A Comparative Study of Neutrosophic Subalgebras in Sheffer Stroke UP-algebras

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Abstract

In this paper, we conduct a comprehensive study of neutrosophic subalgebras of various types within the framework of Sheffer stroke UP-algebras (SUP-algebras). Specifically, we introduce and characterize $(\in, \in)$, $(\in, \in \vee q)$, and $(q, \in \vee q)$-neutrosophic subalgebras based on neutrosophic $\in$-subsets, q-subsets, and $(\in \vee q)$-subsets. Necessary and sufficient conditions are established for these subsets to form subalgebras under the Sheffer stroke operation. Several theorems demonstrate how these types interrelate and differ in their structural properties, with illustrative examples provided. Furthermore, we identify the conditions under which certain canonical subsets, such as $X_0^1 = \{x \in X \mid T(x) > 0, I(x) > 0, F(x) < 1\}$, form subalgebras across different neutrosophic configurations. These results offer a unified perspective and deeper insight into the algebraic behavior of neutrosophic systems in the context of SUP-algebras.

Keywords: Neutrosophic set; Neutrosophic $\in$-subset; Neutrosophic q-subset; Neutrosophic $\in \vee q$-subset; $(\in, \in)$-neutrosophic subalgebra; $(\in, q)$-neutrosophic subalgebra; $(q, \in)$-neutrosophic subalgebra; $(q, \in \vee q)$-neutrosophic subalgebra

1 Introduction

Zadeh[12] introduced fuzzy sets by assigning a degree of membership $t \in [0, 1]$, and Atanassov[1] extended this idea by incorporating a nonmembership degree $f \in [0, 1]$, leading to the framework of intuitionistic fuzzy sets. Smarandache further generalized this setting through the notion of neutrosophic sets,[10] where truth, indeterminacy, and falsity are treated as three independent components (t, i, f), providing a richer platform for uncertainty modeling.

To further generalize uncertainty modeling, Smarandache[9] introduced the concept of neutrosophic sets, extending the intuitionistic framework by incorporating a third independent component: the degree of indeterminacy $i \in [0, 1]$, alongside truth t and falsehood f. This triplet (t, i, f) allows for explicit representation of incomplete or contradictory information, surpassing the binary and intuitionistic paradigms in capturing neutrality and ambiguity.

Neutrosophic sets[9] generalize classical, fuzzy, and intuitionistic fuzzy sets by allowing independent specification of truth (t), indeterminacy (i), and falsehood (f) for each element. This unified framework has enabled broad applications in logic, decision theory, and data-driven domains. For further developments, see the comprehensive repository.[11]

The Sheffer stroke (NAND), introduced by Sheffer,[8] is a functionally complete binary operation: any propositional formula or Boolean axiom can be expressed using only this operator. Its self-sufficiency offers structural simplicity, making it an effective foundation for algebraic formulations of logic, including Boolean and UP-algebraic systems.[2]

Recent studies have shown that Sheffer stroke UP-algebras admit a wide variety of structural and uncertainty-based extensions. The algebraic foundations were strengthened in[3] through the investigation of ideals, while hesitant fuzzy and fuzzy set approaches were incorporated in[4] to handle multi-valued uncertainty. Building on this line, intuitionistic fuzzy structures were developed to capture dual membership and nonmembership aspects.[7] More recently, structural variations such as semidetached SUP-subalgebras were explored in.[6] Collectively, these contributions demonstrate both the algebraic richness and adaptability of Sheffer stroke UP-algebras, motivating the present comparative investigation of neutrosophic subalgebras of several types.

This work establishes algebraic characterizations for $(\in, \in)$ and $(q, \in \vee q)$-neutrosophic subalgebras within SUP-algebras. We investigate the structural conditions under which neutrosophic $\in$, q, and $(\in \vee q)$-subsets induce valid subalgebras, and identify the necessary constraints for a neutrosophic set to qualify as a $(q, \in \vee q)$-subalgebra.

2 Preliminaries

In this section, we recall the foundational concepts necessary for the development of neutrosophic subalgebras in the context of SUP-algebras. We begin with the definition of the Sheffer operation and the algebraic structure known as SUP-algebra. We then present the notions of neutrosophic sets and associated subsets, which serve as the primary framework for defining and analyzing various types of neutrosophic subalgebras introduced later in the paper.

Definition 2.1. [8] Let $X = \langle X, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions: for all $x, y, z \in X$,

$$\begin{aligned}
&\text{(S1) } x|y = y|x\\
&\text{(S2) } (x|x)|(x|y) = x\\
&\text{(S3) } x|((y|z)|(y|z)) = ((x|y)|(x|y))|z\\
&\text{(S4) } (x|((x|x)|(y|y)))|(x|((x|x)|(y|y))) = x.
\end{aligned}$$

Definition 2.2. [5] A Sheffer stroke UP-algebra (briefly, SUP-algebra) is a structure $\langle X, |, 0\rangle$ of type $(2, 0)$ such that 0 is the fixed element in X and the following conditions are satisfied for all $x, y, z \in X$,

$$\begin{aligned}
&\text{(SUP-1) } (((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|((y|(x|x))|\\
&\qquad (z|(y|y)))))|(((z|(x|x))|(z|(x|x)))|(((y|(x|x))|(z|(y|y)))|\\
&\qquad ((y|(x|x))|(z|(y|y))))) = 0\\
&\text{(SUP-2) } x|x = x|(0|0)\\
&\text{(SUP-3) } (x|(y|y))|(x|(y|y)) = 0 \ and \ (y|(x|x))|(y|(x|x)) = 0 \Rightarrow x = y.
\end{aligned}$$

Proposition 2.3. [5] *Let $X = \langle X, |, 0\rangle$ be an SUP-algebra. Then the binary relation $x \leq y$ if and only if $(y|(x|x))|(y|(x|x)) = 0$ is a partial order on X.*

Lemma 2.4. [5] *Let $X = \langle X, |, 0\rangle$ be an SUP-algebra. Then for all $x, y, z \in X$, we have*

(1) $x \leq y \Rightarrow y|(z|z) \leq x|(z|z)$, $z|(x|x) \leq z|(y|y)$

(2) $x \leq y \Leftrightarrow y|y \leq x|x$

(3) $y|(x|x) \leq x$

(4) $y \leq (y|(x|x))|(y|(x|x))$

(5) $x \leq y \Rightarrow x \leq (y|(z|z))|(y|(z|z))$

(6) $z|(y|y) \leq z|(y|(x|x))$

(7) $((z|(y|y))|(z|(y|y)))|(x|x) \leq z|(y|(x|x))$

(8) $x|((y|(z|z))|(y|(z|z))) \leq (x|(y|y))|((x|(z|z))|(x|(z|z)))$.

