On Graded Weakly J_{gr}-classical Prime Submodules

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Abstract

Let $\mho$ be a group, Υ be a $\mho$-graded commutative ring with unity 1 and $\mathcal{M}$ a graded Υ-module. Our goal in this paper, introducing the concept of graded weakly J_{gr}-classical prime submodule as a generalization of graded weakly classical prime submodule and offering several results pertinent of graded weakly J_{gr}-classical prime submodules. For instance, we give characterizations of graded weakly J_{gr}-classical prime submodule. Also, we give some restrictions for graded submodule to be a graded weakly J_{gr}-classical prime submodule. A proper graded submodule V of $\mathcal{M}$ is said to be a graded weakly J_{gr}-classical prime submodule of $\mathcal{M}$ if, whenever $0 \neq abx \in V$ where $a, b \in h(\Upsilon)$ and $x \in h(\mathcal{M})$, then either $ax \in V + J_{gr}(\mathcal{M})$ or $bx \in V + J_{gr}(\mathcal{M})$, The symbol $J_{gr}(\mathcal{M})$ indicates the graded Jacobson radical of Υ-module $\mathcal{M}$.

Keywords: Graded weakly J_{gr}-classical prime submodule; Graded J_{gr}-classical prime submodule; Graded weakly classical prime submodule; Graded classical prime submodule

1 Introduction and Preliminaries

Let $\mho$ be a group, unless otherwise stated, all $\mho$-graded rings are commutative with unity 1 and all graded modules are unitary. Beginning, we present indispensable properties of graded rings and graded modules. We call that a ring Υ is a $\mho$-graded ring if there are additive subgroups Υ_x of Υ, where $x \in \mho$ satisfies $\Upsilon = \oplus_{x\in\mho}\Upsilon_x$ and $\Upsilon_x\Upsilon_y \subseteq \Upsilon_{xy}$ for all $x, y \in \mho$. Likewise, we call that an Υ-module $\mathcal{K}$ is a graded Υ-module if there are additive subgroups $\mathcal{K}_x$ of $\mathcal{K}$ where $x \in \mho$ satisfies $\mathcal{K} = \bigoplus_{x\in\mho} \mathcal{K}_x$ and $\Upsilon_x \mathcal{K}_y \subseteq \mathcal{K}_{xy}$ for all $x, y \in \mho$. We call that an ideal T of Υ is a graded ideal if $T = \oplus_{x\in\mho}(T \cap \Upsilon_x) := \oplus_{g\in\mho}T_x$, and denoted by $T \leq^{id}_{\mho} \Upsilon$. We refer to a proper graded ideal T of Υ by $T <^{id}_{\mho} \Upsilon$. We call that a submodule V of $\mathcal{K}$ is a graded submodule if $V = \oplus_{x\in\mho}(V \cap \mathcal{K}_x) := \oplus_{x\in\mho}V_x$, and denoted by $V \leq^{sub}_{\mho} \mathcal{K}$. We refer to a proper $\mho$-graded submodule V of $\mathcal{K}$ by $V <^{sub}_{\mho} \mathcal{K}$. We call that a $T <^{id}_{\mho} \Upsilon$ is a graded maximal (Gr-maximal) ideal if, there exists $P \leq^{id}_{\mho} \Upsilon$, with $T \subseteq P \subseteq \Upsilon$, and then either $P = \Upsilon$ or $P = T$. Similarly, we call that $V <^{sub}_{\mho} \mathcal{K}$ is a graded maximal (Gr-maximal) submodule if, there exists $Q \leq^{sub}_{\mho} \mathcal{K}$ with $V \subseteq Q \subseteq \mathcal{K}$, and then either $Q = V$ or $Q = \mathcal{K}$, see.[1] Let $a \in \Upsilon$ and $V \leq^{sub}_{\mho} \mathcal{K}$, then $(V :_{\mathcal{K}} a) = \{x \in \mathcal{K} : ax \in V\} \leq^{sub}_{\mho} \mathcal{K}$, (we can replace a by any $T \leq^{id}_{\mho} \Upsilon$ to define $(V :_{\mathcal{K}} T)$). As well as, let $x \in \mathcal{K}$, then $(V :_{\Upsilon} x) = \{a \in \Upsilon : ax \in V\} \leq^{id}_{\mho} \Upsilon$, (we can replace x by any $Q \leq^{sub}_{\mho} \mathcal{K}$ to define $(V :_{\Upsilon} Q)$). The graded Jacobson radical of $\mathcal{K}$ is defined as $J_{gr}(\mathcal{K}) = \cap\{Q : Q$ is a Gr-maximal submodule of $\mathcal{K}\}$. If there is no Gr-maximal submodule of $\mathcal{K}$, then $\mathcal{K} = J_{gr}(\mathcal{K})$, see.[1]

