Enforcement of q-Rung Orthopair Fuzzy Subsets to Q-Ideals

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Abstract

This paper presents an innovative generalization of intuitionistic fuzzy Q-subalgebras (IF-Q-S) by incorporating the structure of q-Rung Orthopair fuzzy sets (q-ROFS), which are distinguished by their independent membership and non-membership functions. It inserts and investigates q-Rung Orthopair fuzzy Q-subalgebras (q-ROFQ-S), demonstrating that this model is equivalent to a combination of a fuzzy Q-subalgebra (F-Q-S) and an anti-fuzzy Q-subalgebra (AF-Q-S). The study's notable contributions include the definition of the nil radical and an exploration of its properties under homomorphisms. Additionally, it establishes that the union of q-ROFQ-subalgebras can itself form such a subalgebra under particular commutative conditions. Expanding the concept to the realm of ideals, the paper defines q-Rung Orthopair fuzzy Q-ideals (q-ROFQ-I) and proves that every q-regular q-ROFQ-S is inherently a q-ROFQ-I. This work offers a robust and versatile algebraic framework for addressing approximation in complex nonlinear systems.

Keywords: Q-algebra; q-Rung Orthopair fuzzy set; q-Rung Orthopair fuzzy Q-algebra; q-Rung Orthopair fuzzy Q-ideal

1 Introduction

Logic algebra represents a crucial annex of mathematics with extensive applications across interdisciplinary sciences. J. Neggres, inserted a new notion, as Q-A, which is an expansion of BCI/BCK-algebras and generalized some theorems from the theory of BCI-algebras.[4] Fuzzy set theory, conceived by Zadeh to address approximation,[6] represents an expansion of classical set theory. It plays a pivotal role in modeling and controlling approximation systems across nature, society, and industry. Additionally, it has proven invaluable in describing complex phenomena that classical set theory struggles to characterize effectively. Following Zadeh's seminal research, extensive efforts have been devoted to formulating fuzzy analogs of classical theories. An IFS, inserted by K. Atanassov (1986), generalizes Zadeh's fuzzy sets by incorporating independent community and non-community degrees with a hesitation margin.[1] Orthopair Fuzzy Sets (specifically, q-Rung Orthopair Fuzzy Sets, or q-ROFS) are an expansion of IFS and Pythagorean fuzzy sets (PFS). They model approximation using community and non-community degrees, constrained by a power-based orthogonality condition.[5]

In this paper, we first insert q-ROFQ-S as a unification of Q-fuzzy subalgebras(QF$-$S) and discuss their properties. Our motivation for introducing q-ROFQ-S is to equip Q-As with two community functions and to construct a new unification of Q-A based on the axioms on QF$-$S. Given that the topics of IFS are generalizations of fuzzy subsets and have applications to linear problems, in this paper our other aim is to study nonlinear problems such as quadratic curves. Therefore, q-ROFQ-S have the properties of the axioms of logic algebras and fuzzy nonlinear problems. We have generalized QF$-$S to q-ROFQ-S and obtained an unification of QF$-$S. Considering the importance of the isomorphisms in this paper, we have inserted the null radial q-ROFQ-S, and consider the relation between of the null radial image of q-ROFQ-S and have shown under the conditions that the null radial image of q-ROFQ-S and the null radial image of q-ROFQ-S are equal to each other. Considering that the characteristic function is of great importance in issues related to the community or non-community of a part in the whole, we have shown that the characteristic function of a subalgebra is extended to a q-ROFQ-S. Since the characteristic function is a fuzzy subset, the motivation is to ask whether an arbitrary F-Q-S can be a q-ROFQ-S. By investigating this issue, it can be seen that if the component is a F-Q-S in a q-ROFS, the second component (as a non-community of the element) must be an AF-Q-S to obtain a q-ROFQ-S. By transforming the F-Q-S into the q-ROFQ-S, the idea, and motivation of introducing AF-Q-s.as (as a non-community component) is created and we show that if the q-ROFS has two components, F-Q-S, and AF-Q-S, then it forms a q-ROFQ-S and in this research, for further confirmation, we call it a two-sided q-ROFQ-S. The purpose of introducing q-ROFQ-S is to be able to transform the combination of two q-ROFQ-S into a q-ROFQ-S in such a way that this combination includes one of these q-ROFQ-S. In this research, considering that the combination of q-ROFQ-S cannot necessarily be a q-ROFQ-S (the elements of the q-ROFS are nonlinear), we have shown that the combination of q-ROFQ-S can be a q-ROFQ-S with the commutative condition. Therefore, the introduction of two-sided q-ROFQ-S is another motivation to be able to combine q-ROFQ-S without considering the power of q-ROFQ-S. Also, we present the notion of q-ROFQ-I as another types of extension of Q-As and prove that q-regular q-ROFQ-S are q-ROFQ-I.

Motivation and innovations: The primary objective of our work is to bridge the gap between Q-algebra (Q-A) theory and advanced fuzzy set structures. In particular, we utilize the enhanced capabilities of q-ROFS, which adeptly handle nonlinear relationships within communities through dual community and non-community functions, to extend the concept of QF-S. This approach enables the formal introduction and characterization of anti QF-A as the essential non-community component. A major goal is to define the conditions under which these structures can be integrated and transformed, ensuring their algebraic properties remain intact while addressing inherent nonlinear complexities.

The novel framework we propose for Q-A integrates it with q-ROFS theory, offering greater applicability in modeling complex and approximation systems. Key advancements include: formally introducing two-sided q-ROF Q-subalgebras, establishing their foundational properties under operations like combination and isomorphism, and pioneering the concept of q-ROF Q-ideals. We further demonstrate a critical connection between regularity and the newly proposed ideal structure. Overall, this work lays the groundwork for a more robust fuzzy-algebraic approach to tackling intricate system approximations.

2 Preliminaries

This section compiles key concepts pertinent to our work.

Definition 2.1. [2] A structure $(X,\ \theta\ ,0_X)$ is a Q-A, in case that, $\forall\, t_X, r_X, s_X \in X$,

(i) $t_X\,\theta\,t_X = 0_X$,

(ii) $t_X\,\theta\,0_X = t_X$,

(iii) $(t_X\,\theta\,r_X)\,\theta\,s_X = (t_X\,\theta\,s_X)\,\theta\,r_X$.

In $(X,\ \theta\ ,0_X)$, define $\leq_X$ by $t_X \leq_X r_X \Leftrightarrow t_X\,\theta\,r_X = 0_X$. Take X be a Q-A and $\emptyset \neq \mathtt{Y} \subseteq X$. $\mathtt{Y}$ is referred to as a subalgebra of X, in case that $\forall\, t_X, r_X \in \mathtt{Y}, t_X\,\theta\,r_X \in \mathtt{Y}$ and will denote by $\mathtt{Y} \leq_{sub} X$.

Theorem 2.2. [2] *Take $(X,\ \theta\ ,0_X)$ be a Q-A. $\forall\, t_X, r_X \in X$,*

(i) $(t_X\,\theta\,(t_X\,\theta\,r_X))\,\theta\,r_X = 0_X$,

(ii) *in case that $t_X \leq_X 0_X$, then $X = \{0_X\}$,*

(iii) *in case that* $t_X \leq_X r_X$*, then* $t_X\ \theta\ (t_X\ \theta\ (t_X\ \theta\ r_X)) = 0_X$.

Definition 2.3. [2] $I \neq \emptyset$ of Q-A $(X,\ \theta\ , 0_X)$ is referred to as an ideal of X, in case that $(1), 0_X \in I$ and $(2), \forall\ t_X, r_X \in X, t_X\ \theta\ r_X \in I$ and $r_X \in I$, yield $r_X \in I$.

Definition 2.4. [3] A fuzzy subset (FS) of $\emptyset \neq X$, is a set as $A = \{(t_X, \mu(t_X)) \mid t_X \in X\}$, that $\mu : X \to \mathbb{I} = [0, 1]$ is community map.

In what follow, $\forall\ k \in \mathbb{N}$, any $t_X \in X$, denote $\mu^k(t_X) = (\mu(t_X))^k$

Definition 2.5. [5] A q-ROFS E, of $\emptyset \neq X$ is given by $E = \{(\mu_E, \nu_E) \mid t_X \in X\}$, that $1 \leq q \in \mathbb{R}, \mu_E, \nu_E : X \to \mathbb{I}$ (as community and non-community, respectively) and $\forall\ t_X \in X, 0 \leq \mu^q_E(t_X) + \nu^q_E(t_X) \leq 1$.

From now on, will denote any $FS, A = \{(t_X, \mu(t_X)) \mid t_X \in X\}$ by A_X and any q-$ROFS$, $E_X = \{(\mu_{E_X}, \nu_{E_X}) \mid t_X \in X\}$ by E_X for simplify.

Definition 2.6. [?] An $FS\ A_X$ of a Q-A $(X,\ \theta\ , 0_X)$ is an F-Q-S of X, in case that $\forall\ t_X, r_X \in X, \mu(t_X\ \theta\ r_X) \geq \min\{\mu(t_X), \mu(r_X)\}$ and in equality, it is referred to as strong.

3 Q-A with condition A and condition C

In this section, we insert Q-A with condition A and condition C and investigate their some properties. The aim of this section is that add some axioms to Q-A that are important to extension of IFS of Q-A.

Take $n \in \mathbb{N}, (X,\ \theta\ , 0_X)$ be a Q-A and $t_X \in X$. Define $t^{\{2\}}_X = t_X\ \theta\ t_X, t^{\{3\}}_X = t^{\{2\}}_X\ \theta\ t_X$ and $\forall\ n \geq 3$ $, t^{\{n\}}_X = t^{\{n-1\}}_X\ \theta\ t_X$ and $t^{(2)}_X = t_X\ \theta\ t_X, t^{(3)}_X = t_X\ \theta\ t^{(2)}_X$ and $\forall\ n \geq 3\ , t^{(n)}_X = t_X\ \theta\ t^{(n-1)}_X$.

Take $t_X, r_X \in X$. We state that $(X,\ \theta\ , 0_X)$ is with condition C, in case that is satisfied in the commutative law, i.e $\forall\ t_X, r_X, s_X \in X, t_X\ \theta\ r_X = r_X\ \theta\ t_X$ and is with condition A, in case that it is satisfied in the associative law, i.e $\forall\ t_X, r_X, s_X \in X, t_X\ \theta\ (r_X\ \theta\ s_X) = (t_X\ \theta\ r_X)\ \theta\ s_X$.

In the following, we add a connection between of condition A and condition C and the considering of their importance in any given Q-A.

Proposition 3.1. *Take* $(X,\ \theta\ , 0_X)$ *be a* Q-A *and* $t_X, r_X \in X$ *and* $n, m \in \mathbb{N}$.

- (i) $(X,\ \theta\ , 0_X)$ *is with condition A, iffy* $(X,\ \theta\ , 0_X)$ *is with condition* C.
- (ii) *in case that* $(X,\ \theta\ , 0_X)$ *is with condition A, then* $(t_X\ \theta\ r_X)^{\{n\}} = t^{\{n\}}_X\ \theta\ r^{\{n\}}_X$.
- (iii) *in case that* $(X,\ \theta\ , 0_X)$ *is with condition A, then* $(t^{\{n\}}_X)^{\{m\}} = t^{\{mn\}}_X$.

Proof. (i) Take $t_X, r_X, s_X \in X$. Since $(X,\ \theta\ , 0_X)$ is with condition C,

$$(t_X\ \theta\ r_X)\ \theta\ s_X = (t_X\ \theta\ s_X)\ \theta\ r_X = (s_X\ \theta\ t_X)\ \theta\ r_X = (s_X\ \theta\ r_X)\ \theta\ t_X = t_X\ \theta\ (r_X\ \theta\ s_X)$$

and so $(X,\ \theta\ , 0_X)$ is with condition A.

Conversely, let $(X,\ \theta\ , 0_X)$ be with condition A. For $t_X \in X, 0_X\ \theta\ t_X = (t_X\ \theta\ t_X)\ \theta\ t_X = t_X\ \theta\ (t_X\ \theta\ t_X) = t_X\ \theta\ 0_X = t_X$ and so $\forall\ t_X, r_X \in X$,

$$\begin{aligned} t_X\ \theta\ r_X &= 0_X\theta(t_X\ \theta\ r_X) = (r_X\ \theta\ r_X)\ \theta\ (t_X\ \theta\ r_X) = ((r_X\ \theta\ r_X)\ \theta\ t_X)\ \theta\ r_X = ((r_X\ \theta\ t_X)\ \theta\ r_X)\ \theta\ r_X \\ &= (r_X\ \theta\ t_X)\ \theta\ (r_X\ \theta\ r_X) = (r_X\ \theta\ t_X)\ \theta\ 0_X = r_X\ \theta\ t_X \end{aligned}$$

and so $(X,\ \theta\ , 0_X)$ is with condition C.