Definition 2.5. [5] A nonempty subset G of an SUP-algebra $X = \langle X, |, 0\rangle$ is called a subalgebra of X if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.

Definition 2.6. [3] A nonempty subset G of an SUP-algebra $X = \langle X, |, 0\rangle$ is called an ideal of X if for all $x, y \in X$,

(1) $y \in G \Rightarrow (y|(x|x))|(y|(x|x)) \in G$

(2) $(y|(x|x))|(y|(x|x)) \in G$ and $x \in G \Rightarrow y \in G$.

Definition 2.7. [11] A neutrosophic set in a nonempty set X is defined to be a structure

$$A := \{(x, A_T(x), A_I(x), A_F(x)) \mid x \in X\}, \tag{2.1}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. The neutrosophic set in (2.1) is simply denoted by $A = (A_T, A_I, A_F)$.

3 Comparative Analysis of Neutrosophic Subalgebras in SUP-Algebras

Building upon the notions introduced in the previous section, we now explore multiple classes of neutrosophic subalgebras within the framework of SUP-algebras. By employing the structure of neutrosophic sets—characterized by independent truth, indeterminacy, and falsity degrees—we define and analyze the behavior of $(\in, \in)$, $(\in, \in \vee q)$, and $(q, \in \vee q)$-neutrosophic subalgebras. Each class is examined through precise algebraic conditions that determine when the corresponding neutrosophic subsets yield valid subalgebras. This section presents a unified and comparative perspective, supported by theorems and illustrative examples.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$T_\in(A, \alpha) = \{x \in X \mid A_T(x) \geq \alpha\}$$
$$I_\in(A, \beta) = \{x \in X \mid A_I(x) \geq \beta\}$$
$$F_\in(A, \gamma) = \{x \in X \mid A_F(x) \leq \gamma\}$$
$$T_q(A, \alpha) = \{x \in X \mid A_T(x) + \alpha > 1\}$$
$$I_q(A, \beta) = \{x \in X \mid A_I(x) + \beta > 1\}$$
$$F_q(A, \gamma) = \{x \in X \mid A_F(x) + \gamma < 1\}$$
$$T_{\in\vee q}(A, \alpha) = \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\}$$
$$I_{\in\vee q}(A, \beta) = \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\}$$
$$F_{\in\vee q}(A, \gamma) = \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.$$

We say $U_\in(A, \alpha)$, $I_\in(A, \beta)$, and $F_\in(A, \gamma)$ are neutrosophic $\in$-subsets of X, and $U_q(A, \alpha)$, $I_q(A, \beta)$, and $F_q(A, \gamma)$ are neutrosophic q-subsets of X, and $U_{\in\vee q}(A, \alpha)$, $I_{\in\vee q}(A, \beta)$, and $F_{\in\vee q}(A, \gamma)$ are neutrosophic $\in \vee q$-subsets of X. For $\Phi \in \{\in, q, q \in \vee q\}$, the element of $T_\Phi(A, \alpha)$ (resp., $I_\Phi(A, \beta)$, $F_\Phi(A, \gamma)$) is called a neutrosophic T_Φ-point (resp., neutrosophic I_Φ-point, neutrosophic F_Φ-point) with value α (resp., β, γ). It is clear that

$$T_{\in\vee q}(A, \alpha) = T_\in(A, \alpha) \cup T_q(A, \alpha)$$
$$I_{\in\vee q}(A, \beta) = I_\in(A, \beta) \cup I_q(A, \beta)$$
$$F_{\in\vee q}(A, \gamma) = F_\in(A, \gamma) \cup F_q(A, \gamma).$$

Definition 3.1. Given $\Phi, \Psi \in \{\in, q, q \in \vee q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in an SUP-algebra X is called an (Φ, Ψ)-neutrosophic subalgebra of X if the following assertions are valid: for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]$, and $\gamma_x, \gamma_y \in [0, 1)$,

$$\begin{array}{l} x \in T_\Phi(A, \alpha_x), y \in T_\Phi(A, \alpha_y) \Rightarrow (x|(y|y))|(x|(y|y)) \in T_\Psi(A, \alpha_x \wedge \alpha_y) \\ x \in I_\Phi(A, \beta_x), y \in I_\Phi(A, \beta_y) \Rightarrow (x|(y|y))|(x|(y|y)) \in I_\Psi(A, \beta_x \wedge \beta_y) \\ x \in F_\Phi(A, \gamma_x), y \in F_\Phi(A, \gamma_y) \Rightarrow (x|(y|y))|(x|(y|y)) \in F_\Psi(A, \gamma_x \vee \gamma_y). \end{array} \tag{3.1}$$

Theorem 3.2. *A neutrosophic set $A = (A_T, A_I, A_F)$ in an SUP-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X if and only if it satisfies*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T((x|(y|y))|(x|(y|y))) \geq A_T(x) \wedge A_T(y) \\ A_I((x|(y|y))|(x|(y|y))) \geq A_I(x) \wedge A_I(y) \\ A_F((x|(y|y))|(x|(y|y))) \leq A_F(x) \vee A_F(y) \end{array} \right). \tag{3.2}$$