The notion of graded prime submodule has been admired by many researchers and has been studied widely, for example, in.[2,3] A $V <^{sub}_{\mho} \mathcal{K}$ is a graded prime submodule provided that, for each $a \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $ax \in V$, implies either $a \in (V :_{\Upsilon} \mathcal{K})$ or $x \in V$. Also, many researchers have given generalizations of graded prime submodules. Among these, in[4] the author introduced the concept of graded weakly prime submodules. Let $V <^{sub}_{\mho} \mathcal{K}$, we call that V is a graded weakly prime submodule of $\mathcal{K}$ if for each $a \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $0 \neq ax \in V$, then either $x \in V$ or $a \in (V :_{\Upsilon} \mathcal{K})$. Also, many authors studied the notion of graded classical prime submodules such as,[5,6] we call that a $V <^{sub}_{\mho} \mathcal{K}$ is a graded classical prime submodule if for each $a, b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $abx \in V$, then either $ax \in V$ or $bx \in V$. Subsequently, in,[7,8] the researchers give a generalization of graded classical prime submodules as follows, we call that a $V <^{sub}_{\mho} \mathcal{K}$ is a graded weakly classical prime submodule if, for each $a, b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $0 \neq abx \in V$, then either $bx \in V$ or $ax \in V$. The aim of this paper, introducing the concept of graded weakly J_{gr}-classical prime submodules as a generalization of graded weakly classical prime submodules and giving important results surrounding it.

2 Results

Definition 2.1. Let $V <^{sub}_{\mho} \mathcal{K}$, we call that V is a graded weakly J_{gr}-classical prime (or simply, Gr-WJCP-Sub) of $\mathcal{K}$ if, whenever $0 \neq abx \in V$ where a, $b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$, then either $ax \in V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$. Accordingly, a graded ideal L of Υ is called a graded weakly J_{gr}-classical prime ideal (or simply, Gr-WJCP-Id) of Υ if L is a graded weakly J_{gr}-classical prime submodule of the graded Υ-module Υ.

Theorem 2.2. *Let V be a Gr-WCP-Sub of $\mathcal{K}$, then V is a Gr-WJCP-Sub*

Proof. Let $0 \neq abx \in V$ where a, $b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$, then either $ax \in V$ or $bx \in V$ as V is a Gr-WCP-Sub. Since $V \subseteq V + J_{gr}(\mathcal{K})$, it follows that either $ax \in V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$. □

The following example illustrates that the converse of the above theorem is not true in general.

Example 2.3. Let $\Upsilon = \mathbb{Z}$ be a $\mho$-graded ring with $\Upsilon_0 = \mathbb{Z}$ and $\Upsilon_1 = \{0\}$, where $\mho = \mathbb{Z}_2$. Let $\mathcal{K} = \mathbb{Z}_{48}$ a graded Υ-module with $\mathcal{K}_0 = \mathbb{Z}_{48}$ and $\mathcal{K}_1 = \{\overline{0}\}$. Afterwards, take $V = \langle\overline{4}\rangle \leq^{sub}_{\mho} \mathcal{K}$. Then V is a Gr-WJCP-Sub of $\mathcal{K}$, where $J_{gr}(\mathbb{Z}_{48}) = \langle\overline{2}\rangle \cap \langle\overline{3}\rangle = \langle\overline{6}\rangle$. However, V is not a Gr-WCP-Sub of $\mathcal{K}$, since there exist $2 \in h(\mathbb{Z})$ and $\overline{1} \in h(\mathbb{Z}_{48})$ with $0 \neq 2 \cdot 2 \cdot \overline{1} \in V$, but $2 \cdot \overline{1} \notin V = \langle\overline{4}\rangle$.