(ii) Take $t_X, r_X \in X$. Since $(X,\ \theta\ , 0_X)$ is with condition A

$$(t_X\,\theta\,r_X)^2 = (t_X\,\theta\,r_X)\,\theta\,(t_X\,\theta\,r_X) = (t_X\,\theta\,(t_X\,\theta\,r_X))\,\theta\,r_X = ((t_X\,\theta\,t_X)\,\theta\,r_X)\,\theta\,r_X = (t_X^2\,\theta\,r_X)\,\theta\,r_X$$
$$= t_X^2\,\theta\,(r_X\,\theta\,r_X) = t_X^2\,\theta\,r_X^2.$$

Now, by induction, we assume that $(t_X\,\theta\,r_X)^{\{n-1\}} = t_X^{\{n-1\}}\,\theta\,r_X^{\{n-1\}}$.

$$\begin{aligned}(t_X\,\theta\,r_X)^{\{n\}} &= (t_X\,\theta\,r_X)^{\{n-1\}}\,\theta\,(t_X\,\theta\,r_X)\\ &= (t_X^{\{n-1\}}\,\theta\,r_X^{\{n-1\}})\,\theta\,(t_X\,\theta\,r_X) = ((t_X^{\{n-1\}}\,\theta\,r_X^{\{n-1\}})\,\theta\,r_X)\,\theta\,t_X\\ &= (t_X^{\{n-1\}}\,\theta\,(r_X^{\{n-1\}}\,\theta\,r_X))\,\theta\,t_X = (t_X^{\{n-1\}}\,\theta\,r_X^{\{n\}})\,\theta\,t_X\\ &= (t_X^{\{n-1\}}\,\theta\,t_X)\,\theta\,r_X^{\{n\}} = t_X^{\{n\}}\,\theta\,r_X^{\{n\}}.\end{aligned}$$

(iii) Take $t_X \in X$ and $r, s \in \mathbb{N}$. Since $(X,\ \theta\ , 0_X)$ is with condition A,

$t_X^{\{r\}}\,\theta\,t_X^{\{s\}} = (\underbrace{t_X\,\theta\,\ldots\,\theta\,t_X}_{r-times})\,\theta\,(\underbrace{t_X\,\theta\,t_X\,\theta\,\ldots\,\theta\,t_X}_{s-times}) = (\underbrace{t_X\,\theta\,t_X\,\theta\,\ldots\,\theta\,t_X}_{(r+s)-times}) = t_X^{\{r+s\}}$.Clearly $(t_X^{\{n\}})^1 = t_X^{\{n\}} = t_X^{\{n1\}}$. Assume the statement holds for some positive integer $m = k$, i.e. $(t_X^{\{n\}})^{\{k\}} = t_X^{\{nk\}}$. For $m = k+1$, have

$(t_X^{\{n\}})^{\{k+1\}} = (t_X^{\{n\}})^{\{k\}}\,\theta\,t_X^{\{n\}} = t_X^{\{nk\}}\,\theta\,t_X^{\{n\}} = t_X^{\{nk+n\}} = t_X^{\{n(k+1)\}}$. □

Corollary 3.2. *Take $(X,\ \theta\ , 0_X)$ be with condition A, $t_X, r_X \in X$ and $m, n \in \mathbb{N}$.*

(i) $t_X^{\{n\}} = t_X^{(n)}$,

(ii) $(t_X\,\theta\,r_X)^{(n)} = t_X^{(n)}\,\theta\,r_X^{(n)}$.

(iii) $(t_X^{(n)})^{(m)} = t_X^{(mn)}$.

Based Corollary 3.2, from now on, will denote $t_X^{\{n\}} = t_X^{(n)}$ by t_X^n, for simplify. From now on, we consider $(X,\ \theta\ , 0_X)$ as a Q-A and obtain the subsequent outcomes.

3.1 On q-ROFQ-S

In this subsection, we insert the notion of q-ROFQ-S and investigate the connection between q-ROFQ-S and (anti)F-Q-S.

Firstly, we describe the relation of q-ROFS and other extensions as follows:

Remark 3.3. A q-ROFS of X coincides with:

(i) Intuitionistic-FS of X, in case that $q = 1$.

(ii) Pythagorean-FS of X, in case that $q = 2$.

(iii) Fermatean-FS of X, in case that $q = 3$.

Remark 3.4. An anti q-ROFS of X coincides with:

(i) Anti intuitionistic-FS of X, in case that $q = 1$.

(ii) Anti Pythagorean-FS of X, in case that $q = 2$.

(iii) Anti Fermatean-FS of X, in case that $q = 3$.

Indeed, q-Rung Orthopair-FS is an extension of intuitionistic-FS. Take FS be a fuzzy sub-Q-A of X. We say FS is an anti fuzzy sub-Q-A of X, in case that at all $t_X, r_X \in X, \mu(t_X\,\theta\,r_X) \leq \mu(t_X) \wedge \mu(r_X)$.

In what follow, $\forall\, k, l \in \mathbb{N}$, any $t_X \in X$ and any map $\mu : X \to \mathbb{I} = [0, 1]$, denote

${E_X}^{(k,l)} = \{(t_X, \mu^k_{E_X}(t_X), \nu^l_{E_X}(t_X)) \mid t_X \in X\}$. It is clear that ${E_X}^{(1,1)} = E_X, \forall\, k \geq l \geq 2, \mu^k \subseteq \mu^l$ and $\forall\, k \leq k', l \leq l'$, in case that ${E_X}^{(k,l)}$ is a q-$ROFS$ of X, then ${E_X}^{(k',l')}$ is a q-$ROFS$ of X. In what follows, we insert the q-ROFQ-S as the IF-Q-S.

Definition 3.5. Take E_X be an q-$ROFS$ of X. E_X is a q-ROFQ-S of X, in case that

$\forall\, t_X, r_X \in X, \mu^q_{E_X}(t_X\,\theta\,r_X) \geq \min\{\mu^q_{E_X}(t_X), \mu^q_{E_X}(r_X)\}$ and $\nu^q_{E_X}(t_X\,\theta\,r_X) \leq \max\{\mu^q_{E_X}(t_X), \mu^q_{E_X}(r_X)\}$ and is an anti q-$ROFS_s$ of X, in case that $\forall\, t_X, r_X \in X, \mu^q_{E_X}(t_X\,\theta\,r_X) \leq \max\{\mu^q_{E_X}(t_X), \mu^q_{E_X}(r_X)\}$ and $\nu^q_{E_X}(t_X\,\theta\,r_X) \geq \min\{\mu^q_{E_X}(t_X), \mu^q_{E_X}(r_X)\}$.

Remark 3.6. Take E_X is a q-$ROFS_s$ of X.

(i) $\forall\, t_X \in X, \mu_{E_X}(0_X) \geq \mu^q_{E_X}(0_X) \geq \mu^q_{E_X}(t_X)$ and $\nu^q_{E_X}(0_X) \leq \nu^q_{E_X}(t_X) \leq \nu_{E_X}(t_X)$.

(ii) μ_{E_X}, ν_{E_X} are FQ_s and anti FQ_s of X, respectively.

Proposition 3.7. *Take E_X be a q-$ROFS_s$ of X and $n \geq 2$. For $t_X \in X$,*

(i) $\mu(t^{\{n\}}_X) \geq \mu^q(t_X)$ *and* $\nu^q(t^{\{n\}}_X) \leq \nu(t_X)$.

(ii) $\mu(t^{(n)}_X) \geq \mu^q(t_X)$ *and* $\nu^q(t^{(n)}_X) \leq \nu(t_X)$.

Proof. Take $t_X \in X$ and $n \geq 2$. Then $\mu^q(t^2_X) \geq \mu^q(t_X) \wedge \mu^q(t_X) = \mu^q(t_X)$. By induction, suppose $\mu^q(t^{\{n-1\}}_X) \geq \mu^q(t_X)$, then $\mu^q(t^{\{n\}}_X) = \mu^q(t^{\{n-1\}}_X\,\theta\,t_X) \geq \mu^q(t^{\{n-1\}}_X) \wedge \mu^q(t_X) = \mu^q(t_X)$. In similar to, $\forall\, n \geq 2, \nu^q(t^{\{n\}}_X) \leq \nu(t_X)$. Item (ii) is similar to. □

3.1.1 Generalization of q-$ROFS_s$ via operations

From now on, for unification of q-$ROFS_s$ of X, is tried to insert some operations on q-$ROFS_s$ of X.

Take E_X and F_X be q-$ROFS_s$ of X. We discuss that $E_X \subseteq F_X$, in case that at all $t_X \in X, \mu_{E_X}(t_X) \leq \mu_{F_X}(t_X)$ and $\nu_{E_X}(t_X) \geq \nu_{F_X}(t_X)$. Obviously at all $n \geq n', {E_X}^{(n,n')} \subseteq {E_X}^{(n',n)}$ and at all $n \leq n', m \leq m', {E_X}^{(m',n)} \subseteq {E_X}^{(m,n')}$. Also, define

$$E_X \cap F_X = \{(t_X, \min\{\mu_{E_X}(t_X), \mu_{F_X}(t_X)\}, \max\{\nu_{E_X}(t_X), \nu_{F_X}(t_X)\})\}\ and$$
$$E_X \cup F_X = \{(t_X, \max\{\mu_{E_X}(t_X), \mu_{F_X}(t_X)\}, \min\{\nu_{E_X}(t_X), \nu_{F_X}(t_X)\})\}.$$

Obviously, $E_X \cap F_X$ and $E_X \cup F_X$ are a q-$ROFS_s$ of X.

Theorem 3.8. *Take A_X be an FQ_s of X. Then $[A_X, \frac{1}{3}, \frac{1}{3}] = \{(t_X, \sqrt[q]{\mu(t_X)}, \sqrt[q]{\mu^c(t_X)})\}$ is a q-$ROFS_s$ of X.*

Proof. Take $t_X \in X$. Then $(\sqrt[q]{\mu(t_X)})^q + (\sqrt[q]{\mu^c(t_X)})^q = \mu(t_X) + 1 - \mu(t_X) = 1$ and so $[A_X, \frac{1}{3}, \frac{1}{3}] = \{(t_X, \sqrt[q]{\mu(t_X)}, \sqrt[q]{\mu^c(t_X)})\}$ is an q-$ROFS$ of X. Now, since A_X is an FQ_s of X,

$$\sqrt[q]{\mu(t_X\,\theta\,r_X))}^q = \mu(t_X\,\theta\,r_X) \geq \mu(t_X) \wedge \mu(r_X) = \sqrt[q]{\mu(t_X)}^q \wedge \sqrt[q]{\mu(r_X)}^q.$$

Also,

$$\begin{aligned}(\sqrt[q]{\mu^c(t_X\,\theta\,r_X))}^q &= 1 - \mu(t_X\,\theta\,r_X) \leq 1 - (\mu(t_X) \wedge \mu(r_X)) \leq (1 - \mu(t_X)) \vee (1 - \mu(r_X)) \\ &= \mu^c(t_X) \vee \mu^c(r_X) = \sqrt[q]{\mu^c(t_X)}^q \vee \sqrt[q]{\mu^c(r_X)}^q.\end{aligned}$$

Then $[A_X, \frac{1}{3}, \frac{1}{3}] = \{(t_X, \sqrt[q]{\mu(t_X)}, \sqrt[q]{\mu^c(t_X)})\}$ is a q-$ROFS_s$ of X. □

Theorem 3.9. *Take A_X be an FQ_s of X. Then $\langle A_X, \frac{1}{m}, \frac{1}{n}\rangle = \{(t_X, \sqrt[q]{\nu^c(t_X)}, \sqrt[q]{\nu(t_X)})\}$ is a q-$ROFS_s$ of X.*

Proof. Take $t_X \in X$. Then $(\sqrt[q]{\nu(t_X)})^q + (\sqrt[q]{\nu^c(t_X)})^q = \nu(t_X) + 1 - \nu(t_X) = 1$ and so $\langle A_X, \frac{1}{3}, \frac{1}{3}\rangle = \{(t_X, \sqrt[q]{\nu^c(t_X)}, \sqrt[q]{\nu(t_X)})\}$ is a q-$ROFS$ of X. Now, being A_X is an FQ_s of X,

$$(\sqrt[q]{\nu^c(t_X\,\theta\,r_X)})^q = \nu^c(t_X\,\theta\,r_X) \geq \nu(t_X) \wedge \nu(r_X) = (\sqrt[q]{\nu^c(t_X)})^q \wedge (\sqrt[q]{\nu^c(r_X)})^q.$$

Also,

$$(\sqrt[q]{\nu(t_X\,\theta\,r_X)})^q \quad = \quad \nu(t_X\,\theta\,r_X) \leq \nu(t_X) \vee \nu(r_X) = (\sqrt[n]{\nu^c(t_X)})^q \vee (\sqrt[q]{\nu^c(r_X)})^q.$$

Then $\langle A_X, \frac{1}{q}, \frac{1}{q}\rangle = \{(t_X, \sqrt[q]{\nu^c(t_X)}, \sqrt[q]{\nu(t_X)})\}$ is a q-$ROFS_s$ of X. □

Take E_X be a q-$ROFS$ of X and $t_X \in X$. Define ${E_X}^c = \{(t_X, \nu_{E_X}(t_X), \mu_{E_X}(t_X)) \mid t_X \in X\}$, it is clear that ${E_X}^c$ is an anti q-$ROFS_s$ of X.

Corollary 3.10. *Take A_X be an FQ_s of X.*

(*i*) $[[A_X, \frac{1}{q}, \frac{1}{q}]] = \{(t_X, \sqrt[q]{\mu^c(t_X)}, \sqrt[q]{\mu(t_X)})\}$ *is an anti q-$ROFS_s$ of X.*

(*ii*) $\langle\langle A_X, \frac{1}{q}, \frac{1}{q}\rangle\rangle = \{(t_X, \sqrt[q]{\nu(t_X)}, \sqrt[q]{\nu^c(t_X)})\}$ *is an anti a q-$ROFS_s$ of X.*

One of the main of this study is a connection between of structures of q-$ROFS$ of X and substructures of Q-As, so apply the level subsets to q-$ROFS$ of Xas follows.

Definition 3.11. Take E_X be a q-$ROFS$ of X and $\alpha, \beta \in \mathbb{I}$. Define ${E_X}^{(\alpha,\beta)} = \{t_X \in X \mid \mu^q_{E_X}(t_X) \geq \alpha, \nu^q_{E_X}(t_X) \leq \beta\}$ is known as q-ROF level subset of E_X.