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. If there exist $x, y \in X$ such that $A_T((x|(y|y))|(x|(y|y))) < A_T(x) \wedge A_T(y)$, then $A_T((x|(y|y))|(x|(y|y))) < \alpha_t \leq A_T(x) \wedge A_T(y)$ for some $\alpha_t \in (0, 1]$. It follows that $x, y \in T_\in(A, \alpha_t)$ but $(x|(y|y))|(x|(y|y)) \notin T_\in(A, \alpha_t)$. Hence, $A_T((x|(y|y))|(x|(y|y))) \geq A_T(x) \wedge A_T(y)$ for all $x, y \in X$. Similarly, we show that $A_I((x|(y|y))|(x|(y|y))) \geq A_I(x) \wedge A_I(y)$ for all $x, y \in X$. Suppose that there exist $a, b \in X$ and $\gamma_s \in [0, 1]$ be such that $A_F((a|(b|b))|(a|(b|b))) > \gamma_s \geq A_F(a) \vee A_F(b)$. Then $a, b \in F_\in(A, \gamma_s)$ and $(a|(b|b))|(a|(b|b)) \notin F_\in(A, \gamma_s)$, which is a contradiction. Therefore, $A_F((x|(y|y))|(x|(y|y))) \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which satisfies the condition (3.2). Let $x, y \in X$ be such that $x \in T_\in(A, \alpha_x)$ and $y \in T_\in(A, \alpha_y)$. Then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$, which imply that $A_T((x|(y|y))|(x|(y|y))) \geq A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y$, that is, $(x|(y|y))|(x|(y|y)) \in T_\in(A, \alpha_x \wedge \alpha_y)$. Similarly, if $x \in I_\in(A, \beta_x)$ and $y \in I_\in(A, \beta_y)$, then $(x|(y|y))|(x|(y|y)) \in I_\in(A, \beta_x \wedge \beta_y)$. Now, let $x \in F_\in(A, \gamma_x)$ and $y \in F_\in(A, \gamma_y)$ for $x, y \in X$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$, and so $A_F((x|(y|y))|(x|(y|y))) \leq A_F(x) \vee A_F(y) \leq \gamma_x \vee \gamma_y$. Hence, $(x|(y|y))|(x|(y|y)) \in F_\in(A, x \vee y)$. Therefore, $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. □

Theorem 3.3. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of an SUP-algebra X, then the subsets $T_q(A, \alpha)$, $I_q(A, \beta)$, and $F_q(A, \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ whenever they are nonempty.*

Proof. Let $x, y \in T_q(A, \alpha)$. Then $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$. It follows that $A_T((x|(y|y))|(x|(y|y))) + \alpha \geq (A_T(x) \wedge A_T(y)) + \alpha = (A_T(x) + \alpha) \wedge (A_T(y) + \alpha) > 1$ and so that $(x|(y|y))|(x|(y|y)) \in T_q(A, \alpha)$. Hence, $T_q(A, \alpha)$ is a subalgebra of X. Similarly, we can prove that $I_q(A, \beta)$ is a subalgebra of X. Now, let $x, y \in F_q(A, \beta)$. Then $A_F(x) + \gamma < 1$ and $A_F(y) + \gamma < 1$, which imply that $A_F((x|(y|y))|(x|(y|y))) + \gamma \leq (A_F(x) \vee A_F(y)) + \gamma = (A_F(x) + \gamma) \vee (A_F(y) + \gamma) < 1$. Hence, $(x|(y|y))|(x|(y|y)) \in F_q(A, \gamma)$, and $F_q(A, \gamma)$ is a subalgebra of X. □

The converse of Theorem 3.3 is not true, as seen in the following example.

Example 3.4. Consider an SUP-algebra $X = \{0, a, b, c\}$ with the following Cayley table:

$\vert$	0	a	b	c
0	a	a	a	a
a	a	0	c	b
b	a	c	c	a
c	a	b	b	b

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X as follows:
If $a \in [0.4, 0.6)$, $b \in [0.5, 0.8)$, and $\gamma \in (0.2, 0.5]$, then $T_q(A, \alpha)$, $I_q(A, \beta)$, and $F_q(A, \gamma)$ are subalgebras of X. But $A = (A_T, A_I, A_F)$ is not an $(\in, \in)$-neutrosophic subalgebra of X.

Theorem 3.5. *If $A = (A_T, A_I, A_F)$ is a $(q, \in \vee q)$-neutrosophic subalgebra of an SUP-algebra X, then neutrosophic q-subsets $T_q(A, \alpha)$, $I_q(A, \beta)$, and $F_q(A, \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.8	0.2
a	0.4	0.5	0.7
b	0.4	0.5	0.6
c	0.4	0.5	0.5

Proof. Let $x, y \in T_q(A, \alpha)$. Then $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A, \alpha)$, and so $(x|(y|y))|(x|(y|y)) \in T_\in(A, \alpha)$ or $(x|(y|y))|(x|(y|y)) \in T_q(A, \alpha)$. If $(x|(y|y))|(x|(y|y)) \in T_\in(A, \alpha)$, then $A_T((x|(y|y))|(x|(y|y))) \geq \alpha > 1 - \alpha$ since $\alpha > 0.5$. Hence, $(x|(y|y))|(x|(y|y)) \in T_q(A, \alpha)$. Therefore, $T_q(A, \alpha)$ is a subalgebra of X. Similarly, we prove that $I_q(A, \beta)$ is a subalgebra of X. Let $x, y \in F_q(A, \gamma)$. Then $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A, \gamma)$, and so $(x|(y|y))|(x|(y|y)) \in F_\in(A, \alpha)$ or $(x|(y|y))|(x|(y|y)) \in F_q(A, \alpha)$. If $(x|(y|y))|(x|(y|y)) \in F_\in(A, \gamma)$, then $A_F((x|(y|y))|(x|(y|y))) \leq \gamma < \gamma - 1$ since $\gamma \in [0, 0.5)$. Hence, $(x|(y|y))|(x|(y|y)) \in F_q(A, \gamma)$, and therefore $F_q(A, \gamma)$ is a subalgebra of X. □