Theorem 2.4. *If V is a Gr-maximal submodule of $\mathcal{K}$, then V is a Gr-WJCP-Sub*

Proof. Let $0 \neq abx \in V$ where $a, b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$. Since V is a Gr-maximal submodule of $\mathcal{K}$, then V is a Gr-prime submodule of $\mathcal{K}$ it follows that $a\mathcal{K} \subseteq V$ or $bx \in V$. In particular, either $ax \in V$ or $bx \in V$. Since $V \subseteq V + J_{gr}(\mathcal{K})$, then $ax \in V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$. Therefore, V is a Gr-WJCP-Sub of $\mathcal{K}$. □

The next example illustrates that the converse of Theorem 2.4 is not true in general.

Example 2.5. Let $\Upsilon = \mathbb{Z}$ be a $\mho$-graded ring with $\Upsilon_0 = \mathbb{Z}$, $\Upsilon_1 = \{0\}$ where $\mho = \mathbb{Z}_2$. Let $\mathcal{K} = \mathbb{Z}_{48}$ be a graded Υ-module with $\mathcal{K}_0 = \mathbb{Z}_{48}$ and $\mathcal{K}_1 = \{\overline{0}\}$. Afterwards, take $V = \langle\overline{4}\rangle \leq^{sub}_{\mho} \mathcal{K}$, by Example 2.3, V is a Gr-WJCP-Sub of $\mathcal{K}$. However, V is not a Gr-maximal submodule of $\mathcal{K}$ since $\langle\overline{4}\rangle \subsetneq \langle\overline{2}\rangle \subsetneq \mathbb{Z}_{48}$.

Theorem 2.6. *Every Gr-WP-Sub V of $\mathcal{K}$ is a Gr-WJCP-Sub of $\mathcal{K}$.*

Proof. Let $0 \neq abx \in V$ where a, $b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$, then either $a \in (V :_{\Upsilon} \mathcal{K})$ or $bx \in V$ as V is a Gr-WP-Sub. But $V \subseteq V + J_{gr}(\mathcal{K})$, So implies either $a\mathcal{K} \subseteq V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$. In particular, $ax \in V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$, so V is a Gr-WJCP-Sub of $\mathcal{K}$. □

The following example illustrate that the converse of Theorem 2.6 is not true in general.

Example 2.7. Let $\mho = \mathbb{Z}_2$ and $\Upsilon = \mathbb{Z}$ be a $\mho$-graded ring with $\Upsilon_0 = \mathbb{Z}$, $\Upsilon_1 = \{0\}$. Let $\mathcal{K} = \mathbb{Z}_{48}$ be a graded Υ-module with $\mathcal{K}_0 = \mathbb{Z}_{48}$ and $\mathcal{K}_1 = \{\overline{0}\}$. Afterwards, take $V = \langle\overline{8}\rangle \leq_{\mho}^{sub} \mathcal{K}$, then V is a Gr-WJCP-Sub where $J_{gr}(\mathcal{K}) = \langle\overline{6}\rangle$. However, $V = \langle\overline{8}\rangle$ is not a Gr-WP-Sub of $\mathcal{K}$, since there exist $2 \in h(\mathbb{Z})$ and $\overline{4} \in h(\mathbb{Z}_{48})$ and $0 \neq 2 \cdot \overline{4} \in \langle\overline{8}\rangle$, but $2 \notin (\langle\overline{8}\rangle :_{\mathbb{Z}} \mathbb{Z}_{48}) = 8\mathbb{Z}$ and $\overline{4} \notin \langle\overline{8}\rangle$.

Next, we introduced the following theorems that tell us some equivalent characterizations of the Gr-WJCP-Sub.

Theorem 2.8. *Let $V <_{\mho}^{sub} \mathcal{K}$. Then V is a Gr-WJCP-Sub of $\mathcal{K}$ if and only if $(V :_{\mathcal{K}} tu) \subseteq (0 :_{\mathcal{K}} tu) \cup (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} t) \cup (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} u)$ for each t, $u \in h(\Upsilon)$.*

Proof. ($\Longrightarrow$) Let $x \in (V :_{\mathcal{K}} tu) \cap h(\mathcal{K})$ where t, $u \in h(\Upsilon)$, then $tux \in V$. If $tux = 0$, then $x \in (0 :_{\mathcal{K}} tu)$. If $0 \neq tux \in V$, then either $tx \in V + J_{gr}(\mathcal{K})$ or $ux \in V + J_{gr}(\mathcal{K})$ as V is a Gr-WJCP-Sub of $\mathcal{K}$, thus $x \in (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} t) \cup (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} u)$. Therefore, $(V :_{\mathcal{K}} tu) \subseteq (0 :_{\mathcal{K}} tu) \cup (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} t) \cup (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} u)$.