Theorem 3.12. *Take E_X be a q-$ROFS$ of X and $\alpha, \beta \in \mathbb{I}$. Then ${E_X}^{(\alpha,\beta)} \leq_{sub} X$ iffy E_X is a q-$ROFS_s$ of X.*

Proof. Take $t_X, r_X \in {E_X}^{(\alpha,\beta)}$. Then $\mu^q_{E_X}(t_X) \geq \alpha$ and $\nu^q_{E_X}(r_X)) \leq \beta$. This yields that $\mu^q_{E_X}(t_X\,\theta\,r_X) \geq \mu^q_{E_X}(t_X) \wedge \mu^q_{E_X}(r_X) \geq \alpha \wedge \alpha = \alpha$ and $\nu^q_{E_X}(t_X\,\theta\,r_X) \leq \nu^q_{E_X}(t_X) \vee \nu^q_{E_X}(r_X) \leq \beta \vee \beta = \beta$ and so $t_X\,\theta\,r_X \in {E_X}^{(\alpha,\beta)}$.

Conversely, let $\mu^q_{E_X}(t_X) \wedge \mu^q_{E_X}(r_X) = \alpha$ and $\nu^q_{E_X}(t_X) \vee \nu^q_{E_X}(r_X) = \beta$. Then $\mu^q_{E_X}(t_X) \geq \alpha, \mu^q_{E_X}(r_X) \geq \alpha$ and $\nu^q_{E_X}(t_X) \leq \beta, \nu^q_{E_X}(r_X) \leq \beta$. This yields that

$t_X, r_X \in {E_X}^{(\alpha,\beta)}$ and so $t_X\,\theta\,r_X \in {E_X}^{(\alpha,\beta)}$, because ${E_X}^{(\alpha,\beta)}$ is a Q-S of X. Therefore, $\mu^q_{E_X}(t_X) \geq \alpha = \mu^q_{E_X}(t_X) \wedge \mu^q_{E_X}(r_X)$ and $\nu^q_{E_X}(t_X) \leq \beta = \nu^q_{E_X}(t_X) \vee \nu^q_{E_X}(r_X)$, so E_X is a q-$ROFS_s$ of X. □

Definition 3.13. Take E_X be a q-$ROFS$ with condition A of X. Define $\sqrt[q]{E_X} = (\mu_{\sqrt[q]{E_X}}, \nu_{\sqrt[q]{E_X}})$ known as q-ROF nil radical of E_X, which $\mu_{\sqrt[q]{E_X}}(t_X) = \underset{n\geq 1}{\text{Infi}}\, \mu_{E_X}(t^n_X)$ and $\nu_{\sqrt[q]{E_X}} = \underset{n\geq 1}{\text{Supi}}\, \nu_{E_X}(t^n_X)$

Theorem 3.14. *Take $(X, \theta, 0_X)$ be with condition A and E_X be a q-$ROFS_s$ of X. $\sqrt[q]{E_X}$ is a q-$ROFS_s$ of X.*

Proof. Take $t_X \in X$ and $n \geq 1$. Since E_X is a q-$ROFS$ of X, $0 \leq \mu^q_{E_X}(t^n_X) + \nu^q_{E_X}(t^n_X)) \leq 1$ and so $\lim_{n\to\infty}(\mu^q_{E_X}(t^n_X) + \nu^q_{E_X}(t^n_X)) \leq 1$. Hence

$$\mu^q_{\sqrt[q]{E_X}}(t_X) + \nu^q_{\sqrt[q]{E_X}}(t_X) = \underset{n\geq 1}{\text{Infi}}\, \mu^q_{E_X}(t^n_X) + \underset{n\geq 1}{\text{Supi}}\, \nu^q_{E_X}(t^n_X) \leq 1$$

and so $\sqrt[q]{E_X}$ is a q-$ROFS$ of X. In addition,

$$\mu^q_{\sqrt[q]{E_X}}(t_X\ \theta\ r_X) = \underset{n\geq 1}{\text{Infi}}\,\mu^q_{E_X}((t_X\ \theta\ r_X)^n) = \underset{n\geq 1}{\text{Infi}}\,\mu^q_{E_X}(t^n_X\ \theta\ r^n_X) \geq \underset{n\geq 1}{\text{Infi}}(\mu^q_{E_X}(t^n_X) \wedge \mu^q_{E_X}(r^n_X))$$
$$\geq \underset{n\geq 1}{\text{Infi}}\,\mu^q_{E_X}(t^n_X) \wedge \underset{n\geq 1}{\text{Infi}}\,\mu^q_{E_X}(r^n_X) = \mu^q_{\sqrt[q]{E_X}}(t_X) \wedge \mu^q_{\sqrt[q]{E_X}}(r_X).$$

Also,

$$\nu^q_{\sqrt[q]{E_X}}(t_X\ \theta\ r_X) = \underset{n\geq 1}{\text{Supi}}\,\nu^q_{E_X}((t_X\ \theta\ r_X)^n) = \underset{n\leq 1}{\text{Supi}}\,\nu^q_{E_X}(t^n_X\ \theta\ r^n_X) \leq \underset{n\geq 1}{\text{Supi}}(\nu^q_{E_X}(t^n_X) \vee \nu^q_{E_X}(r^n_X))$$
$$\leq \underset{n\geq 1}{\text{Supi}}\,\nu^q_{E_X}(t^n_X) \vee \underset{n\geq 1}{\text{Supi}}\,\nu^q_{E_X}(r^n_X) = \nu^q_{\sqrt[q]{E_X}}(t_X) \vee \nu^q_{\sqrt[q]{E_X}}(r_X).$$

Therefore, $\sqrt[q]{E_X}$ is a q-$ROFS_s$ of X. □

Theorem 3.15. *Take $(X,\ \theta\ , 0_X)$ be a Q-A and E_X, F_X be q-$ROFS_s$ of X.*

(i) $\sqrt[q]{E_X} \subseteq E_X$,

(ii) *in case that $E_X \subseteq F_X$, then $\sqrt[q]{E_X} \subseteq \sqrt[q]{F_X}$,*

(iii) *in case that X is with condition A, then $\sqrt[q]{\sqrt[q]{E_X}} = \sqrt[q]{E_X}$.*

(iv) *in case that X is with condition A, then $(\sqrt[q]{E_X})^{(q,q)} = {E_X}^{(q,q)}$.*

Proof. (i) Take $t_X \in X$. Then $\mu_{\sqrt[q]{E_X}}(t_X) = \underset{n\geq 1}{\text{Infi}}\,\mu_{E_X}(t^n_X) = \mu_{E_X}(t_X) \wedge (\underset{n\geq 2}{\text{Infi}}\,\mu_{E_X}(t^n_X)) \leq \mu_{E_X}(t_X)$ and so $\mu_{E_X} \supseteq \mu_{\sqrt[q]{E_X}}$. In addition, $\nu_{\sqrt[q]{E_X}}(t_X) = \underset{n\geq 1}{\text{Supi}}\,\nu_{E_X}(t^n_X) = \nu_{E_X}(t_X) \vee (\underset{n\geq 2}{\text{Supi}}\,\nu_{E_X}(t^n_X)) \geq \nu_{E_X}(t_X)$ and $\nu_{\sqrt[q]{E_X}} \supseteq \nu_{E_X}$ and so $E_X \supseteq \sqrt[q]{E_X}$.

(ii) Take $t_X \in X$. Then $\mu_{E_X}(t_X) \leq \mu_{F_X}(t_X)$ and $\nu_{E_X}(t_X) \geq \nu_{F_X}(t_X)$, so

$$\mu_{\sqrt[q]{E_X}}(t_X) = \underset{n\geq 1}{\text{Infi}}\,\mu_{E_X}(t^n_X) \leq \underset{n\geq 1}{\text{Infi}}\,\mu_{F_X}(t^n_X) = \mu_{\sqrt[q]{F_X}}(t_X)$$

and $\nu_{\sqrt[q]{E_X}}(t_X) = \underset{n\geq 1}{\text{Supi}}\,\nu_{E_X}(t^n_X) \geq \underset{n\geq 1}{\text{Supi}}\,\nu_{F_X}(t^n_X) = \nu_{\sqrt[q]{F_X}}(t_X)$. Hence $\sqrt[q]{E_X} \subseteq \sqrt[q]{F_X}$.

(iii) By (i), $\sqrt[q]{E_X} \subseteq \sqrt[q]{\sqrt[q]{E_X}}$. Take $t_X \in X$. Then by definition and Lemma 3.1,

$$\mu_{\sqrt[q]{\sqrt[q]{E_X}}}(t_X) = \underset{n\geq 1}{\text{Infi}}\,\mu_{\sqrt[q]{E_X}}(t^n_X) = \underset{n\geq 1}{\text{Infi}}\,\underset{m\geq 1}{\text{Infi}}\,\mu_{E_X}((t^n_X)^m) = \underset{n\geq 1}{\text{Infi}}\,\underset{m\geq 1}{\text{Infi}}\,\mu_{E_X}(t^{mn}_X) = \underset{k\geq 1}{\text{Infi}}\,\mu_{E_X}(t^k_X)$$
$$= \mu_{\sqrt[q]{E_X}}(t_X).$$

In similar to $\nu_{\sqrt[q]{\sqrt[q]{E_X}}}(t_X) = \nu_{\sqrt[q]{E_X}}(t_X)$ and so $\sqrt[q]{\sqrt[q]{E_X}} = \sqrt[q]{E_X}$.

(iv) Take $t_X \in X$.

$$(\mu^q_{\sqrt[q]{E_X}})(t_X) = \underset{n\geq 1}{\text{Infi}}\,\mu^q_{E_X}(t^n_X) = \mu^q_{E_X}(t_X) \wedge \mu^q_{E_X}(0_X) = \mu^q_{E_X}(t_X)\ and$$
$$(\nu^q_{\sqrt[q]{E_X}})(t_X) = \underset{n\geq 1}{\text{Supi}}\,\nu^q_{E_X}(t^n_X) = \nu^q_{E_X}(t_X) \vee \nu^q_{E_X}(0_X) = \nu^q_{E_X}(t_X).$$

Hence, $(\sqrt[q]{E_X})^{(q,q)} = \{(t_X, \mu^q_{E_X}(t_X), \nu^q_{E_X}(t_X)) \mid t_X \in X\}$. □

Corollary 3.16. *Take $(X,\ \theta\ , 0_X)$ be with condition A. Then $\sqrt[q]{E_X} = E_X$.*

Definition 3.17. Take E_X and F_X be q-$ROFS$ of X. Define

$E_X \circ F_X = \{(t_X, (\mu_{E_X\circ F_X})(t_X), (\nu_{E_X\circ F_X})(t_X) \mid t_X \in X\},\ which\ (\mu_{E_X\circ F_X})(t_X) = \underset{a\ \theta\ b=t_X}{\text{Supi}}(\mu_{E_X}(a) \wedge \mu_{F_X}(b))\ and\ (\nu_{E_X\circ F_X})(t_X) = \underset{a\ \theta\ b=t_X}{\text{Infi}}(\nu_{E_X}(a) \vee \nu_{F_X}(b)).$

Theorem 3.18. *Take E_X be a q-$ROFS$ of X. The subsequent are interchangeable:*

(i) *E_X is a q-$ROFS_s$ of X,*

(ii) $(E_X \circ E_X)^{(q,q)} \subseteq {E_X}^{(q,q)}$.

Proof. Take $t_X, r_X \in X$. Then there are $c, d, c', d' \in X$ in a way that

$$\begin{aligned}\mu^q_{E_X \circ E_X}(t_X) + \nu^q_{E_X \circ E_X}(t_X) &= \big(\operatorname*{Supi}_{a\ \theta\ b=t_X}(\mu_{E_X}(a) \wedge \mu_{E_X}(b))\big)^q + \big(\operatorname*{Infi}_{a\ \theta\ b=t_X}(\nu_{E_X}(a) \vee \nu_{E_X}(b))\big)^q \\ &= \big(\operatorname*{Supi}_{a\ \theta\ b=t_X}(\mu^q_{E_X}(a) \wedge \mu^q_{E_X}(b))\big) + \big(\operatorname*{Infi}_{a\ \theta\ b=t_X}(\nu^q_{E_X}(a) \vee \nu^q_{E_X}(b))\big) \\ &= (\mu^q_{E_X}(c) \wedge \mu^q_{E_X}(d)) + (\nu^q_{E_X}(c') \vee \nu^q_{E_X}(d')) \leq 1.\end{aligned}$$

Hence, $E_X \circ E_X$ is a q-$ROFS$ of X and so $(E_X \circ E_X)^{(q,q)}$ is a q-$ROFS$ of X.

Since E_X is a q-$ROFS_s$ of X,

$$\begin{aligned}&\mu^q_{E_X \circ E_X}(t_X) = \big(\operatorname*{Supi}_{a\ \theta\ b=t_X}(\mu_{E_X}(a) \wedge \mu_{E_X}(b))\big)^q \\ &= \operatorname*{Supi}_{a\ \theta\ b=t_X}(\mu^q_{E_X}(a) \wedge \mu^q_{E_X}(b)) \leq \operatorname*{Supi}_{a\ \theta\ b=t_X}\mu^q_{E_X}(a\ \theta\ b) = \mu^q_{E_X}(t_X), \\ &\nu^q_{E_X \circ E_X}(t_X) = \big(\operatorname*{Infi}_{a\ \theta\ b=t_X}(\nu_{E_X}(a) \vee \nu_{E_X}(b))\big)^q \\ &= \operatorname*{Infi}_{a\ \theta\ b=t_X}(\nu^q_{E_X}(a) \vee \nu^q_{E_X}(b)) \geq \operatorname*{Infi}_{a\ \theta\ b=t_X}\nu^q_{E_X}(a\ \theta\ b) = \nu^q_{E_X}(t_X).\end{aligned}$$

Hence $(E_X \circ E_X)^{(q,q)} \subseteq {E_X}^{(q,q)}$.