Theorem 3.6. *A neutrosophic set $A = (A_T, A_I, A_F)$ in an SUP-algebra X is an $(\in, \in \vee q)$-neutrosophic subalgebra of X if and only if it satisfies*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T((x|(y|y))|(x|(y|y))) \geq \bigwedge\{A_T(x) \wedge A_T(y), 0.5\} \\ A_I((x|(y|y))|(x|(y|y))) \geq \bigwedge\{A_I(x) \wedge A_I(y), 0.5\} \\ A_F((x|(y|y))|(x|(y|y))) \leq \bigvee\{A_F(x) \vee A_F(y), 0.5\} \end{array} \right). \tag{3.3}$$

Proof. Suppose that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of X and let $x, y \in X$. If $A_T(x) \wedge A_T(y) < 0.5$, then $A_T((x|(y|y))|(x|(y|y))) \geq A_T(x) \wedge A_T(y)$. For, assume that $A_T((x|(y|y))|(x|(y|y))) < A_T(x) \wedge A_T(y)$ and choose α_t such that $A_T((x|(y|y))|(x|(y|y))) < \alpha_t < A_T(x) \wedge A_T(y)$. Then $x \in T_\in(A, \alpha_t)$ and $y \in T_\in(A, \alpha_t)$ but $(x|(y|y))|(x|(y|y)) \notin T_\in(A, \alpha_t)$. Also $A_T((x|(y|y))|(x|(y|y))) + \alpha_t < 1$, that is, $(x|(y|y))|(x|(y|y)) \in T_q(A, \alpha_t)$. Thus, $(x|(y|y))|(x|(y|y)) \notin T_{\in\vee q}(A, \alpha_t)$, a contradiction. Therefore, $A_T((x|(y|y))|(x|(y|y))) \geq \bigvee\{A_T(x), A_T(y), 0.5\}$ whenever $A_T(x) \wedge A_T(y) < 0.5$. Now suppose that $A_T(x) \wedge A_T(y) \geq 0.5$. Then $x \in T_\in(A, 0.5)$ and $y \in T_\in(A, 0.5)$, which imply that $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A, 0.5)$. Hence, $A_T((x|(y|y))|(x|(y|y))) \geq 0.5$. Otherwise, $A_T((x|(y|y))|(x|(y|y))) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently, $A_T((x|(y|y))|(x|(y|y))) \geq \bigvee\{A_T(x), A_T(y), 0.5\}$ for all $x, y \in X$. Similarly, we know that $A_I((x|(y|y))|(x|(y|y))) \geq \bigvee\{A_I(x), A_I(y), 0.5\}$ for all $x, y \in X$. Suppose that $A_F(x)_A \vee F(y) > 0.5$. If $A_F((x|(y|y))|(x|(y|y))) > A_F(x) \vee A_F(y) = \gamma_s$, then $x, y \in F_\in(A, \gamma_s)$, not $(x|(y|y))|(x|(y|y)) \notin F_\in(A, \gamma)$ and $A_F((x|(y|y))|(x|(y|y))) + \gamma > 2\gamma > 1$, that is, $(x|(y|y))|(x|(y|y)) \notin F_q(A, \gamma)$. This is a contradiction. Hence, $A_F((x|(y|y))|(x|(y|y))) \leq \bigwedge\{A_F(x), A_F(y), 0.5\}$ whenever $A_F(x) \vee A_F(y) > 0.5$. Now, assume that $A_F(x) \vee A_F(y) \leq 0.5$. Then $x, y \in F_\in(A, 0.5)$ and so $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A, 0.5)$. Thus, $A_F((x|(y|y))|(x|(y|y))) \leq 0.5$ or $A_F((x|(y|y))|(x|(y|y))) + 0.5 < 1$. If $A_F((x|(y|y))|(x|(y|y))) > 0.5$, then $A_F((x|(y|y))|(x|(y|y))) + 0.5 > 0.5 + 0.5 = 1$, a contradiction. Thus, $A_F((x|(y|y))|(x|(y|y))) \leq 0.5$, and so $A_F((x|(y|y))|(x|(y|y))) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ whenever $A_F(x) \vee A_F(y) \leq 0.5$. Therefore, $A_F((x|(y|y))|(x|(y|y))) \leq \bigwedge\{A_F(x), A_F(y), 0.5\}$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which satisfies the condition (3.3). Let $x, y \in X$ and $\alpha_x, \alpha_y, \beta_x, \beta_y, \gamma_x, \gamma_y \in [0, 1]$. If $x \in T_\in(A, \alpha_x)$ and $y \in T_\in(A, \alpha_y)$, then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \beta_y$. If $A_T((x|(y|y))|(x|(y|y))) < \alpha_x \wedge \alpha_y$, then $A_T(x) \wedge A_T(y) \geq 0.5$. Otherwise, we have $A_T((x|(y|y))|(x|(y|y))) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} = A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y$, a contradiction. It follows that $A_T((x|(y|y))|(x|(y|y))) + \alpha_x \wedge \alpha_y > 2A_T((x|(y|y))|(x|(y|y))) \geq 2\bigwedge\{A_T(x), A_T(y), 0.5\} = 1$ and so that $(x|(y|y))|(x|(y|y)) \in T_q(A, \alpha_x \wedge \alpha_y) \subseteq T_{\in\vee q}(A, \alpha_x \wedge \alpha_y)$. Similarly, if $x \in I_\in(A, \alpha_x)$ and $y \in I_\in(A, \alpha_y)$, then $(x|(y|y))|(x|(y|y)) \in I_{\in\vee q}(A, \alpha_x \wedge \alpha_y)$. Now, let $x \in F_\in(A, \alpha_x)$ and $y \in F_\in(A, \alpha_y)$. Then $A_F(x) \leq \alpha_x$ and $A_F(y) \leq \alpha_y$. If $A_F((x|(y|y))|(x|(y|y))) > \alpha_x \vee \alpha_y$, then $A_F(x) \vee A_F(y) \leq 0.5$ because if not, then $A_F((x|(y|y))|(x|(y|y))) \leq \vee\{A_F(x), A_F(y), 0.5\} \leq A_F(x) \vee A_F(y) \leq \alpha_x \vee \alpha_y$, which is a contradiction. Hence, $A_F((x|(y|y))|(x|(y|y))) + \alpha_x \vee \alpha_y < 2A_F((x|(y|y))|(x|(y|y))) \leq 2\bigvee\{A_F(x), A_F(y), 0.5\} = 1$; and so $(x|(y|y))|(x|(y|y)) \in F_q(A, \gamma_x \vee \gamma_y) \subseteq F_{\in\vee q}(A, \gamma_x \vee \gamma_y)$. Therefore, $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of X. □