($\Longleftarrow$) Let $0 \neq tux \in V$ where t, $u \in h(\Upsilon)$ and $x \in h(\mathcal{K})$, then $x \in (V :_{\mathcal{K}} tu)$ since $(V :_{\mathcal{K}} tu) \subseteq (0 :_{\mathcal{K}} tu) \cup (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} t) \cup (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} u)$ and $tux \neq 0$ it follows that $x \in (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} t) \cup (V + J_{gr}(\mathcal{K}) :_{\mathcal{K}} u)$, hence either $tx \in V + J_{gr}(\mathcal{K})$ or $ux \in V + J_{gr}(\mathcal{K})$. Therefore, V is a Gr-WJCP-Sub of $\mathcal{K}$. □

Theorem 2.9. *Let $V <_{\mho}^{sub} \mathcal{K}$. Then V is a Gr-WJCP-Sub of $\mathcal{K}$ if and only if for each $a \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $ax \notin V + J_{gr}(\mathcal{K})$, implies $(V :_{\Upsilon} ax) \subseteq (0 :_{\Upsilon} ax) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \mathcal{K})$.*

Proof. ($\Longrightarrow$) Assume that $a \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $ax \notin V + J_{gr}(\mathcal{K})$. Let $t \in (V :_{\Upsilon} ax) \cap h(\Upsilon)$, then $atx \in V$. If $atx = 0$ thus $t \in (0 :_{\Upsilon} ax)$. If $0 \neq atx \in V$, since V is a Gr-WJCP-Sub of $\mathcal{K}$ and $ax \notin V + J_{gr}(\mathcal{K})$ implies $tx \in V + J_{gr}(\mathcal{K})$, thus $t \in (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \mathcal{K})$. Hence, $(V :_{\Upsilon} am) \subseteq (0 :_{\Upsilon} am) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \mathcal{K})$.

($\Longleftarrow$) Let $0 \neq abx \in V$ with $ax \notin V + J_{gr}(\mathcal{K})$ where a, $b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$, implies $b \in (V :_{\Upsilon} ax)$. Since $(V :_{\Upsilon} ax) \subseteq (0 :_{\Upsilon} ax) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \mathcal{K})$ and $0 \neq abx \in V$, we have $b \in (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \mathcal{K})$. That is, $bx \in V + J_{gr}(\mathcal{K})$. Hence, V is a Gr-WJCP-Sub of $\mathcal{K}$. □

Corollary 2.10. *Let $V <_{\mho}^{sub} \mathcal{K}$. Then V is a Gr-WJCP-Sub of $\mathcal{K}$ if and only if for each $t \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $t\langle x\rangle \nsubseteq V + J_{gr}(\mathcal{K})$, implies $(V :_{\Upsilon} t\langle x\rangle) \subseteq (\langle 0\rangle :_{\Upsilon} t\langle x\rangle) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$.*

Proof. ($\Longrightarrow$) Let $t \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $t\langle x\rangle \nsubseteq V + J_{gr}(\mathcal{K})$. Let $a \in (V :_{\Upsilon} t\langle x\rangle) \cap h(\Upsilon)$, then $at\langle x\rangle \subseteq V$. If $at\langle x\rangle = \{0\}$, then $a \in (\langle 0\rangle :_{\Upsilon} t\langle x\rangle)$. If $\{0\} \neq at\langle x\rangle \subseteq V$, and since $t\langle x\rangle \nsubseteq V + J_{gr}(\mathcal{K})$, then $0 \neq atx \in V$ and $tx \notin V + J_{gr}(\mathcal{K})$. But V is a Gr-WJCP-Sub, by Theorem 2.9. we get $(V :_{\Upsilon} tx) \subseteq (\langle 0\rangle :_{\Upsilon} tx) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \mathcal{K})$. Since $a \in (V :_{\Upsilon} tx)$ and $atx \neq 0$, it follows that $a \in (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \mathcal{K})$. Hence, $a \in (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$. Therefore, $(V :_{\Upsilon} t\langle x\rangle) \subseteq (\langle 0\rangle :_{\Upsilon} t\langle x\rangle) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$.