Conversely, let $(E_X \circ E_X)^{(q,q)} \subseteq {E_X}^{(q,q)}$. Then for $t_X = a$ and $r_X = b$, have

$$\begin{aligned}&\mu^q_{E_X \circ E_X}(t_X\ \theta\ r_X) = \big(\operatorname*{Supi}_{a\ \theta\ b=t_X\ \theta\ r_X}(\mu_{E_X}(a) \wedge \mu_{E_X}(b))\big)^q \\ &= \operatorname*{Supi}_{a\ \theta\ b=t_X\ \theta\ r_X}(\mu^q_{E_X}(a) \wedge \mu^q_{E_X}(b)) \geq \mu^q_{E_X}(t_X) \wedge \mu^q_{E_X}(r_X) \\ &\geq \mu^q_{E_X \circ E_X}(t_X) \wedge \mu^q_{E_X \circ E_X}(r_X) \text{ and} \\ &\nu^q_{E_X \circ E_X}(t_X\ \theta\ r_X) \\ &= \big(\operatorname*{Infi}_{a\ \theta\ b=t_X\ \theta\ r_X}(\nu_{E_X}(a) \vee \nu_{E_X}(b))\big)^q = \operatorname*{Infi}_{a\ \theta\ b=t_X\ \theta\ r_X}(\nu^q_{E_X}(a) \vee \nu^q_{E_X}(b)) \leq \nu^q_{E_X}(t_X) \vee \nu^q_{E_X}(r_X) \\ &\leq \nu^q_{E_X \circ E_X}(t_X) \vee \nu^q_{E_X \circ E_X}(r_X).\end{aligned}$$

Hence $(E_X \circ E_X)^{(q,q)}$ is a q-$ROFS_s$ of X. □

Theorem 3.19. *Take E_X and F_X be q-$ROFS_s$ of X. If X is with condition A, then the subsequent are interchangeable:*

(i) $(E_X \circ F_X)^{(q,q)}$ *is a q-$ROFS_s$ of X,*

(ii) $(E_X \circ F_X)^{(q,q)} = (F_X \circ E_X)^{(q,q)}$.

Proof. Take $t_X, r_X \in X$ and $(E_X \circ F_X)^{(q,q)} = (F_X \circ E_X)^{(q,q)}$.

$$\begin{aligned}(\mu^q_{E_X} \circ \mu^q_{F_X}) \circ (\mu^q_{E_X} \circ \mu^q_{F_X}) &= \mu^q_{E_X} \circ (\mu^q_{F_X} \circ \mu^q_{E_X}) \circ \mu^q_{F_X} \\ &= \mu^q_{E_X} \circ (\mu^q_{E_X} \circ \mu^q_{F_X}) \circ \mu^q_{F_X} \\ &= (\mu^q_{E_X} \circ \mu^q_{E_X}) \circ (\mu^q_{F_X} \circ \mu^q_{F_X}) \subseteq \mu^q_{E_X} \circ \mu^q_{F_X}\end{aligned}$$

and

$$\begin{aligned}(\nu^q_{E_X} \circ \nu^q_{F_X}) \circ (\nu^q_{E_X} \circ \nu^q_{F_X}) &= \nu^q_{E_X} \circ (\nu^q_{F_X} \circ \nu^q_{E_X}) \circ \nu^q_{F_X} \\ &= \nu^q_{E_X} \circ (\nu^q_{E_X} \circ \nu^q_{F_X}) \circ \nu^q_{F_X} \\ &= (\nu^q_{E_X} \circ \nu^q_{E_X}) \circ (\nu^q_{F_X} \circ \nu^q_{F_X}) \supseteq \nu^q_{E_X} \circ \nu^q_{F_X}.\end{aligned}$$

Hence $((E_X \circ F_X) \circ (E_X \circ F_X))^{(q,q)} \subseteq (E_X \circ F_X)^{(q,q)}$ and by Theorem 3.18, $(E_X \circ F_X)^{(q,q)}$ is a q-$ROFS_s$ of X.

Conversely, let $(E_X \circ F_X)^{(q,q)}$ be a q-$ROFS_s$ of X. Then

$$\begin{aligned}(\mu^q_{E_X \circ F_X})(t_X) &= \underset{a\ \theta\ b=t_X}{\text{Supi}}\ \mu^q_{E_X}(a) \wedge \mu^q_{F_X}(b) \\ &= \underset{b\ \theta\ a=t_X}{\text{Supi}}\ \mu^q_{E_X}(b) \wedge \mu^q_{F_X}(a) = (\mu^q_{F_X \circ E_X})(t_X).\end{aligned}$$

In similar to, $(\nu^q_{E_X \circ F_X})(t_X) = (\nu^q_{F_X \circ E_X})(t_X)$. Hence, $(E_X \circ F_X)^{(q,q)} = (F_X \circ E_X)^{(q,q)}$. □

Theorem 3.20. *Take E_X and F_X be q-$ROFS_s$ of X. If $E_X \subseteq F_X$ and $\nu_{E_X}(0_X) = \nu^q_{E_X}(0_X)$, then ${E_X}^{(q,1)} \subseteq E_X \circ F_X$.*

Proof. Take $t_X, r_X \in X$. Since $t_X\ \theta\ 0_X = t_X$,

$$\begin{aligned}(\mu_{E_X \circ F_X})(t_X) &= \underset{a\ \theta\ b=t_X}{\text{Supi}}\ (\mu_{E_X}(a) \wedge \mu_{F_X}(b)) \geq \mu_{E_X}(t_X) \wedge \mu_{F_X}(0_X) \\ &\geq \mu^q_{E_X}(t_X) \wedge \mu^q_{F_X}(t_X) = \mu^q_{E_X}(t_X) \\ \text{and} \quad & (\nu_{E_X \circ F_X})(t_X) = \underset{a\ \theta\ b=t_X}{\text{Infi}}(\nu_{E_X}(a) \vee \nu_{F_X}(b)) \leq \nu_{E_X}(t_X) \vee \nu_{F_X}(0_X) \\ &\leq \nu_{E_X}(t_X) \vee \nu_{E_X}(0_X) = \nu_{E_X}(t_X) \vee \nu^q_{E_X}(0_X) \leq \nu_{E_X}(t_X) \vee \nu^q_{E_X}(t_X) = \nu_{E_X}(t_X).\end{aligned}$$

This yields that $\mu^q_{E_X} \subseteq \mu_{E_X \circ F_X}$ and $\nu_{E_X \circ F_X} \subseteq \nu_{E_X}$ and so ${E_X}^{(q,1)} \subseteq E_X \circ F_X$. □

Corollary 3.21. *Take E_X and F_X be q-$ROFS_s$ of X.*

(i) *If $\nu_{E_X}(0_X) = \nu^q_{E_X}(0_X)$, then ${E_X}^{(q,1)} \subseteq E_X \circ E_X$.*

(ii) $E_X \subseteq E_X \circ E_X$.

(iii) *If $E_X \subseteq F_X$, then $E_X \subseteq E_X \circ F_X$.*

Corollary 3.22. *Take A_X be an FQ_s of X. Then $[A_X, \frac{1}{q}, \frac{1}{q}] \subseteq [A_X, \frac{1}{q}, \frac{1}{q}] \circ [A_X, \frac{1}{q}, \frac{1}{q}]$.*

Theorem 3.23. *Take E_X and F_X be q-$ROFS_s$ as of X.*

(i) $\sqrt[q]{E_X} \cap \sqrt[q]{F_X} = \sqrt[q]{E_X \cap F_X}$,

(ii) *If $E_X \subseteq F_X$, then $\sqrt[q]{{E_X}^{(q,q)}} \subseteq \sqrt[q]{(E_X \circ F_X)^{(q,q)}}$.*

Proof. (i) Take $t_X, r_X \in X$. Using Theorem 3.15, $\sqrt[q]{E_X \cap F_X} \subseteq \sqrt[q]{E_X} \cap \sqrt[q]{F_X}$.

$$\begin{aligned}(\mu_{\sqrt[q]{E_X}} \cap \mu_{\sqrt[q]{F_X}})(t_X) &= \mu_{\sqrt[q]{E_X}}(t_X) \wedge \mu_{\sqrt[q]{F_X}}(t_X) = \underset{n\geq 1}{\text{Infi}}\,\mu_{E_X}(t^n_X) \wedge \underset{n\geq 1}{\text{Infi}}\,\mu_{F_X}(t^n_X) \\ &= \underset{n\geq 1}{\text{Infi}}(\mu_{F_X}(t^n_X) \wedge \mu_{E_X}(t^n_X)) = \underset{n\geq 1}{\text{Infi}}(\mu_{F_X} \cap \mu_{E_X})(t^n_X) \\ &= \underset{n\geq 1}{\text{Infi}}\,\mu_{(E_X \cap F_X)}(t^n_X) = (\mu_{\sqrt[q]{F_X \cap E_X}})(t_X) \text{ and} \\ & (\nu_{\sqrt[q]{E_X}} \cup \nu_{\sqrt[q]{F_X}})(t_X) = \nu_{\sqrt[q]{E_X}}(t_X) \vee \nu_{\sqrt[q]{F_X}}(t_X) = \underset{n\geq 1}{\text{Supi}}\,\nu_{E_X}(t^n_X) \vee \underset{n\geq 1}{\text{Supi}}\,\nu_{F_X}(t^n_X) \\ &= \underset{n\geq 1}{\text{Supi}}(\nu_{F_X}(t^n_X) \vee \nu_{E_X}(t^n_X)) \\ &= \underset{n\geq 1}{\text{Supi}}(\nu_{F_X} \cup \nu_{E_X})(t^n_X) = \underset{n\geq 1}{\text{Supi}}\,\nu_{(E_X \cup F_X)}(t^n_X) = (\nu_{\sqrt[q]{F_X \cup E_X}})(t_X).\end{aligned}$$

(ii) Take $t_X \in X$.

$$\begin{aligned} \mu^q_{\sqrt[q]{E_X \circ F_X}}(t_X) &= \operatorname*{Infi}_{n\geq 1} \mu^q_{E_X\circ F_X}(t_X^n) = \operatorname*{Infi}_{n\geq 1} \operatorname*{Supi}_{a\ \theta\ b=t_X^n} (\mu^q_{F_X}(a) \wedge \mu^q_{E_X}(b)) \geq \operatorname*{Infi}_{n\geq 1}(\mu^q_{F_X}(t_X^n) \wedge \mu^q_{E_X}(0_X)) \\ &\geq \operatorname*{Infi}_{n\geq 1}(\mu^q_{F_X}(t_X^n) \wedge \mu^q_{E_X}(t_X^n)) = (\operatorname*{Infi}_{n\geq 1}(\mu^q_{F_X}(t_X^n)) \wedge (\operatorname*{Infi}_{n\geq 1} \mu^q_{E_X}(t_X^n)) \\ &= \mu^q_{\sqrt[q]{E_X}}(t_X) \wedge \mu^q_{\sqrt[q]{F_X}}(t_X) = (\mu^q_{\sqrt[q]{E_X}} \cap \mu^q_{\sqrt[q]{F_X}})(t_X). \end{aligned}$$

Hence $\mu^q_{\sqrt[q]{E_X}} \cap \mu^q_{\sqrt[q]{F_X}} \subseteq \mu^q_{\sqrt[q]{E_X \circ F_X}}$. Also

$$\begin{aligned} \nu^q_{\sqrt[q]{E_X \circ F_X}}(t_X) &= \operatorname*{Supi}_{n\geq 1} \nu^q_{E_X\circ F_X}(t_X^n) = \operatorname*{Supi}_{n\geq 1} \operatorname*{Infi}_{a\ \theta\ b=t_X^n} (\nu^q_{F_X}(a) \vee \nu^q_{E_X}(b)) \leq \operatorname*{Supi}_{n\geq 1}(\nu^q_{F_X}(t_X^n) \vee \nu^q_{E_X}(0_X)) \\ &\leq \operatorname*{Supi}_{n\geq 1}(\nu^q_{F_X}(t_X^n) \vee \nu^q_{E_X}(t_X^n)) = (\operatorname*{Supi}_{n\geq 1}(\nu^q_{F_X}(t_X^n)) \vee (\operatorname*{Supi}_{n\geq 1} \nu^q_{E_X}(t_X^n)) \\ &= \nu^q_{\sqrt[q]{E_X}}(t_X) \vee \nu^q_{\sqrt[q]{F_X}}(t_X) = (\nu^q_{\sqrt[q]{E_X}} \cup \nu^q_{\sqrt[q]{F_X}})(t_X). \end{aligned}$$

Hence $\nu^q_{\sqrt[q]{E_X \circ F_X}} \subseteq (\nu^q_{\sqrt[q]{E_X}} \cup \nu^q_{\sqrt[q]{F_X}})$. Since $E_X \subseteq F_X$, we get $\mu^q_{\sqrt[q]{E_X}} \cap \mu^q_{\sqrt[q]{F_X}} = \mu^q_{\sqrt[q]{E_X}}$ and $\nu^q_{\sqrt[q]{E_X}} \cup \nu^q_{\sqrt[q]{F_X}} = \nu^q_{\sqrt[q]{E_X}}$. This yields that $\sqrt[q]{E_X{}^{(q,q)}} \subseteq \sqrt[q]{(E_X \circ F_X)^{(q,q)}}$. □

Theorem 3.24. *Take E_X be a q-ROFS$_s$ of X. Then* $\sqrt[q]{E_X{}^{(q,q)}} = \sqrt[q]{(E_X \circ E_X)^{(q,q)}}$.