Theorem 3.7. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of an SUP-algebra X, then neutrosophic q-subsets $T_q(A, \alpha)$, $I_q(A, \beta)$, and $F_q(A, \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

Proof. Let $x, y \in T_q(A,\alpha)$. Then $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$. It follows from Theorem 3.6 that $A_T((x|(y|y))|(x|(y|y))) + \alpha \geq \wedge\{A_T(x), A_T(y), 0.5\} + \alpha = \wedge\{A_T(x) + \alpha, A_T(y) + \alpha, 0.5 + \alpha\} > 1$, that is, $(x|(y|y))|(x|(y|y)) \in T_q(A,\alpha)$. Hence, $T_q(A,\alpha)$ is a subalgebra of X. In a similar way, we can induce that $I_q(A,\beta)$ is a subalgebra of X. Now, let $x, y \in F_q(A,\gamma)$. Then $A_F(x)+\gamma < 1$ and $A_F(y)+\gamma < 1$. Using Theorem 3.6, we have $A_F((x|(y|y))|(x|(y|y)))+\gamma \leq \bigvee\{A_F(x), A_F(y), 0.5\}+\gamma = \vee\{A_F(x)+\gamma, A_F(y)+\gamma, 0.5+\gamma\} < 1$, and so $(x|(y|y))|(x|(y|y)) \in F_q(A,\gamma)$. Therefore, $F_q(A,\gamma)$ is a subalgebra of X. □

Theorem 3.8. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of an SUP-algebra X, then neutrosophic $\in$-subsets $T_\in(A,\alpha)$, $I_\in(A,\beta)$, and $F_\in(A,\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ whenever they are nonempty.*

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of X. For any $x, y \in X$, let $\alpha \in (0, 0.5]$ be such that $x, y \in T_\in(A,\alpha)$. Then $A_T(x) \geq \alpha$ and $A_T(y) \geq \alpha$. It follows from (3.3) that $A_T((x|(y|y))|(x|(y|y))) \geq \wedge\{A_T(x), A_T(y), 0.5\} \geq \alpha \wedge 0.5 = \alpha$ and so that $(x|(y|y))|(x|(y|y)) \in T_\in(A,\alpha)$. Hence, $T_\in(A,\alpha)$ is a subalgebra of X. By a similar way, we can induce that $I_\in(A,\beta)$ is a subalgebra of X for all $\in \beta \in (0, 0.5]$. Now, let $\gamma \in [0.5, 1)$ be such that $x, y \in T_F \in (A,\gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. It follows from (3.3) that $A_F((x|(y|y))|(x|(y|y))) \leq \vee\{A_F(x), A_F(y), 0.5\} \leq \gamma \vee 0.5 = \gamma$ and so that $(x|(y|y))|(x|(y|y)) \in F_\in(A,\gamma)$. Hence, $F_\in(A,\alpha)$ is a subalgebra of X. □

Theorem 3.9. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in an SUP-algebra X, if the neutrosophic $\in \vee q$-subsets $T_{\in\vee q}(A,\alpha)$, $I_{\in\vee q}(A,\beta)$, and $F_{\in\vee q}(A,\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of X whenever they are nonempty.*

Proof. Let $T_{\in\vee q}(A,\gamma)$ be a subalgebra of X and assume that $A_T((x|(y|y))|(x|(y|y))) < \bigwedge\{A_T(x), A_T(y), 0.5\}$ for some $x, y \in X$. Then there exists $\alpha \in (0, 0.5]$ such that $A_T((x|(y|y))|(x|(y|y))) < \alpha \leq \bigwedge\{A_T(x), A_T(y), 0.5\}$. It follows that $x, y \in T_\in(A,\alpha) \subseteq T_{\in\vee q}(A,\alpha)$, and so that $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Hence, $A_T((x|(y|y))|(x|(y|y))) \geq \alpha$ or $A_T((x|(y|y))|(x|(y|y))) + \alpha > 1$. This is a contradiction, and so $A_T((x|(y|y))|(x|(y|y))) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$ for all $x, y \in X$. Similarly, we show that $A_I((x|(y|y))|(x|(y|y))) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$ for all $x, y \in X$. Now, let $F_{\in\vee q}(A,\alpha)$ be a subalgebra of X and assume that $A_F((x|(y|y))|(x|(y|y))) > \bigvee\{A_F(x), A_F(y), 0.5\}$ for some $x, y \in X$. Then

$$A_F((x|(y|y))|(x|(y|y))) > \gamma \geq \bigvee\{A_F(x), A_F(y), 0.5\} \tag{3.4}$$

for some $\gamma \in [0.5, 1)$, which implies that $x, y \in F_\in(A,\gamma) \subseteq F_{\in\vee q}(A,\gamma)$. Thus, $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. From (3.4), we have $(x|(y|y))|(x|(y|y)) \notin F_\in(A,\gamma)$ and $A_F((x|(y|y))|(x|(y|y))) + \gamma > 2 \geq 1$, that is, $(x|(y|y))|(x|(y|y)) \notin F_q(A,\gamma)$. This is a contradiction, and hence $A_F((x|(y|y))|(x|(y|y))) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ for all $x, y \in X$. Using Theorem 3.6, we know that $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of X. □

Theorem 3.10. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of an SUP-algebra X, then neutrosophic $\in \vee q$-subsets $T_{\in\vee q}(A,\alpha)$, $I_{\in\vee q}(A,\beta)$, and $F_{\in\vee q}(A,\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ whenever they are nonempty.*

Proof. Assume that $T_{\in\vee q}(A,\alpha)$, $I_{\in\vee q}(A,\beta)$, and $F_{\in\vee q}(A,\gamma)$ are nonempty for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Let $x, y \in I_{\in\vee q}(A,\beta)$. Then $x \in I_\in(A,\beta)$ or $x \in I_q(A,\beta)$, and $y \in I_\in(A,\beta)$ or $y \in I_q(A,\beta)$. Hence, we have the following four cases:

(1) $x \in I_\in(A,\beta)$ and $y \in I_\in(A,\beta)$

(2) $x \in I_\in(A,\beta)$ and $y \in I_q(A,\beta)$

(3) $x \in I_q(A,\beta)$ and $y \in I_\in(A,\beta)$

(4) $x \in I_q(A,\beta)$ and $y \in I_q(A,\beta)$.