($\Longleftarrow$) Let $0 \neq abx \in V$ where a, $b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$. Suppose $ax \notin V + J_{gr}(\mathcal{K})$, then $a\langle x\rangle \nsubseteq V + J_{gr}(\mathcal{K})$. Since $0 \neq abx \in V$, then $\{0\} \neq ab\langle x\rangle \subseteq V$. That is, $b \in (V :_{\Upsilon} a\langle x\rangle)$, by assumption, $b \in (V :_{\Upsilon} a\langle x\rangle) \subseteq (\langle 0\rangle :_{\Upsilon} a\langle x\rangle) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$ but $ab\langle x\rangle \neq \{0\}$ implies $b \in (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$. Hence, $bx \in V + J_{gr}(\mathcal{K})$. Therefore, V is a Gr-WJCP-Sub of $\mathcal{K}$. □

Corollary 2.11. *Let $V <_{\mho}^{sub} \mathcal{K}$. Then V is a Gr-WJCP-Sub of $\mathcal{K}$ if and only if for each $a \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $\langle a\rangle\langle x\rangle \nsubseteq V + J_{gr}(\mathcal{K})$, implies $(V :_{\Upsilon} \langle a\rangle\langle x\rangle) \subseteq (\langle 0\rangle :_{\Upsilon} \langle a\rangle\langle x\rangle) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$.*

Proof. ($\Longrightarrow$) Let $t \in (V :_{\Upsilon} \langle a\rangle\langle x\rangle) \cap h(\Upsilon)$ with $\langle a\rangle\langle x\rangle \nsubseteq V + J_{gr}(\mathcal{K})$ where $a \in h(\Upsilon)$ and $x \in h(\mathcal{K})$, then $tax \in V$ and $ax \notin V + J_{gr}(\mathcal{K})$. Since V is a Gr-WJCP-Sub of $\mathcal{K}$, by Theorem 2.9, we get $(V :_{\Upsilon} ax) \subseteq (\langle 0\rangle :_{\Upsilon} ax) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \mathcal{K})$ since $t \in (V :_{\Upsilon} ax)$ then either $tax = 0$ or $tx \in V + J_{gr}(\mathcal{K})$. Thus either $t\langle a\rangle\langle x\rangle = \{0\}$ or $t\langle x\rangle \subseteq V + J_{gr}(\mathcal{K})$. Hence, $t \in (\langle 0\rangle :_{\Upsilon} \langle a\rangle\langle x\rangle) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$. Therefore, $(V :_{\Upsilon} \langle a\rangle\langle x\rangle) \subseteq (\langle 0\rangle :_{\Upsilon} \langle a\rangle\langle x\rangle) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$.

($\Longleftarrow$) Let $0 \neq abx \in V$ where $a, b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$ with $ax \notin V + J_{gr}(\mathcal{K})$, thus $\langle a\rangle\langle x\rangle \nsubseteq V + J_{gr}(\mathcal{K})$, then $\{0\} \neq b\langle a\rangle\langle x\rangle \subseteq V$ by assumption, $b \in (V :_{\Upsilon} \langle a\rangle\langle x\rangle) \subseteq (\langle 0\rangle :_{\Upsilon} \langle a\rangle\langle x\rangle) \cup (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$, but $b\langle a\rangle\langle x\rangle \neq \{0\}$, thus $b \in (V + J_{gr}(\mathcal{K}) :_{\Upsilon} \langle x\rangle)$. This implies that $bx \in V + J_{gr}(\mathcal{K})$. Hence, V is a Gr-WJCP-Sub of $\mathcal{K}$. □

Theorem 2.12. *Let $V, Q \leq_{\mho}^{sub} \mathcal{K}$ with $V \subsetneq Q$ and $J_{gr}(\mathcal{K}) \subseteq J_{gr}(Q)$. If V is a Gr-WJCP-Sub of $\mathcal{K}$, then V is a Gr-WJCP-Sub of Q.*