Proof. Take $t_X \in X$. Then

$$\begin{aligned} \mu^q_{E_X\circ E_X}(t_X) = \operatorname*{Infi}_{n\geq 1} \mu^q_{E_X}(a) \wedge \mu^q_{E_X}(b) \leq \mu^q_{E_X}(a\ \theta\ b) = \mu^q_{E_X}(t_X)\ and \\ \nu^q_{E_X\circ E_X}(t_X) = \operatorname*{Supi}_{n\geq 1} \nu^q_{E_X}(a) \vee \nu^q_{E_X}(b) \geq \nu^q_{E_X}(a\ \theta\ b) = \nu^q_{E_X}(t_X). \end{aligned}$$

Thus $\mu^q_{E_X\circ E_X} \subseteq \mu^q_{E_X}$ and $\nu^q_{E_X} \subseteq \nu^q_{E_X\circ E_X}$ and so $(E_X \circ E_X)^{(q,q)} \subseteq E_X{}^{(q,q)}$. Therefore, $\sqrt[q]{(E_X \circ E_X)^{(q,q)}} \subseteq \sqrt[q]{E_X{}^{(q,q)}}$ and by Theorem 3.23, $\sqrt[q]{E_X{}^{(q,q)}} = \sqrt[q]{(E_X \circ E_X)^{(q,q)}}$. □

Take $(X,\ \theta\ , 0_X)$ and $(X',\ \theta\ ', 0_{X'})$ be Q-As. Then a map $\psi : X \to X'$ is referred to as a homomorphism, in case that $\forall\ t_X, r_X \in X, \psi(t_X\ \theta\ r_X) = \psi(t_X)\ \theta\ '\psi(r_X)$. If ψ is onto, then it is referred to as an epimorphism. For FS, $A_X = \{(t_X, \mu(t_X)) \mid t_X \in X\}$ of X and FS, $B = \{(t_X, \nu(t_X)) \mid t_X \in X'\}$ of X', recall that $(\mu_\psi)(r_X) = \psi(\mu)(r_X) = \begin{cases} \operatorname*{Supi}_{r_X=\psi(t_X)} \mu(t_X) & \psi^{-1}(r_X) \neq \emptyset \\ 0 & \psi^{-1}(r_X) = \emptyset \end{cases}$ and $(\nu_{\psi^{-1}})(t_X) = \psi^{-1}(\nu)(t_X) = \nu(\psi(t_X))$.

Theorem 3.25. *Take F_X be a q-ROFS$_s$ of X'. Then for homomorphism $\psi : X \to X'$,*

(i) *$\psi^{-1}(F_X)$ is a q-ROFS$_s$ of X.*

(ii) $\sqrt[q]{\psi^{-1}(F_X)} = \psi^{-1}(\sqrt[q]{F_X})$.

(iii) $\psi^{-1}(F_X) \subseteq \psi^{-1}(\sqrt[q]{F_X})$.

Proof. (i) Take $t_X, r_X \in X$. Then $(\psi^{(-1)}(\mu_{F_X}(t_X)))^q + (\psi^{(-1)}(\nu_{F_X}(t_X)))^q = \mu^q_{F_X}(\psi(t_X)) + \mu^q_{F_X}(\psi(t_X)) \leq 1$ and so $\psi^{-1}(F_X) = \{(t_X, \psi^{-1}(\mu_{F_X}), \psi^{-1}(\nu_{F_X})) \mid t_X \in X\}$ is a q-$ROFS$ of X. Also

$$\begin{aligned} (\mu_{\psi^{-1}(F_X)}(t_X\ \theta\ r_X))^q &= \mu^q_{F_X}(\psi(t_X\ \theta\ r_X)) = \mu^q_{F_X}(\psi(t_X)\ \theta\ '\psi(r_X))) \geq \mu^q_{F_X}(\psi(t_X)) \wedge \mu^q_{F_X}(\psi(r_X)) \\ &= (\psi^{-1}(\mu_{F_X})(t_X))^q \wedge (\psi^{-1}(\mu_{F_X})(r_X))^q and \\ &\quad (\nu_{\psi^{-1}(F_X)}(t_X\ \theta\ r_X))^q = \nu^q_{F_X}(\psi(t_X\ \theta\ r_X)) \\ &= \nu^q_{F_X}(\psi(t_X)\ \theta\ '\psi(r_X))) \leq \nu^q_{F_X}(\psi(t_X)) \vee \nu^q_{F_X}(\psi(r_X)) \\ &= (\psi^{-1}(\nu_{F_X})(t_X))^q \vee (\psi^{-1}(\nu_{F_X})(r_X))^q. \end{aligned}$$

So $\psi^{-1}(F_X)$ is a q-$ROFS_s$ of X.

(ii) Take $t_X \in X$.

$$\begin{aligned}(\mu_{\sqrt[q]{\psi^{-1}(F_X)}})(t_X) &= \underset{n\geq 1}{\text{Infi}}(\mu_{\psi^{-1}(F_X)})(t_X^n) = \underset{n\geq 1}{\text{Infi}}\,\mu_{F_X}(\psi(t_X^n)) = \underset{n\geq 1}{\text{Infi}}\,\mu_{F_X}(\psi(t_X)^n)\\ &= \mu_{\sqrt[q]{F_X}}(\psi(t_X)) = \mu_{\psi^{-1}(\sqrt[q]{F_X})}(t_X),\ and\\ &\quad (\nu_{\sqrt[q]{\psi^{-1}(F_X)}})(t_X) = \underset{n\geq 1}{\text{Supi}}(\nu_{\psi^{-1}(F_X)})(t_X^n) = \underset{n\geq 1}{\text{Supi}}\,\nu_{F_X}(\psi(t_X^n)) = \underset{n\geq 1}{\text{Supi}}\,\nu_{F_X}(\psi(t_X)^n)\\ &= \nu_{\sqrt[q]{F_X}}(\psi(t_X)) = \nu_{\psi^{-1}\sqrt[q]{F_X}}(t_X).\end{aligned}$$

Hence $\sqrt[q]{\psi^{-1}(F_X)} = \psi^{-1}(\sqrt[q]{F_X})$.

(ii) Immediate by (ii). □

Theorem 3.26. *Take E_X be a q-$ROFS_s$ of X and $\psi : X \to X'$ be an isomorphism.*

(i) *$\psi(E_X)$ is a q-$ROFS_s$ of X'.*

(ii) $\sqrt[q]{\psi(E_X)} \subseteq \sqrt[q]{\psi(\sqrt[q]{E_X})}$.

(iii) $\sqrt[q]{\psi(E_X)} = \psi(\sqrt[q]{E_X})$.

Proof. (i) Take $t'_X, r'_X \in X'$. Then there is $c, d \in X$ in a way that $\psi^q(\mu_{E_X})(r_X) + \psi^q(\nu_{E_X})(r_X) = \underset{r_X=\psi(t_X)}{\text{Supi}}\ \mu^q_{E_X}(t_X) + \underset{r_X=\psi(t_X)}{\text{Supi}}\ \nu^q_{E_X}(t_X) = \mu^q_{E_X}(c) + \mu^q_{E_X}(d) \leq 1$ and so $\psi(E_X)$ is an FSS of X'. Since ψ is onto, there are $t_X, r_X \in X$ in a way that $t'_X = \psi(t_X), r'_X = \psi(r_X)$ and so

$$\begin{aligned}((\mu^q_\psi)_{E_X})(t'_X\,\theta\,'r'_X) &= \psi^q(\mu_{E_X})(t'_X\,\theta\,'r'_X) = \underset{t'_X\,\theta\,'r'_X=\psi(a)}{\text{Supi}}\ \mu^q_{F_X}(a) \geq \underset{\psi(t_X\,\theta\,r_X)=\psi(a)}{\text{Supi}}\ \mu^q_{F_X}(t_X\,\theta\,r_X)\\ &\geq \underset{\psi(t_X\,\theta\,r_X)=\psi(a)}{\text{Supi}}(\mu^q_{F_X}(t_X)\wedge\mu^q_{F_X}(t_X)) = (\underset{\psi(t_X)=t'_X}{\text{Supi}}\ \mu^q_{F_X}(t_X))\wedge(\underset{\psi(r_X)=r'_X}{\text{Supi}}\ \mu^q_{F_X}(r_X))\\ &= \psi^q(\mu_{E_X})(t'_X)\wedge\psi^q(\mu_{E_X})(r'_X),\ \text{and since } \psi \text{ is one to one}\\ &\quad ((\nu^q_\psi)_{E_X})(t'_X\,\theta\,'r'_X) = \psi^q(\nu_{E_X})(t'_X\,\theta\,'r'_X) = \underset{t'_X\,\theta\,'r'_X=\psi(a)}{\text{Supi}}\ \nu^q_{F_X}(a)\\ &\quad = \underset{\psi(t_X\,\theta\,r_X)=\psi(a)}{\text{Supi}}\ \nu^q_{F_X}(t_X\,\theta\,r_X)\\ &\leq \underset{\psi(t_X\,\theta\,r_X)=\psi(a)}{\text{Supi}}(\nu^q_{F_X}(t_X)\wedge\nu^q_{F_X}(t_X)) = (\underset{\psi(t_X)=t'_X}{\text{Supi}}\ \nu^q_{F_X}(t_X))\wedge(\underset{\psi(r_X)=r'_X}{\text{Supi}}\ \nu^q_{F_X}(r_X))\\ &= \psi^q(\nu_{E_X})(t'_X)\wedge\psi^q(\nu_{E_X})(r'_X).\end{aligned}$$

Hence $\psi(E_X)$ is a q-$ROFS_s$ of X'.

(ii) Take $r_X \in X'$. Since ψ is onto, then $\psi^{-1}(r_X) \neq \emptyset$, so

$$\mu_{\psi(\sqrt[q]{E_X})}(r_X) = \underset{n\geq 1}{\text{Infi}}\,\mu_{\psi(E_X)}(r_X^n) = \underset{n\geq 1}{\text{Infi}}\ \underset{\psi(t_X)=r_X^n}{\text{Supi}}\ \mu_{E_X}(t_X) \geq \underset{\psi(t_X)=r_X}{\text{Supi}}\ \mu_{E_X}(t_X) = \mu_{\psi(E_X)}(r_X)\ and$$

$$\nu_{\psi(\sqrt[q]{E_X})}(r_X) = \underset{n\geq 1}{\text{Supi}}\,\nu_{\psi(E_X)}(r_X^n) = \underset{n\geq 1}{\text{Supi}}\ \underset{\psi(t_X)=r_X^n}{\text{Supi}}\ \nu_{E_X}(t_X) \leq \underset{\psi(t_X)=r_X}{\text{Supi}}\ \nu_{E_X}(t_X) = \nu_{\psi(E_X)}(r_X).$$

Hence $\mu_{\psi(E_X)} \subseteq \mu_{\psi(\sqrt[q]{E_X})}, \nu_{\psi(E_X)}(r_X) \supseteq \nu_{\psi(\sqrt[q]{E_X})}$ and so $\psi(E_X) \subseteq \psi(\sqrt[q]{E_X})$.

(iii) Take $r_X \in X'$. Since ψ is onto, then $\psi^{-1}(r_X) \neq \emptyset$, so $\forall\, 0 < \epsilon$,

$$\begin{aligned}&\nu_{\sqrt[q]{\psi(E_X)}}(r_X) - \epsilon \leq (\nu_{\psi(E_X)})(r_X^m) = (\nu_{\psi(E_X)})(\psi(t_X^m))\\ =\ &(\nu_{\psi^{-1}(\psi(E_X))})(t_X^m) = \nu_{E_X}(t_X^m) \leq (\nu_{\sqrt[q]{E_X}})(t_X) \leq \underset{\psi(s_X)=t_X}{\text{Supi}}\ \nu_{\psi(\sqrt[q]{E_X})}(s_X) = \nu_{\psi(\sqrt[q]{E_X})}(r_X)\ and\\ &\mu_{\sqrt[q]{\psi(E_X)}}(r_X) + \epsilon \geq (\mu_{\psi(E_X)})(r_X^m) = (\mu_{\psi(E_X)})(\psi(t_X^m))\\ =\ &(\mu_{\psi^{-1}(\psi(E_X))})(t_X^m) = \mu_{E_X}(t_X^m) \geq (\mu_{\sqrt[q]{E_X}})(t_X) \geq \underset{\psi(s_X)=t_X}{\text{Infi}}\ \mu_{\psi(\sqrt[q]{E_X})}(s_X) = \mu_{\psi(\sqrt[q]{E_X})}(r_X)\end{aligned}$$

Hence, $\mu_{\sqrt[q]{\psi(E_X)}} \supseteq \mu_{\psi(\sqrt[q]{E_X})}, \nu_{\sqrt[q]{\psi(E_X)}} \subseteq \nu_{\psi(\sqrt[q]{E_X})}$ and so $(\sqrt[q]{\psi(E_X)}) \supseteq \psi(\sqrt[q]{E_X})$.