The first case implies that $(x|(y|y))|(x|(y|y)) \in I_{\in\vee q}(A,\beta)$. For the second case, $y \in I_q(A,\beta)$ induces $A_I(y) > 1-\beta \geq \beta$, that is, $y \in I_{\in}(A,\beta)$. Thus, $(x|(y|y))|(x|(y|y)) \in I_{\in\vee q}(A,\beta)$. Similarly, the third case implies $(x|(y|y))|(x|(y|y)) \in I_{\in\vee q}(A,\beta)$. The last case induces $A_I(x) > 1-\beta \geq \beta$ and $A_I(y) > 1-\beta \geq \beta$, that is, $x \in I_{\in}(A,\beta)$ and $y \in I_{\in}(A,\beta)$. Hence, $(x|(y|y))|(x|(y|y)) \in I_{\in\vee q}(A,\beta)$. Therefore, $I_{\in\vee q}(A,\beta)$ is a subalgebra of X for all $\beta \in (0,0.5]$. By the similar way, we show that $T_{\in\vee q}(A,\alpha)$ is a subalgebra of X for all $\alpha \in (0,0.5]$. Let $x,y \in F_{\in\vee q}(A,\gamma)$. Then $A_F(x) \leq \gamma$ or $A_F(x)+\gamma < 1$, and $A_F(y) \leq \gamma$ or $A_F(y)+\gamma < 1$. If $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$, then $A_F((x|(y|y))|(x|(y|y))) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \vee\{\gamma, 0.5\} = \gamma$ by Theorem 3.6, and so $(x|(y|y))|(x|(y|y)) \in F_{\in}(A,\gamma) \subseteq F_{\in\vee q}(A,\gamma)$. If $A_F(x) \leq \gamma$ and $A_F(y)+\gamma < 1$, then $A_F((x|(y|y))|(x|(y|y))) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{\gamma, 1-\gamma, .5\} = \gamma$ by Theorem 3.6. Thus, $(x|(y|y))|(x|(y|y)) \in F_{\in}(A,\gamma) \subseteq F_{\in\vee q}(A,\gamma)$. Similarly, if $A_F(x)+\gamma < 1$ and $A_F(y) \leq \gamma$, then $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. Finally, assume that $A_F(x)+\gamma < 1$ and $A_F(y)+\gamma < 1$. Then $A_F((x|(y|y))|(x|(y|y))) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{1-\gamma, 0.5\} = 0.5 < \gamma$ by Theorem 3.6. Hence, $(x|(y|y))|(x|(y|y)) \in F_{\in}(A,\gamma) \subseteq F_{\in\vee q}(A,\gamma)$. Consequently, $F_{\in\vee q}(A,\gamma)$ is a subalgebra of X for all $\gamma \in [0.5,1)$. □

Theorem 3.11. *If $A = (A_T, A_I, A_F)$ is an $(q, \in\vee q)$-neutrosophic subalgebra of an SUP-algebra X, then neutrosophic $\in\vee q$-subsets $T_{\in\vee q}(A,\alpha)$, $I_{\in\vee q}(A,\beta)$, and $F_{\in\vee q}(A,\gamma)$ are subalgebras of X for all $\alpha,\beta \in (0,0.5]$ and $\gamma \in [0.5,1)$ whenever they are nonempty.*

Proof. Let $x,y \in T_{\in\vee q}(A,\alpha)$. Then $x \in T_{\in}(A,\alpha)$ or $x \in T_q(A,\alpha)$, and $y \in T_{\in}(A,\alpha)$ or $y \in T_q(A,\alpha)$. If $x \in T_q(A,\alpha)$ and $y \in T_q(A,\alpha)$, then obviously $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Suppose that $x \in T_{\in}(A,\alpha)$ and $y \in T_q(A,\alpha)$. Then $A_T(x)+\alpha \geq 2\alpha > 1$, that is, $x \in T_q(A,\alpha)$. It follows that $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Similarly, if $x \in T_q(A,\alpha)$ and $y \in T_{\in}(A,\alpha)$, then $(x|(y|y))|(x|(y|y)) \in T_{\in\vee q}(A,\alpha)$. Now, let $x,y \in F_{\in\vee q}(A,\gamma)$. Then $x \in F_{\in}(A,\gamma)$ or $x \in F_{\in}(A,\gamma)$, and $y \in F_{\in}(A,\gamma)$ or $y \in F_{\in}(A,\gamma)$. If $x \in F_{\in}(A,\gamma)$ and $y \in F_{\in}(A,\gamma)$, then clearly $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. If $x \in F_{\in}(A,\gamma)$ and $y \in F_{\in}(A,\gamma)$, then $A_F(x)+\gamma \leq 2\gamma < 1$, that is, $x \in F_q(A,\gamma)$. It follows that $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. Similarly, if $x \in F_q(A,\gamma)$ and $y \in F_{\in}(A,\gamma)$, then $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. Finally, assume that $x \in F_{\in}(A,\gamma)$ and $y \in F_{\in}(A,\gamma)$. Then $A_F(x)+\gamma \leq 2\gamma < 1$ and $A_F(y)+\gamma \leq 2\gamma < 1$, that is, $x \in F_q(A,\gamma)$ and $y \in F_q(A,\gamma)$. Therefore, $(x|(y|y))|(x|(y|y)) \in F_{\in\vee q}(A,\gamma)$. Consequently, $T_{\in\vee q}(A,\alpha)$, $I_{\in\vee q}(A,\beta)$, and $F_{\in\vee q}(A,\gamma)$ are subalgebras of X for all $\alpha,\beta \in (0.5,1]$ and $\gamma \in [0,0.5)$. □