Proof. Let $0 \neq abq \in V$ where $a, b \in h(\Upsilon)$ and $q \in Q \cap h(\mathcal{K})$, then either $aq \in V + J_{gr}(\mathcal{K})$ or $bq \in V + J_{gr}(\mathcal{K})$, since V is a Gr-WJCP-Sub of $\mathcal{K}$. But, $J_{gr}(\mathcal{K}) \subseteq J_{gr}(Q)$, so either $aq \in V + J_{gr}(Q)$ or $bq \in V + J_{gr}(Q)$. Thus, V is a Gr-WJCP-Sub of Q. □

The intersection of two Gr-WJCP-Subs is not necessarily a Gr-WJCP-Sub, see the next example.

Example 2.13. Let $\Upsilon = \mathbb{Z}$ be a $\mho$-graded ring with $\Upsilon_0 = \mathbb{Z}$ and $\Upsilon_1 = \{0\}$ where $\mho = \mathbb{Z}_2$. Let $\mathcal{K} = \mathbb{Z}_{48}$ be a graded Υ-module with $\mathcal{K}_0 = \mathbb{Z}_{48}$ and $\mathcal{K}_1 = \{\overline{0}\}$. Let $\langle\overline{3}\rangle, \langle\overline{4}\rangle \leq_{\mho}^{sub} \mathcal{K}$. Clearly $\langle\overline{3}\rangle$ and $\langle\overline{4}\rangle$ are Gr-WJCP-Subs of $\mathcal{K}$ where $J_{gr}(\mathcal{K}) = \langle\overline{6}\rangle$. However, $\langle\overline{3}\rangle \cap \langle\overline{4}\rangle = \langle\overline{12}\rangle$ is not a Gr-WJCP-Sub of $\mathcal{K}$, since $3, 4 \in h(\mathbb{Z})$ and $\overline{1} \in h(\mathbb{Z}_{48})$ such that $0 \neq 3 \cdot 4 \cdot \overline{1} \in \langle\overline{12}\rangle$ but $3 \cdot \overline{1} \notin \langle\overline{12}\rangle + \langle\overline{6}\rangle = \langle\overline{6}\rangle$ and $4 \cdot \overline{1} \notin \langle\overline{12}\rangle + \langle\overline{6}\rangle = \langle\overline{6}\rangle$.

Next theorem shows that the intersection of two Gr-WJCP-Subs is a Gr-WJCP-Sub under some conditions.

Theorem 2.14. *Let $V, Q \leq_{\mho}^{sub} \mathcal{K}$ with either $V \subseteq J_{gr}(\mathcal{K})$ or $Q \subseteq J_{gr}(\mathcal{K})$. If V and Q are Gr-WJCP-Subs of $\mathcal{K}$, then $V \cap Q$ is a Gr-WJCP-Sub of $\mathcal{K}$.*

Proof. Obviously, $V \cap Q <_{\mho}^{sub} \mathcal{K}$ since $V \cap Q \subseteq Q \subsetneq \mathcal{K}$. Let $0 \neq abx \in V \cap Q$ where $a, b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$. If $V \subseteq J_{gr}(\mathcal{K})$, since $0 \neq abx \in V \cap Q \subseteq V$ and V is a Gr-WJCP-Sub of $\mathcal{K}$, then either $ax \in V + J_{gr}(\mathcal{K}) = J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K}) = J_{gr}(\mathcal{K})$. But $J_{gr}(\mathcal{K}) \subseteq (V \cap Q) + J_{gr}(\mathcal{K})$. This implies that either $ax \in (V \cap Q) + J_{gr}(\mathcal{K})$ or $bx \in (V \cap Q) + J_{gr}(\mathcal{K})$. Hence $V \cap Q$ is a Gr-WJCP-Sub of $\mathcal{K}$. As a similar manner, for $Q \subseteq J_{gr}(\mathcal{K})$, we get $V \cap Q$ is a Gr-WJCP-Subof $\mathcal{K}$. □

Theorem 2.15. *Let $V, Q \leq_{\mho}^{sub} \mathcal{K}$ with $V \subsetneq Q$ and $J_{gr}(\mathcal{K}) \subseteq J_{gr}(Q)$. If V is a Gr-WJCP-Sub of $\mathcal{K}$, then $V \cap Q$ is a Gr-WJCP-Sub of Q.*