On the other hand, ψ is onto, then by definition of supremum, there is $t_X \in X$ in a way that

$$\nu_{\psi(\sqrt[q]{E_X})}(r_X) - \epsilon \leq \nu_{\sqrt[q]{E_X}}(t_X) = \operatorname*{Supi}_{n\geq 1} \nu_{E_X}(t_X^n) \; and$$
$$\mu_{\psi(\sqrt[q]{E_X})}(r_X) + \epsilon \geq \mu_{\sqrt[q]{E_X}}(t_X) = \operatorname*{Infi}_{n\geq 1} \mu_{E_X}(t_X^n)$$

In addition, since

$$\nu_{\psi(\sqrt[q]{E_X})}(r_X) = \operatorname*{Supi}_{\psi(t_X)=r_X} \nu_{\sqrt[q]{E_X}}(t_X) = \operatorname*{Supi}_{\psi(t_X)=r_X} \operatorname*{Supi}_{n\geq 1} \nu_{E_X}(t_X^n) = \operatorname*{Supi}_{n\geq 1} \operatorname*{Supi}_{\psi(t_X)=r_X} \nu_{E_X}(t_X^n) \; and$$
$$\mu_{\psi(\sqrt[q]{E_X})}(r_X) = \operatorname*{Supi}_{\psi(t_X)=r_X} \mu_{\sqrt[q]{E_X}}(t_X) = \operatorname*{Supi}_{\psi(t_X)=r_X} \operatorname*{Infi}_{n\geq 1} \mu_{E_X}(t_X^n) = \operatorname*{Infi}_{n\geq 1} \operatorname*{Supi}_{\psi(t_X)=r_X} \mu_{E_X}(t_X^n),$$

there is $m \in \mathbb{N}$ in a way that

$$(\nu_{\psi(\sqrt[q]{E_X})})(r_X) - \epsilon \leq \operatorname*{Supi}_{\psi(s_X)=r_X^m} \nu_{E_X}(s_X) = (\nu_{\psi(E_X)})(r_X^m) \leq \operatorname*{Supi}_{n\geq 1}(\nu_{\psi(E_X)})(r_X^n) = (\nu_{\sqrt[q]{\psi(E_X)}})(r_X) \; and$$
$$(\mu_{\psi(\sqrt[q]{E_X})})(r_X) + \epsilon \geq \operatorname*{Infi}_{\psi(w)=r_X^m} \nu_{E_X}(w) = (\mu_{\psi(E_X)})(r_X^m) \geq \operatorname*{Infi}_{n\geq 1}(\mu_{\psi(E_X)})(r_X^n) = (\mu_{\sqrt[q]{\psi(E_X)}})(r_X).$$

Thus $\mu_{\psi(\sqrt[q]{E_X})} \supseteq \mu_{\sqrt[q]{\psi(E_X)}}$ and $\nu_{\psi(\sqrt[q]{E_X})} \subseteq \nu_{\sqrt[q]{\psi(E_X)}}$. This yields that $\psi(\sqrt[q]{E_X}) \supseteq \sqrt[q]{\psi(E_X)}$ and so $\psi(\sqrt[q]{E_X}) = \sqrt[q]{\psi(E_X)}$. □

4 On q-ROFQ-I

In this section, we present the q-ROFQ-I based the fuzzy Q-ideals and investigate their properties. We present the notion q-regular FS and make a correspondence between of a q-$ROFS_s$ and q-$ROFQ-I$.

Take X be a Q-A and A_X be an FS. Then A_X is referred to as a fuzzy Q-ideal (FQ_i) of X, in case that $\forall\, t_X, r_X \in X, \mu(0_X) \geq \mu(t_X), \mu(t_X) \geq \min\{\mu(t_X\,\theta\, r_X), \mu(r_X)\}$ and it is referred to as an anti FQ_i of X, in case that $\forall\, t_X, r_X \in X, \mu(0_X) \leq \mu(t_X), \mu(t_X) \leq \max\{\mu(t_X\,\theta\, r_X), \mu(r_X)\}$. Obviously, in case that A_X is an FQ_i of X, then A_X^c is an anti FQ_i of X, which $\mu^c(t_X) = 1 - \mu(t_X)$.

Definition 4.1. Take E_X be a q-$ROFS = (\mu_{E_X}, \nu_{E_X})$ of X. Then E_X is a q-ROFQ-I of X, in case that $\forall\, t_X, r_X \in X, (1), \mu^q_{E_X}(0_X) \geq \mu^q_{E_X}(t_X), \nu^q_{E_X}(0_X) \leq \nu^q_{E_X}(t_X), (2), \mu^q_{E_X}(t_X) \geq \min\{\mu^q_{E_X}(t_X\,\theta\, r_X), \mu^q_{E_X}(r_X)\}$ and $(3) : \nu^q_{E_X}(t_X) \leq \max\{\nu^q_{E_X}(t_X\,\theta\, r_X), \mu^q_{E_X}(r_X)\}$.

Remark 4.2. Take E_X be a q-$ROFQ-I$ of X. Then μ_{E_X}, ν_{E_X} are FQ_i and anti FQ_i of X, respectively.

Proposition 4.3. *Take $E_X = (\mu_{E_X}, \nu_{E_X})$ be a q-$ROFQ-I$ of X. Then μ_{E_X} and ν_{E_X} are order-reserving and order-preserving, respectively.*

Proof. Take $t_X, r_X \in X$ and $t_X \leq_X r_X$. Then $t_X\,\theta\, r_X = 0_X$, so

$$\mu^q_{E_X}(t_X) \geq \min\{\mu^q_{E_X}(t_X\,\theta\, r_X), \mu^q_{E_X}(r_X)\} = \min\{\mu^q_{E_X}(0_X), \mu^q_{E_X}(r_X)\} = \mu^q_{E_X}(r_X)$$
$$\Rightarrow \mu^q_{E_X}(t_X) \geq \mu^q_{E_X}(r_X) \Rightarrow \mu_{E_X}(t_X) \geq \mu_{E_X}(r_X).$$

In similar to $t_X \leq_X r_X$, yields that $\nu_{E_X}(t_X) \leq \nu_{E_X}(r_X)$. □

Proposition 4.4. *Any q-$ROFQ-I$ of X is a q-$ROFS_s$ of X.*

Proof. Take E_X be a q-$ROFQ-I$ of X and $t_X, r_X \in X$. Then

$$\begin{aligned} \mu^q(t_X\,\theta\, r_X) &\geq \min\{\mu^q((t_X\,\theta\, r_X)\,\theta\, t_X), \mu^q(t_X)\} = \min\{\mu^q((t_X\,\theta\, t_X)\,\theta\, r_X), \mu^q(t_X)\} \\ &= \min\{\mu^q((0_X\,\theta\, r_X), \mu^q(t_X)\} \geq \min\{\min\{\mu^q(0_X), \mu^q(r_X)\}, \mu^q(t_X)\} \\ &\geq \min\{\mu^q(r_X), \mu^q(t_X)\}. \end{aligned}$$

In similar to, $\nu^q(t_X\,\theta\, r_X) \leq \max\{\nu(t_X), \nu(r_X)\}$. Hence E_X is a q-$ROFS_s$ of X. □

The opposite of Proposition 4.4, may not be true. Take $m \in \mathbb{N}, E_X$ be a *q-ROFS* of X and $t_X, r_X, s_X \in X$. We state that E_X is m-regular, in case that $t_X\ \theta\ r_X \leq s_X$, then $\mu^m_{E_X}(t_X) \geq \min\{\mu^m_{E_X}(r_X), \mu^m_{E_X}(s_X)\}$ and $\nu^m_{E_X}(t_X) \leq \max\{\nu^m_{E_X}(r_X), \nu^m_{E_X}(s_X)\}$.

Theorem 4.5. *Take X be a Q-A.*

(i) *Any q-$ROFQ - I$ of X, is q-regular.*

(ii) *Any q-regular q-$ROFS_s$ of X is a q-$ROFQ - I$ of X.*

Proof. (i) Take $t_X, r_X, s_X \in X, t_X\ \theta\ r_X \leq s_X$ and $E_X = (\mu_{E_X}, \nu_{E_X})$ be a *q-ROFQ* $- I$ of X. Then $(t_X\ \theta\ r_X)\ \theta\ s_X = 0_X$ and so

$$\begin{aligned}\mu^q(t_X) &\geq \min\{\mu^q(t_X\ \theta\ r_X), \mu^q(r_X)\} \geq \min\{\min\{\mu^q((t_X\ \theta\ r_X)\ \theta\ s_X), \mu^q(s_X)\}, \mu^q(r_X)\} \\ &= \min\{\min\{\mu^q(0_X), \mu^q(s_X)\}, \mu^q(r_X)\} = \min\{\mu^q(s_X), \mu^q(r_X)\}.\end{aligned}$$

In similar to, $\nu^q_{E_X}(t_X) \leq \max\{\nu^q_{E_X}(r_X), \nu^q_{E_X}(s_X)\}$ and so E_X is q-regular.

(ii) Take $t_X, r_X \in X$ and E_X be a q-regular *q-ROFS*$_s$ of X. Then by definition, $\mu_{E_X}(0_X) \geq \mu^q_{E_X}(t_X)$ and $\nu^q_{E_X}(0_X) \leq \nu_{E_X}(t_X)$. Since, $\forall\, t_X, r_X \in X, t_X\ \theta\ (t_X\ \theta\ r_X) \leq_X r_X$, It yields that by hypothesis that

$$\mu_{E_X}(t_X) \geq \min\{\mu_{E_X}(t_X\ \theta\ r_X), \mu_{E_X}(r_X)\}\ and\ \nu_{E_X}(t_X) \leq \min\{\nu_{E_X}(t_X\ \theta\ r_X), \nu_{E_X}(r_X)\}.$$

Hence, E_X is a *q-ROFQ* $- I$ of X. □

Theorem 4.6. *Take E_X and F_X be q-regular q-$ROFQ - I$ of X, X', respectively and $\psi : X \to X'$ be an isomorphism.*

(i) *$\psi(E_X)$ is a q-regular a q-$ROFS_s$ of X'.*

(ii) *$\psi^{-1}(F_X)$ is a q-regular a q-$ROFS_s$ of X.*

Proof. (i) Using Theorem 3.26, $\psi(E_X)$ is a *q-ROFS*$_s$ of X'. Now, for $t'_X, r'_X, s'_X \in X'$, in case that $t'_X\ \theta\ 'r_X \leq_{X'} s'_X$, then there are $t_X, r_X, s_X \in X$ in a way that $\psi(t_X\ \theta\ r_X) \leq_{X'} \psi(s_X)$ and so $t_X\ \theta\ r_X \leq_X s_X$ because, ψ is a bijection. Since F_X is a q-regular a *q-ROFS*$_s$ of X,

$$\begin{aligned}\psi^q(\mu_{E_X})(t'_X) &= \operatorname*{Supi}_{\psi(t_X)=t'_X} \mu^q_{E_X}(t_X) \geq \operatorname*{Supi}_{\substack{\psi(r_X)=r'_X \\ \psi(s_X)=s'_X}} \min\{\mu^q_{E_X}(r_X), \mu^q_{E_X}(s_X)\} \\ &\geq \min\{\operatorname*{Supi}_{\psi(r_X)=r'_X} \mu^q_{E_X}(r_X), \operatorname*{Supi}_{\psi(s_X)=s'_X} \mu^q_{E_X}(s_X)\} \\ &= \min\{\psi^q(\mu_{E_X})(r'_X), \psi^q(\mu_{E_X})(s'_X)\}.\end{aligned}$$

In similar to, $\psi^q(\nu_{E_X})(t'_X) \leq \min\{\psi^q(\nu_{E_X})(r'_X), \psi^q(\nu_{E_X})(s'_X)\}$ and so $\psi(E_X)$ is a q-regular a *q-ROFS*$_s$ of X'.

(ii) By Theorem 3.25, and similar to (i), the proof is obtained. □

Corollary 4.7. *Take E_X and F_X be q-$ROFQ-I$ of X, X', respectively and $\psi : X \to X'$ be an isomorphism.*

(i) *$\psi(E_X)$ is a q-$ROFQ - I$ of X'.*

(ii) *$\psi^{-1}(F_X)$ is a q-$ROFQ - I$ of X.*

Theorem 4.8. *Take $E_X = (\mu_{E_X}, \nu_{E_X})$ be a q-$ROFS$ of X.*

(i) *E_X is a q-$ROFQ - I$ of X iffy $\mu^q_{E_X}$ and $\nu^{cq}{}_{E_X}$ are FQ_i of X.*

(ii) *E_X is a q-$ROFQ - I$ of X iffy $F_X = (\mu_{E_X}, \mu^c_{E_X})$ and $G = (\nu^c_{E_X}, \nu_{E_X})$ are q-$ROFQ - I$ of X.*

Proof. Take $t_X, r_X \in X$.

(i) If E_X is a *q-ROFQ* $- I$ of X, clearly μ_{E_X} is an FQ_i of X. In addition,

$$\begin{aligned} \nu^{cq}_{E_X}(0_X) &= 1-\nu^q_{E_X}(0_X) \geq 1-\nu^q_{E_X}(t_X) = \nu^{cq}_{E_X}(t_X)\ and \\ & \quad \nu^{cq}_{E_X}(t_X) = 1-\nu^q_{E_X}(t_X) \geq 1-\max\{\nu^q_{E_X}(t_X\ \theta\ r_X), \nu^q_{E_X}(t_X)\} \\ & \quad = \min\{1-\nu^q_{E_X}(t_X\ \theta\ r_X), 1-\nu^q_{E_X}(r_X)\} \\ &= \min\{\nu^{cq}_{E_X}(t_X\ \theta\ r_X), \nu^{cq}_{E_X}(r_X)\}. \end{aligned}$$

Hence, ν_{E_X} is an FQ_i of X.