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, we consider the following set:

$$X_0^1 = \{x \in X \mid A_T(x) > 0, A_I(x) > 0, A_F(x) < 1\}.$$

Theorem 3.12. *If $A = (A_T, A_I, A_F)$ is an $(\in,\in)$-neutrosophic subalgebra of an SUP-algebra X, then the set X_0^1 is a subalgebra of X.*

Proof. Proof. Let $x,y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$, and $A_F(y) < 1$. Suppose that $A_T((x|(y|y))|(x|(y|y))) = 0$. Note that $x \in T_{\in}(A, A_T(x))$ and $y \in T_{\in}(A, A_T(y))$. But $(x|(y|y))|(x|(y|y)) \notin T_{\in}(A, A_T(x) \wedge A_T(y))$ because $A_T((x|(y|y))|(x|(y|y))) = 0 < A_T(x) \wedge A_T(y)$. This is a contradiction, and thus $A_T((x|(y|y))|(x|(y|y))) > 0$. In the similar way, we show that $A_I((x|(y|y))|(x|(y|y))) > 0$. Note that $x \in F_{\in}(A, A_F(x))$ and $y \in F_{\in}(A, A_F(y))$. If $A_F((x|(y|y))|(x|(y|y))) = 1$, then $A_F((x|(y|y))|(x|(y|y))) = 1 > A_F(x) \vee A_F(y)$, and so $(x|(y|y))|(x|(y|y)) \notin F_{\in}(A, A_F(x) \vee A_F(y))$. This is impossible. Hence, $(x|(y|y))|(x|(y|y)) \in X_0^1$, and hence X_0^1 is a subalgebra of X. □

Theorem 3.13. *If $A = (A_T, A_I, A_F)$ is an $(\in, q)$-neutrosophic subalgebra of an SUP-algebra X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x,y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$, and $A_F(y) < 1$. If $A_T((x|(y|y))|(x|(y|y))) = 0$, then $A_T((x|(y|y))|(x|(y|y))) + A_T(x) \wedge A_T(y) = A_T(x) \wedge A_T(y) \leq 1$. Hence, $(x|(y|y))|(x|(y|y)) \notin T_q(A, A_T(x) \wedge A_T(y))$, which is a contradiction since $x \in T_{\in}(A, A_T(x))$ and $y \in T_{\in}(A, A_T(y))$. Thus, $A_T((x|(y|y))|(x|(y|y))) > 0$. Similarly, we get $A_I((x|(y|y))|(x|(y|y))) > 0$. Assume that $A_F((x|(y|y))|(x|(y|y))) = 1$. Then $A_F((x|(y|y))|(x|(y|y))) + A_F(x) \vee A_F(y) = 1 + A_F(x) \vee A_F(y) \geq 1$, that is, $(x|(y|y))|(x|(y|y)) \notin F_q(A, A_F(x) \vee A_F(y))$. This is a contradiction because of $x \in F_{\in}(A, A_F(x))$ and $y \in F_{\in}(A, A_F(y))$. Hence, $A_F((x|(y|y))|(x|(y|y))) < 1$. Consequently, $(x|(y|y))|(x|(y|y)) \in X_0^1$ and X_0^1 is a subalgebra of X. □

Theorem 3.14. *If $A = (A_T, A_I, A_F)$ is a $(q, \in)$-neutrosophic subalgebra of an SUP-algebra X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$, and $A_F(y) < 1$. It follows that $A_T(x) + 1 > 1$, $A_T(y) + 1 > 1$, $A_I(x) + 1 > 1$, $A_I(y) + 1 > 1$, $A_F(x) + 0 < 1$, and $A_F(y) + 0 < 1$. Hence, $x, y \in T_q(A, 1) \cap I_q(A, 1) \cap F_q(A, 0)$. If $A_T((x|(y|y))|(x|(y|y))) = 0$ or $A_I((x|(y|y))|(x|(y|y))) = 0$, then $A_T((x|(y|y))|(x|(y|y))) < 1 = 1 \wedge 1$ or $A_I((x|(y|y))|(x|(y|y))) < 1 = 1 \wedge 1$. Thus, $(x|(y|y))|(x|(y|y)) \notin T_q(A, 1 \wedge 1)$ or $(x|(y|y))|(x|(y|y)) \notin I_q(A, 1 \wedge 1)$ which is a contradiction. Hence, $A_T((x|(y|y))|(x|(y|y))) > 0$ and $A_I((x|(y|y))|(x|(y|y))) > 0$. If $A_F((x|(y|y))|(x|(y|y))) = 1$, then $(x|(y|y))|(x|(y|y)) \notin F_q(A, 0 \vee 0)$ which is a contradiction. Thus, $A_F((x|(y|y))|(x|(y|y))) < 1$. Therefore, $(x|(y|y))|(x|(y|y)) \in X_0^1$ and X_0^1 is a subalgebra of X. □