Proof. Since $V \subsetneq Q$, then $V \cap Q <_{\mho}^{sub} \mathcal{K}$. Let $0 \neq abq \in V \cap Q$ where $a, b \in h(\Upsilon)$ and $q \in Q \cap h(\mathcal{K})$, since V is a Gr-WJCP-Sub of $\mathcal{K}$, implies either $aq \in V + J_{gr}(\mathcal{K})$ or $bq \in V + J_{gr}(\mathcal{K})$, but $J_{gr}(\mathcal{K}) \subseteq J_{gr}(Q)$ it follows that either $aq \in (V + J_{gr}(Q)) \cap Q$ or $bq \in (V + J_{gr}(Q)) \cap Q$. By modular law, either $aq \in (V \cap Q) + J_{gr}(Q)$ or $bq \in (V \cap Q) + J_{gr}(Q)$. Hence, $V \cap Q$ is a Gr-WJCP-Sub of Q. □

It is known that, a $V <_{\mho}^{sub} \mathcal{K}$ is called a graded small (Gr-small) submodule of $\mathcal{K}$ if $L + V \neq \mathcal{K}$ for every $L <_{\mho}^{sub} \mathcal{K}$, see.[9] A $V \leq_{\mho}^{sub} \mathcal{K}$ is called a graded coclosed of $\mathcal{K}$, if V/L is not Gr-small of $\mathcal{K}/L$ for any $L <_{\mho}^{sub} V$.

Lemma 2.16. *Let V be a graded coclosed submodule of $\mathcal{K}$, then $J_{gr}(V) = J_{gr}(\mathcal{K}) \cap V$.*

Proof. Clearly, $J_{gr}(V) \subseteq J_{gr}(\mathcal{K}) \cap V$. So, we prove that $J_{gr}(\mathcal{K}) \cap N \subseteq J_{gr}(V)$. Let $Q \leq_{\mho}^{sub} V$ and Q is a Gr-small of $\mathcal{K}$. we claim Q is a Gr-small of V. Suppose on the contrary, Suppose Q is not Gr-small of V, then there exists a $L <_{\mho}^{sub} V$ such that $V = Q + L$. Since Q is a graded coclosed then V/L is not Gr-small of $\mathcal{K}/L$, so there exists $H <_{\mho}^{sub} \mathcal{K}$ such that $\mathcal{K}/L = (V/L) + (H/L)$. Thus $Q + L + H = V + H = \mathcal{K}$, but Q is a Gr-small of $\mathcal{K}$ and $L \leq_{\mho}^{sub} H$. Hence, $\mathcal{K} = H$ a contradiction. So, Q is a Gr-small of V. For $u \in J_{gr}(\mathcal{K}) \cap V$, we have $\Upsilon u \subseteq V$ and Υu is a Gr-small of $\mathcal{K}$. Hence Υu is a Gr-small of V. So, $u \in J_{gr}(V)$. Therefore, $J_{gr}(\mathcal{K}) \cap V \subseteq J_{gr}(V)$. □

Theorem 2.17. *Let $V, Q \leq_{\mho}^{sub} \mathcal{K}$ with $Q \nsubseteq V$ and K is a graded coclosed of $\mathcal{K}$ such that $J_{gr}(\mathcal{K}) \subseteq Q$. If V is a Gr-WJCP-Sub of $\mathcal{K}$, then $V \cap Q$ is a Gr-WJCP-Sub of Q.*

Proof. Since $Q \nsubseteq V$, then there exists $x \in Q - V$ thus $x \in Q - (V \cap Q)$ so $V \cap Q <_{\mho}^{sub} Q$. Let $0 \neq abq \in V \cap Q$ where $a, b \in h(\Upsilon)$ and $q \in Q \cap h(\mathcal{K})$, since V is a Gr-WJCP-Sub of $\mathcal{K}$ and $0 \neq abq \in V \cap Q \subseteq V$ then either $aq \in V + J_{gr}(\mathcal{K})$ or $bq \in V + J_{gr}(\mathcal{K})$. Thus, either $aq \in (V + J_{gr}(\mathcal{K})) \cap Q$ or $bq \in (V + J_{gr}(\mathcal{K})) \cap Q$. Since $J_{gr}(\mathcal{K})) \subseteq Q$, by modular law, either $aq \in (V \cap Q) + J_{gr}(\mathcal{K})$ or $bq \in (V \cap Q) + J_{gr}(\mathcal{K})$. But $J_{gr}(\mathcal{K}) = J_{gr}(\mathcal{K}) \cap Q = J_{gr}(Q)$, by Lemma 2.16 and $J_{gr}(\mathcal{K}) \subseteq Q$. This yields that either $aq \in (V \cap Q) + J_{gr}(Q)$ or $bq \in (V \cap Q) + J_{gr}(Q)$. Therefore, $V \cap Q$ is a Gr-WJCP-Sub of Q. □