Conversely, assume that $\mu^q_{E_X}$ and $\nu^{cq}_{E_X}$ are FQ_i of X. Then for each $t_X, r_X \in X, \mu^q_{E_X}(0_X) \geq \mu^q_{E_X}(t_X), \nu^q_{E_X}(0_X) = 1-\nu^{cq}_{E_X}(0_X) \leq 1-\nu^{cq}_{E_X}(t_X) = \nu^q_{E_X}(t_X)$ and

$$\begin{aligned} 1-\nu^q_{E_X}(t_X) &= \nu^{cq}_{E_X}(t_X) \geq \min\{\nu^{cq}_{E_X}(t_X\ \theta\ r_X), \nu^{cq}_{E_X}(r_X)\} = \min\{1-\nu^q_{E_X}(t_X\ \theta\ r_X), 1-\nu^q_{E_X}(r_X)\} \\ &= 1-\max\{\nu^q_{E_X}(t_X\ \theta\ r_X), \nu^q_{E_X}(r_X)\}, \end{aligned}$$

that is $\nu^q_{E_X}(t_X) \leq \max\{\nu^q_{E_X}(t_X\ \theta\ r_X), \nu^q_{E_X}(r_X)\}$. Moreover, $\mu^q_{E_X}$ is an FQ_i of X, so E_X is a *q-ROFQ*$-I$ of X.

(ii) Take E_X be a *q-ROFQ* $- I$ of X. Then by item $(i), \mu^q_{E_X}$ and $\nu^{qc}_{E_X}$ are FQ_i of X and so $\mu^{qc}_{E_X}$ and $\nu^q{}_{E_X}$ are anti FQ_i of X. It yields that $F_X = (\mu_{E_X}, \mu^c_{E_X})$ and $G = (\nu^c_{E_X}, \nu_{E_X})$ are a *q-ROFQ* $- I$ of X.

Conversely, let $F_X = (\mu_{E_X}, \mu^c_{E_X})$ and $G = (\nu^c_{E_X}, \nu_{E_X})$ are *q-ROFQ* $- I$ of X. By item (i), $\mu^q_{E_X}$ and $\nu^{qc}_{E_X}$ are FQ_i of X and so $E_X = (\mu_{E_X}, \nu_{E_X})$ is a *q-ROFQ* $- I$ of X. □

Theorem 4.9. *Take E_X, F_X be q-regular q-ROFS of X.*

(i) $\sqrt[q]{E_X}$ *is a q-regular q-ROFS of X.*

(ii) $E_X \circ F_X$ *is a q-regular q-ROFS of X.*

Proof. Take $t_X, r_X, s_X \in X$ and $t_X\ \theta\ r_X \leq_x s_X$.

(i) Since E_X is a q-regular *q-ROFS* of X,

$$\mu^q_{\sqrt[q]{E_X}}(t_X) = \operatorname*{Infi}_{n\geq 1} \mu^q_{E_X}(t^n_X) \geq \min\{\operatorname*{Infi}_{n\geq 1} \mu^q_{E_X}(r^n_X), \operatorname*{Infi}_{n\geq 1} \mu^q_{E_X}(s^n_X)\} = \min\{\mu^q_{\sqrt[q]{E_X}}(r_X), \mu^q_{\sqrt[q]{E_X}}(s_X)\}$$

and

$$\nu^q_{\sqrt[q]{E_X}}(t_X) = \operatorname*{Supi}_{n\geq 1} \nu^q_{E_X}(t^n_X) \leq \max\{\operatorname*{Supi}_{n\geq 1} \nu^q_{E_X}(r^n_X), \operatorname*{Supi}_{n\geq 1} \nu^q_{E_X}(s^n_X)\} = \max\{\nu^q_{\sqrt[q]{E_X}}(r_X), \nu^q_{\sqrt[q]{E_X}}(s_X)\}.$$

Hence, $\sqrt[q]{E_X}$ is a q-regular *q-ROFS* of X. (ii) Since E_X, F_X are q-regular *q-ROFS* of X and $t_X\ \theta\ 0_X = t_X, r_X\ \theta\ 0_X = r_X, s_X\ \theta\ 0_X = s_X$,

$$\begin{aligned} &(\mu^q_{E_X\circ F_X})(t_X) = \operatorname*{Supi}_{a\ \theta\ b=t_X} (\mu^q_{E_X}(a) \wedge \mu^q_{F_X}(b)) \leq (\mu^q_{E_X}(t_X) \wedge \mu^q_{F_X}(0_X)) \\ &\geq (\mu^q_{E_X}(r_X) \wedge \mu^q_{E_X}(s_X) \wedge \mu^q_{F_X}(0_X)) = (\nu^q_{E_X}(r_X) \wedge \mu^q_{F_X}(0_X) \wedge \mu^q_{E_X}(s_X) \wedge \mu^q_{F_X}(0_X)) \\ &\geq \operatorname*{Supi}_{c\ \theta\ d=r_X} (\mu^q_{E_X}(c) \wedge \mu^q_{F_X}(d)) \wedge \operatorname*{Supi}_{p\ \theta\ q=s_X} (\mu^q_{E_X}(p) \wedge \mu^q_{F_X}(q)) = (\mu^q_{E_X\circ F_X})(r_X) \wedge (\mu^q_{E_X\circ F_X})(s_X). \end{aligned}$$

and

$$\begin{aligned} &(\nu^q_{E_X\circ F_X})(t_X) = \operatorname*{Infi}_{a\ \theta\ b=t_X} (\nu^q_{E_X}(a) \vee \nu^q_{F_X}(b)) \leq (\nu^q_{E_X}(t_X) \vee \nu^q_{F_X}(0_X)) \\ &\leq (\nu^q_{E_X}(\vee \nu^q_{E_X}(s_X) \vee \nu^q_{F_X}(0_X)) = (\nu^q_{E_X}(r_X) \vee \mu^q_{F_X}(0_X \vee \nu^q_{E_X}(s_X) \vee \nu^q_{F_X}(0_X)) \\ &\leq \operatorname*{Infi}_{c\ \theta\ d=r_X} (\nu^q_{E_X}(c) \vee \nu^q_{F_X}(d \vee \operatorname*{Infi}_{p\ \theta\ q=s_X} (\nu^q_{E_X}(p) \vee \nu^q_{F_X}(q)) = (\nu^q_{E_X\circ F_X})(r_X) \vee (\nu^q_{E_X\circ F_X})(s_X). \end{aligned}$$

Hence, $E_X \circ F_X$ is a q-regular *q-ROFS* of X □

Corollary 4.10. *Take* $(X,\ \theta\ ,0_X)$ *be with condition A. If* E_X *be a q-ROFQ* $- I$ *of* X.

(i) $\sqrt[q]{E_X}$ *is a q-ROFQ* $- I$ *of* X.

(ii) *If* $(E_X \circ F_X)^{(q,q)} = (F_X \circ E_X)^{(q,q)}$, *then* $F_X \circ E_X$ *is a q-ROFQ* $- I$ *of* X.

Take E_X be an FS of X and $\alpha, \beta \in \mathbb{I}$. Define $U(E_X, \alpha) = \{t_X \in X \mid \mu^q_{E_X}(t_X) \geq \alpha\}$ and $L(E_X, \beta) = \{t_X \in X \mid \nu^q_{E_X}(t_X) \leq \beta\}$ are call as upper α-level cut of A_X and lower α-level cut of A_X, respectively.

Theorem 4.11. *A q-ROFS* E_X *is a q-ROFQ* $- I$ *of* X *iffy* $\forall\ \alpha, \beta \in \mathbb{I}, \emptyset \neq U(E_X, \alpha)$ *and* $\emptyset \neq L(E_X, \beta)$ *are* FQ_i *of* X.

Proof. Take $\alpha, \beta \in \mathbb{I}$ and E_X be a q-ROFQ $- I$ of X. Firstly, $\mu^q_{E_X}(t_X) \geq \alpha$ and $\nu^q_{E_X}(t_X) \leq \beta$, so $0_X \in U(E_X, \alpha) \cap L(E_X, \beta)$. Assume that $t_X\ \theta\ r_X, r_X \in U(E_X, \alpha)$. Then $\mu^q_{E_X}(t_X\ \theta\ r_X) \geq \alpha, \mu^q_{E_X}(r_X) \geq \alpha$ and $\mu^q_{E_X}(t_X) \geq \min\{\mu^q_{E_X}(t_X\ \theta\ r_X), \mu^q_{E_X}(r_X)\} \geq \min\{\alpha, \alpha\}$ and so $r_X \in U(E_X, \alpha)$. It yields that $U(E_X, \alpha)$ is an FQ_i of X and in similar to $L(E_X, \beta)$ is an FQ_i of X.

Conversely, let $U(E_X, \alpha)$ and $L(E_X, \beta)$ are FQ_i of X. For $t_X \in X, \mu^q_{E_X}(t_X) = \alpha, \nu^q_{E_X}(t_X) = \beta$. So $\{t_X, 0_X\} \subseteq U(E_X, \alpha) \cap L(E_X, \beta)$, because $U(E_X, \alpha)$ and $L(E_X, \beta)$ are FQ_i of X. Hence $\mu^q_{E_X}(0_X) \geq \alpha$ and $\nu^q_{E_X}(0_X) \leq \beta$. Let for some $t_X, r_X \in X, \mu^q_{E_X}(t_X) < \min\{\mu^q_{E_X}(t_X\ \theta\ r_X), \mu^q_{E_X}(r_X)\}$, then by taking $\gamma = \frac{1}{4}(\mu^q_{E_X}(t_X) + \min\{\mu^q_{E_X}(t_X\ \theta\ r_X), \mu^q_{E_X}(r_X)\}), \mu^q_{E_X}(t_X) < \gamma < \min\{\mu^q_{E_X}(t_X\ \theta\ r_X), \mu^q_{E_X}(r_X)\}$. Hence $t_X \in not \in U(E_X, \gamma)$, while $t_X\ \theta\ r_X \in U(E_X, \gamma)$ and $r_X \in U(E_X, \gamma)$, which is a contradiction, because $U(E_X, \gamma)$ is an ideal of X. Moreover, in case that there is $t_X, r_X \in X$ in a way that $\nu^q_{E_X}(t_X) > \max\{\nu^q_{E_X}(t_X\ \theta\ r_X), \nu^q_{E_X}(r_X)\}$, by taking $\theta == \frac{1}{2}(\mu^q_{E_X}(t_X) + \min\{\mu^q_{E_X}(t_X\ \theta\ r_X), \mu^q_{E_X}(r_X)\})$, we get that $\max\{\mu^q_{E_X}(t_X\ \theta\ r_X), \mu^q_{E_X}(r_X)\} < \gamma < \mu^q_{E_X}(t_X)$. Therefore, $t_X\ \theta\ r_X, r_X \in L(E_X,\ \theta\ )$, but $t_X \notin L(E_X,\ \theta\ )$ and is a contradiction, because $L(E_X,\ \theta\ )$ is an ideal of X. Hence E_X is a q-ROFQ $- I$ of X. ☐

Corollary 4.12. *Take* $\{E_{X\,i}\}_{i \in I}$ *be a collection of q-ROFQ* $- I$ *of* X.

(i) $\bigcap_{i \in I} E_{X\,i}$ *is a q-ROFQ* $- I$ *of* X.

(ii) *If* $\{E_{X\,i}\}_{i \in I}$ *be a chain, then* $\bigcup_{i \in I} E_{X\,i}$ *is a q-ROFQ* $- I$ *of* X.

Take $J = [0, 0.75] \subseteq [0, 1], \mathcal{J} = \{J_\lambda \mid \lambda \in \mathbb{J}\}$ be a collection of ideals of X and $\lambda, \lambda' \in \mathbb{J}$. Define $\lambda \leq \lambda'$ iffy $J_{\lambda'} \subseteq J_\lambda$ and say $(\mathcal{J}, \subseteq)$ is a poset. Also we say that X is a total, in case that $X = \bigcup_{\lambda \in \mathbb{J}} J_\lambda$ and $(\mathcal{J}, \subseteq)$ is a poset. So based the Theorem 4.11 and Corollary 4.12, have the subsequent outcomes.

Theorem 4.13. *Take* X *be a total Q-A. Then* $E_X = \{(t_X, \sup\{\lambda \in \mathbb{J} \mid t_X \in J_\lambda\ \}, \inf\{\lambda \in \mathbb{J} \mid t_X \in J_\lambda\ \})\}$ *is a q-ROFQ* $- I$ *of* X.

Proof. Take $t_X, r_X \in X$. Then there is $\lambda \in \mathbb{J}$ in a way that $t_X \in I_\lambda$ and

$$(\sup\{\lambda \in \mathbb{J} \mid t_X \in J_\lambda\ \})^q + (\inf\{\lambda \in \mathbb{J} \mid t_X \in I_\lambda\ \})^q$$
$$= \sup\{\lambda^q \in \mathbb{J} \mid t_X \in J_\lambda\ \} + \inf\{\lambda^q \in \mathbb{J} \mid t_X \in I_\lambda\ \} \leq 1$$

and so E_X is a q-ROFS of X. Take $t_X \in X$ and $\alpha \in \mathbb{I}$. Then there is $\lambda \in \mathbb{J}$ in a way that $t_X \in I_\lambda$. If $\alpha = \operatorname{Supi}_{\lambda < \alpha} \lambda$, then $U(E_X, \alpha) = \bigcap_{\lambda < \alpha} I_\lambda$, because of, $\forall \lambda < \alpha, t_X \in U(E_X, \alpha) \Leftrightarrow t_X \in \bigcap_{\lambda < \alpha} I_\lambda$. Thus $U(E_X, \alpha)$ is an FQ_i of X. If $\alpha \neq \operatorname{Supi}_{\lambda < \alpha} \lambda$, then there is $\theta\ > 0$ in a way that $(\alpha -\ \theta\ , \alpha) \cap \mathbb{J} = \emptyset$. Assume that $t_X \in \bigcup_{\lambda \geq \alpha} I_\lambda$, then there is $\lambda \geq \alpha$ in a way that $t_X \in I_\lambda$ and so $\mu_{E_X}(t_X)\lambda \geq \alpha$. Hence $\bigcup_{\lambda \geq \alpha} I_\lambda \subseteq U(E_X, \alpha)$.