Theorem 3.15. *If $A = (A_T, A_I, A_F)$ is a (q, q)-neutrosophic subalgebra of an SUP-algebra X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Hence, $A_T(x) + 1 > 1$, $A_T(y) + 1 > 1$, $A_I(x) + 1 > 1$, $A_I(y) + 1 > 1$, $A_F(x) + 0 < 1$ and $A_F(y) + 0 < 1$. Hence, $x, y \in T_q(A, 1) \cap I_q(A, 1) \cap F_q(A, 0)$. If $A_T((x|(y|y))|(x|(y|y))) = 0$ or $A_I((x|(y|y))|(x|(y|y))) = 0$, then $A_T((x|(y|y))|(x|(y|y))) + 1 \wedge 1 = 0 + 1 = 1$ or $A_I((x|(y|y))|(x|(y|y))) + 1 \wedge 1 = 0 + 1 = 1$, and so $(x|(y|y))|(x|(y|y)) \notin T_q(A, 1 \wedge 1)$ or $(x|(y|y))|(x|(y|y)) \notin I_q(A, 1 \wedge 1)$. This is impossible, and thus $A_T((x|(y|y))|(x|(y|y))) > 0$ and $A_I((x|(y|y))|(x|(y|y))) > 0$. If $A_F((x|(y|y))|(x|(y|y))) = 1$, then $A_F((x|(y|y))|(x|(y|y))) + 0 \vee 0 = 1$, that is, $(x|(y|y))|(x|(y|y)) \notin F_q(A, 0 \vee 0)$. This is a contradiction, and so $A_F((x|(y|y))|(x|(y|y))) < 1$. Therefore, $(x|(y|y))|(x|(y|y)) \in X_0^1$ and X_0^1 is a subalgebra of X. □

Theorem 3.16. *For a subalgebra S of an SUP-algebra X, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X such that*

$$(\forall x \in S)(A_T(x) \geq 0.5, A_I(x) \geq 0.5, A_F(x) \leq 0.5) \tag{3.5}$$

$$(\forall x \in X \backslash S)(A_T(x) = 0, A_I(x) = 0, A_F(x) = 1). \tag{3.6}$$

Then it is a $(q, \in \vee q)$-neutrosophic subalgebra of X.

Proof. Assume that $x \in I_q(A, \alpha_x)$ and $y \in I_q(A, \alpha_y)$ for all $x, y \in X$ and $\alpha_x, \alpha_y \in [0, 1]$. Then $A_I(x) + \alpha_x > 1$ and $A_I(y) + \alpha_y > 1$. If $(x|(y|y))|(x|(y|y)) \notin S$, then $x \in X \backslash S$ or $y \in X \backslash S$ since S is a subalgebra of X. Hence, $A_I(x) = 0$ or $A_I(y) = 0$, which imply that $\alpha_x > 1$ or $\alpha_y > 1$. This is a contradiction, and so $(x|(y|y))|(x|(y|y)) \in S$. If $\alpha_x \wedge \alpha_y > 0.5$, then $A_I((x|(y|y))|(x|(y|y))) + \alpha_x \wedge \alpha_y > 1$, that is, $(x|(y|y))|(x|(y|y)) \in I_q(A, \alpha_x \wedge \alpha_y)$. If $\alpha_x \wedge \alpha_y \leq 0.5$, then $A_I((x|(y|y))|(x|(y|y))) \geq 0.5 \geq \alpha_x \wedge \alpha_y$, that is, $(x|(y|y))|(x|(y|y)) \in I_2(A, \alpha_x \wedge \alpha_y)$. Hence, $(x|(y|y))|(x|(y|y)) \in I_{\in \vee q}(A, \alpha_x \wedge \alpha_y)$. Similarly, if $x \in T_q(A, \alpha_x)$ and $y \in T_q(A, \alpha_y)$ for all $x, y \in X$ and $\alpha_x, \alpha_y \in [0, 1]$, then $(x|(y|y))|(x|(y|y)) \in T_{\in \vee q}(A, \alpha_x \wedge \alpha_y)$. Now, let $x, y \in X$ and $\gamma_x, \gamma_y \in [0, 1]$ be such that $x \in F_q(A, \gamma_x)$ and $y \in F_q(A, \gamma_y)$. Then $A_F(x) + \gamma_x < 1$ and $A_F(y) + \gamma_y < 1$. It follows that $(x|(y|y))|(x|(y|y)) \in S$. In fact, if not then $x \in X \backslash S$ or $y \in X \backslash S$ since S is a subalgebra of X. Hence, $A_F(x) = 1$ or $A_F(y) = 1$, which imply that $\gamma_x < 0$ or $\gamma_y < 0$. This is a contradiction, and so $(x|(y|y))|(x|(y|y)) \in S$. If $\gamma_x \vee \gamma_y \geq 0.5$, then $A_F((x|(y|y))|(x|(y|y))) \leq 0.5 \leq \gamma_x \vee \gamma_y$, that is, $(x|(y|y))|(x|(y|y)) \in F_{\in}(A, \gamma_x \vee \gamma_y)$. If $\gamma_x \vee \gamma_y < 0.5$, then $A_F((x|(y|y))|(x|(y|y))) + \gamma_x \vee \gamma_y < 1$, that is, $(x|(y|y))|(x|(y|y)) \in F_q(A, \gamma_x \vee \gamma_y)$. Hence, $(x|(y|y))|(x|(y|y)) \in F_{\in \vee q}(A, \gamma_x \vee \gamma_y)$, and consequently $A = (A_T, A_I, A_F)$ is a $(q, \in \vee q)$-neutrosophic subalgebra of X. □

Combining Theorems 3.5 and 3.16, we have the following corollary.

Corollary 3.17. *For a subalgebra S of X, if $A = (A_T, A_I, A_F)$ is a neutrosophic set in X satisfying conditions* (3.5) *and* (3.6)*, then $T_q(A, \alpha)$, $I_q(A, \beta)$, and $F_q(A, \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

4 Conclusion

In this paper, we presented a comparative study of neutrosophic subalgebras in SUP-algebras, focusing on the $(\in, \in)$, $(\in, \in \vee q)$, and $(q, \in \vee q)$-cases. By employing the framework of neutrosophic subsets, we established necessary and sufficient conditions under which these subsets form valid subalgebras, and we clarified the structural relationships among different classes. Illustrative examples further demonstrated how these conditions operate in practice, highlighting the distinctions between various neutrosophic configurations.

The results provide not only explicit characterizations but also a unified algebraic perspective on neutrosophic substructures within SUP-algebras. This contributes to a deeper understanding of how truth, indeterminacy, and falsity interact in algebraic systems, and suggests avenues for further exploration in related logical algebras and generalized uncertainty models.

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