (Hyperstructure of Hyperpolar Soft Sets). *Let $\mathcal{H}$ be the collection of all hyperpolar soft sets $(F_1, \dots, F_n, E)$ on (U, E). Define a hyperoperation*

$$\star :\ \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{P}(\mathcal{H}), \qquad H_1 \star H_2 := \Big\{\, K \in \mathcal{H} \,\Big|\, \forall e \in E,\ K(e) \in \{H_1(e),\ H_2(e)\} \Big\},$$

where $H_i(e) = (F_1^i(e), \dots, F_n^i(e))$. Then $(\mathcal{H}, \star)$ is a hyperstructure.

Proof. If $K \in H_1 \star H_2$, then for each e, $K(e) \in \mathcal{P}(U)^n$, so $K \in \mathcal{H}$. Thus $H_1 \star H_2 \subseteq \mathcal{H}$. Closure under the induced union on subsets yields a valid hyperoperation on $\mathcal{P}(\mathcal{H})$. □

Theorem 2.26 (Closure under Componentwise Union and Intersection). *Let $H = (F_1, \dots, F_n, E)$ and $G = (G_1, \dots, G_n, E)$ be hyperpolar soft sets on (U, E). Define, for all $e \in E$,*

$$(H \vee G)(e) := \big(F_1(e) \cup G_1(e),\ \dots,\ F_n(e) \cup G_n(e)\big),$$

$$(H \wedge G)(e) := \big(F_1(e) \cap G_1(e),\ \dots,\ F_n(e) \cap G_n(e)\big).$$

Then $H \vee G$ and $H \wedge G$ are hyperpolar soft sets on (U, E).

Proof. For each e and k, $F_k(e) \cup G_k(e) \subseteq U$ and $F_k(e) \cap G_k(e) \subseteq U$. Hence the resulting n-tuples lie in $\mathcal{P}(U)^n$. □

Theorem 2.27 (Complement). *Let $H = (F_1, \dots, F_n, E)$ be a hyperpolar soft set on (U, E). Define its complement $H^c = (F_1^c, \dots, F_n^c, E)$ by*

$$F_k^c(e) := U \setminus F_k(e) \qquad (k = 1, \dots, n;\ e \in E).$$

Then H^c is a hyperpolar soft set on (U, E).

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Proof. Each $F_k^c(e) \subseteq U$; thus $\big(F_1^c(e), \ldots, F_n^c(e)\big) \in \mathcal{P}(U)^n$ for all e. □

Theorem 2.28 (Projection to a Single Polarity). *Let $H = (F_1, \ldots, F_n, E)$ be a hyperpolar soft set. For any fixed $k \in \{1, \ldots, n\}$, the pair (F_k, E) is an ordinary soft set on (U, E).*

Proof. By definition, $F_k : E \to \mathcal{P}(U)$ associates to each e the subset $F_k(e) \subseteq U$. □

Theorem 2.29 (Partial Order). *Let $\mathcal{H}$ be the set of all hyperpolar soft sets on (U, E). Define, for $H = (F_1, \ldots, F_n, E)$ and $G = (G_1, \ldots, G_n, E)$,*

$$H \leq G \iff \forall e \in E,\ \forall k \in \{1, \ldots, n\},\ F_k(e) \subseteq G_k(e).$$

Then $(\mathcal{H}, \leq)$ is a partially ordered set.

Proof. Reflexivity is immediate. If $H \leq G$ and $G \leq H$, then $F_k(e) = G_k(e)$ for all e, k, hence $H = G$ (antisymmetry). Transitivity follows from transitivity of $\subseteq$ on $\mathcal{P}(U)$. □

Theorem 2.30 (Lattice Structure). *Under $\leq$, $(\mathcal{H}, \leq)$ is a lattice. For any $H, G \in \mathcal{H}$, the join $H \vee G$ and meet $H \wedge G$ of Theorem 2.26 are, respectively, the least upper bound and greatest lower bound of $\{H, G\}$.*

Proof. By construction $H \leq H \vee G$ and $G \leq H \vee G$; minimality of $H \vee G$ among upper bounds is by componentwise inclusion. Dually, $H \wedge G \leq H, G$ and maximality among lower bounds follows componentwise. □

Theorem 2.31 (Absorption Laws). *For any $H, G \in \mathcal{H}$,*

$$H \vee (H \wedge G) = H, \qquad H \wedge (H \vee G) = H.$$

Proof. Componentwise in $\mathcal{P}(U)$, $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$. Apply to each polarity and parameter. □

Theorem 2.32 (De Morgan's Laws). *For $H, G \in \mathcal{H}$ with complement defined in Theorem 2.27,*

$$(H \vee G)^c = H^c \wedge G^c, \qquad (H \wedge G)^c = H^c \vee G^c.$$

Proof. Componentwise in $\mathcal{P}(U)$, $\big(A \cup B\big)^c = A^c \cap B^c$ and $\big(A \cap B\big)^c = A^c \cup B^c$. □

3 Conclusion

In this paper, we have further extended these notions and explored Hyperpolar Fuzzy Sets, Hyperpolar Neutrosophic Sets, and Hyperpolar Soft Sets. In the future, we expect further research on extended systems of these concepts, studies on newly defined operations, and the design of algorithms related to these frameworks.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Use of Generative AI and AI-Assisted Tools

I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

DOI: https://doi.org/10.54216/GJMSA.120202

Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025

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DOI: https://doi.org/10.54216/GJMSA.120202
Received: February 13, 2025 Revised: June 05, 2025 Accepted: August 10, 2025