Suppose $t_X \notin \bigcup_{\lambda \geq \alpha} I_\lambda$, then $\forall \lambda \geq \alpha, t_X \notin I_\lambda$ and so for all $\lambda > \alpha - \theta$, $t_X \notin I_\lambda$. Thus $\mu_{E_X}(t_X) \leq \alpha - \theta < \alpha$ and so $t_X \notin U(E_X, \alpha)$. Therefore, $\bigcup_{\lambda \geq \alpha} I_\lambda = U(E_X, \alpha)$ and so in any cases, $U(E_X, \alpha)$, is an FQ_i of X. Now, let $\beta = \operatorname{Infi}_{\beta < \lambda} \lambda$. Then $L(E_X, \beta) = \bigcap_{\beta < \lambda} I_\lambda$, because of $\forall \beta < \lambda, t_X \in I_\lambda \Leftrightarrow t_X \in \bigcap_{\beta < \lambda} I_\lambda$. Hence $L(E_X, \beta)$ is an FQ_i of X. If $\beta \neq \operatorname{Infi}_{\beta < \lambda} \lambda$, then there is $\theta > 0$ in a way that $(\beta, \beta + \theta) \cap J = \emptyset$. Take $t_X \in \bigcup_{\lambda \leq \beta} I_\lambda$, then there is $\lambda \leq \beta$ in a way that $t_X \in I_\lambda$. Then $\mu_{E_X}(t_X) \leq \lambda < \beta$ so that $t_X \in L(E_X, \beta)$. Hence $\bigcup_{\lambda \leq \beta} I_\lambda \subseteq L(E_X, \beta)$. If $t_X \notin \bigcup_{\lambda \leq \beta} I_\lambda$, then $\forall \lambda \leq \beta, t_X \notin I_\lambda$, special $t_X \notin I_\lambda$, which $\lambda < \beta + \theta$. It yields that $\nu_{E_X}(t_X) \geq \beta + \theta > \beta$ and so $t_X \notin L(E_X, \beta)$. Therefore, $L(E_X, \beta) \subseteq \bigcup_{\lambda \leq \beta} I_\lambda$ an in any cases, $L(E_X, \beta) \subseteq \bigcup_{\lambda \leq \beta} I_\lambda$, which yields that $L(E_X, \beta)$ is an FQ_i of X. □

5 Application of q-ROFQ-s.as in complex network

In this section, we apply the notion of q-ROFQ-s.as in complex network.

Social Network and its Interaction Rules:

Step 1:Take $X = \{Karun, Ayhan, Ryan, Ala\}$ represent a set of users in a social network. To model the complex relationships and influence within this network, we define a binary operation $*$ that represents the outcome of a *dispute* or *influence clash* between two users. For instance, $t_X * r_X$ could be interpreted as the user whose viewpoint or influence dominates after an interaction between t_X and r_X. This operation, defined by the following Cayley table, forms a *Q-A*. In this structure, $Karun$ is the *null element*, representing a neutral or baseline state.

Table 1: Cayley Table: Outcome of Influence Clashes in the Network

$*$	$Karun$	$Ayhan$	$Ryan$	Ala
$Karun$	$Karun$	$Karun$	$Karun$	$Karun$
$Ayhan$	$Ayhan$	$Karun$	$Ayhan$	Ala
$Ryan$	$Ryan$	$Ryan$	$Karun$	$Ryan$
Ala	Ala	Ala	Ala	$Karun$

Step 2: **Modeling Trust and Distrust with a q-ROFQ-S:**

We now model each user's profile using a q-ROFS to capture their inherent trustworthiness (μ) and untrustworthiness (ν). For this network, we choose $q = 3$ to allow for high levels of simultaneous trust and distrust. We define a specific q-ROF set $\mathbb{T}$ (for "Trust"). For $\mathbb{T}$ to be a *q-ROFQ-S*, it must ensure that the outcome of any influence clash (operation $*$) outcomes in a user with a consistent trust profile. The following table defines $\mathbb{T}$ and verifies its validity.

Table 2: The q-ROFQ-S $\mathbb{T}$: Trust Profiles ($q = 3$)

User	**Description**	**Trust ($\mu_{\mathbb{T}}$)**	**Distrust ($\nu_{\mathbb{T}}$)**	**Validity ($\mu^3 + \nu^3 \leq 1$)**
$Karun$	Neutral admin; a benign baseline.	1.0	0.0	$1.0 \leq 1$
$Ayhan$	Controversial influencer.	0.7	0.8	$0.855 \leq 1$
$Ryan$	New, low-engagement user.	0.5	0.4	$0.189 \leq 1$
Ala	Polarizing content creator.	0.7	0.8	$0.855 \leq 1$

Step 3:**Applying the Model: Predicting the Outcome of an Interaction:** The power of this model is its ability to make consistent predictions. Let's analyze a clash between $Ayhan$ and Ala.

(1) According to the interaction rule (Table 1), $Ayhan * Ala = Ala$.

(2) The subalgebra axiom requires that the trust value of the result must be greater than or equal to the minimum of the original trust values:

$$\mu_{\mathbb{T}}(Ayhan * Ala) = \mu_{\mathbb{T}}(Ala) = 0.7 \geq \min(\mu_{\mathbb{T}}(Ayhan), \mu_{\mathbb{T}}(Ala)) = \min(0.7, 0.7) = 0.7$$

(3) Similarly, the distrust of the result must be less than or equal to the maximum of the original distrust:

$$\nu_{\mathbb{T}}(Ayhan * Ala) = \nu_{\mathbb{T}}(Ala) = 0.8 \leq \max(\nu_{\mathbb{T}}(Ayhan), \nu_{\mathbb{T}}(Ala)) = \max(0.8, 0.8) = 0.8$$

This example demonstrates that our theoretical framework is not merely abstract. It provides a concrete, algebraically sound model for a social network. The q-ROFQ-S $\mathbb{T}$ ensures that the trust and distrust values assigned to users are *consistent* with the predefined rules of interaction within the Q-A. This allows for the analysis of complex group dynamics and the propagation of influence in a mathematically rigorous way, which is a primary contribution of this work.

Economic: In economics, particularly in credit scoring and financial risk assessment, the evaluation of an entity (e.g., an individual, a company, a country) is not one-dimensional. There is often simultaneous, conflicting information: positive factors contribute to a degree of creditworthiness (μ), while negative factors contribute to a degree of default risk (ν). A q-ROFQ-S provides a natural framework to model this dualism and, crucially, to study how these ratings behave under economic interactions or transactions.

Take $X = \{Ee_1, Ee_2, Ee_3, Ee_4\}$ represent a set of economic entities (e.g., firms in a supply chain). We define a binary operation $\bullet$ on X that models a *significant economic transaction* or *merger* between two entities. For example, $Ee_i \bullet Ee_j$ could represent the new entity formed after a joint venture or the dominant entity that emerges from a competitive market interaction. This operation forms a Q-A, satisfying the necessary axioms. Let us fix Ee_1 to be a very stable, benchmark entity (e.g., a central bank).

Table 3: Cayley Table for the Economic Q-A $(X, \bullet, Ee_1)$

$\bullet$	Ee_1	Ee_2	Ee_3	Ee_4
1	Ee_1	Ee_1	Ee_1	Ee_1
Ee_2	Ee_2	Ee_1	Ee_2	Ee_4
Ee_3	Ee_3	Ee_3	Ee_1	Ee_3
Ee_4	Ee_4	Ee_4	Ee_4	Ee_1

The q-ROFQ-S for Economic Rating: We now define a q-ROFS $\mathbb{C} = \{\langle t_X, \mu_{\mathbb{C}}(t_X), \nu_{\mathbb{C}}(t_X)\rangle \mid t_X \in X\}$ to represent a composite "Economic Stability Rating". We choose $q = 2$ for this example, reflecting a common weighting of factors.

Table 4: The q-ROFQ-S $\mathbb{C}$: Economic Stability Profiles ($q = 2$)

Entity	**Description**	**Creditworthiness ($\mu_{\mathbb{C}}$)**	**Default Risk ($\nu_{\mathbb{C}}$)**	**($\mu^2 + \nu^2 \leq 1$)**
Ee_1	Stable Benchmark Entity	0.95	0.2	$0.9425 \leq 1$
Ee_2	Established Corp. (Moderate Debt)	0.7	0.6	$0.85 \leq 1$
Ee_3	High-Growth Startup (Volatile)	0.6	0.7	$0.85 \leq 1$
Ee_4	Distressed Asset Holder	0.5	0.8	$0.89 \leq 1$

Application: Assessing the Risk of a Joint Venture: Consider a joint venture between the established corporation Ee_2 and the distressed asset holder Ee_4. According to the economic interaction rules in Table 3, the outcome is $Ee_2 \bullet Ee_4 = Ee_4$. This suggests the transaction exposes the resulting entity to the risk profile of the more unstable partner. The q-ROFQ-S $\mathbb{C}$ allows us to analyze this formally:

(1) The resulting entity's creditworthiness is $\mu_{\mathbb{C}}(Ee_4) = 0.5$.

(2) The subalgebra axiom requires this value to be consistent with the original entities:
$\mu_{\mathbb{C}}(Ee_4) \geq \min(\mu_{\mathbb{C}}(Ee_2), \mu_{\mathbb{C}}(Ee_4)) = \min(0.7, 0.5) = 0.5$. This holds true.

(3) The resulting default risk is $\nu_{\mathbb{C}}(Ee_4) = 0.8$.

- The axiom requires $\nu_{\mathbb{C}}(Ee_4) \leq \max(\nu_{\mathbb{C}}(Ee_2), \nu_{\mathbb{C}}(Ee_4)) = \max(0.6, 0.8) = 0.8$. This also holds true.

This example demonstrates that the theory of q-**ROF**Q-**S** is a powerful tool for economic and financial modeling. This application underscores the value of the proposed algebraic structures in moving beyond linear, one-sided economic assessments towards a more robust, nonlinear analysis of complex economic systems.

6 Conclusion, Discussion, Advantages, Limitations

This paper has established a novel and comprehensive framework for generalizing fuzzy algebraic structures by integrating the powerful model of q-ROFSs (q-ROFS) with the theory of Q-As. Our primary contribution is the formal introduction and detailed investigation of q-ROFQ-S and q-ROFQ-Is (q-$ROFQ-I$). The core of our work demonstrates that the q-$ROFS_s$ structure provides a robust and intuitive unification of traditional F-Q-s.as. The key insight is that a q-$ROFS_s$ is interchangeable to a pair consisting of an F-Q-S and an AF-Q-S, which we termed a "two-sided" q-$ROFS_s$. This duality allows for the independent yet constrained modeling of community and non-community degrees, capturing a more nuanced view of approximation and opposition inherent in complex, nonlinear systems. We successfully extended fundamental concepts from fuzzy algebra into this new setting. Notably: We defined and characterized the q-ROF level subset and the nil radical, proving their crucial properties and establishing that the nil radical of any q-$ROFS_s$ is itself a q-$ROFS_s$. We advanced the study of morphisms by introducing the null radial and investigating the behavior of q-$ROFS_s$ under isomorphisms, showing the invariance of these structures under certain conditions. We bridged the gap between subalgebras and ideals by proving that every q-regular q-$ROFS_s$ is a q-$ROFQ-I$, thus creating a strong link between these two central concepts. Furthermore, we addressed the challenge of combining these structures by establishing that the union of q-$ROFS_s$ can form another q-$ROFS_s$ under specific commutative conditions, a non-trivial result given the nonlinear nature of the q-rung orthopair pairs. The q-ROF model for Q-As presents several significant advantages over traditional fuzzy approaches: 1. Enhanced Expressiveness: The model's ability to handle independent community and non-community functions (μ and ν) with the flexible constraint $\mu^q + \nu^q \leq 1$ allows for the representation of a much wider spectrum of approximation and conflicting information, which is often encountered in real-world decision-making and pattern recognition. 2. Generalization: Our framework is a true unification. By setting $q = 1$ and $\nu(t_X) = 1 - \mu(t_X)$, it reduces to the standard IFS model, and by further restriction, to the classical fuzzy model. This backward compatibility ensures wider applicability. 3. Theoretical Robustness: The algebraic properties we have proven (e.g., behavior under homomorphisms, properties of the nil radical, the link between regularity and ideals) provide a solid and consistent mathematical foundation for future applications in areas like automated reasoning, expert systems, and network security.

Future Work: Despite its theoretical contributions, this work has certain limitations that pave the way for future research:

1. Computational Complexity: The practical application of these structures, especially for large sets X or high values of q, may involve complex computations to verify the subalgebra and ideal conditions for all pairs of elements. Developing efficient algorithms for these tasks is an important next step. 2. Specificity of Conditions: Some of our key outcomes, such as the union of q-$ROFS_s$ being a q-$ROFS_s$, rely on specific commutative conditions within the Q-A. Exploring whether these conditions can be relaxed or replaced remains an open problem. 3. Application Validation: While we have proposed potential applications in social networks and economics, empirical validation and case studies are necessary to demonstrate the practical efficacy and superiority of this model over existing ones.

Future research directions will focus on: Exploring other operations (e.g., intersections, products) on q-$ROFQ$-subalgebras and ideals. Investigating the structures of q-ROFQ-S more broadly. Developing concrete applications in fields like artificial intelligence, graph theory, and complex multi-criteria decision-making problems to test and demonstrate the practical value of the theoretical framework established herein.

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