Theorem 2.18. *If $J_{gr}(\mathcal{K})$ is a Gr-WJCP-Sub of $\mathcal{K}$ and $V \leq^{sub}_{\mho} \mathcal{K}$ with $V \subseteq J_{gr}(\mathcal{K})$, then V is a Gr-WJCP-Sub of $\mathcal{K}$.*

Proof. Clearly, $V <^{sub}_{\mho} \mathcal{K}$, since $V \subseteq J_{gr}(\mathcal{K}) \subsetneq \mathcal{K}$. Now, let $0 \neq abx \in V$ where $a, b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$. It follows that, $0 \neq abx \in J_{gr}(\mathcal{K})$ as $V \subseteq J_{gr}(\mathcal{K})$. Since $J_{gr}(\mathcal{K})$ is a Gr-WJCP-Sub of $\mathcal{K}$, we have either $ax \in J_{gr}(\mathcal{K}) = V + J_{gr}(\mathcal{K})$ or $bx \in J_{gr}(\mathcal{K}) = V + J_{gr}(\mathcal{K})$. Thus, V is a Gr-WJCP-Sub of $\mathcal{K}$. □

Corollary 2.19. *Let $J_{gr}(\mathcal{K})$ be a Gr-WJCP-Sub of $\mathcal{K}$ with V is a Gr-small submodule of $\mathcal{K}$, then V is a Gr-WJCP-Sub of $\mathcal{K}$.*

Proof. Since $J_{gr}(\mathcal{K}) = \sum\{A : A$ is a Gr-small submodule of $\mathcal{K}\}$, by [10, Theorem 2.10] and V is a Gr-small of $\mathcal{K}$ we have $V \subseteq J_{gr}(\mathcal{K})$, hence V is a Gr-WJCP-Sub of $\mathcal{K}$, by Theorem 2.18. □

Let $\mathcal{K}, \mathcal{K}'$ be two graded Υ-module, then $\psi : \mathcal{K} \to \mathcal{K}'$ is called a graded homomorphism (Gr-homomorphism) if $\psi(\mathcal{K}_x) \subseteq \mathcal{K}'_x$ for every $x \in \mho$, see.[1]

Theorem 2.20. *Let $\mathcal{K}$, $\mathcal{K}'$ be two graded Υ-modules such that $\psi : \mathcal{K} \to \mathcal{K}'$ be a Gr-homomorphism.*

(i) If ψ is a Gr-epimorphism and V is a Gr-WJCP-Sub of $\mathcal{K}$ with $Ker\psi \subseteq V$, then $\psi(V)$ is a Gr-WJCP-Sub of $\mathcal{K}'$.

(ii) If ψ is a Gr-isomorphism and V' is a Gr-WJCP-Sub of $\mathcal{K}'$ with $Ker\psi$ is a Gr-small submodule of $\mathcal{K}$, then $\psi^{-1}(V)$ is a Gr-WJCP-Sub of $\mathcal{K}$.

Proof. (i) $\psi(V) <^{sub}_{\mho} \mathcal{K}'$, (suppose $\psi(V) = \mathcal{K}'$, since ψ is a Gr-epimorphism, then $\psi(V) = \psi(\mathcal{K})$, let $q \in \mathcal{K}$, there exists $p \in V \cap h(\mathcal{K})$ such that $\psi(q) = \psi(p)$ thus $\psi(q - p) = 0$, so, $q - p \in Ker\psi \subseteq V$ implies $q + V = V$ it follows that $q \in V$, hence $\mathcal{K} = V$, a contradiction). Let $0 \neq abx' \in \psi(V)$ where a, $b \in h(\Upsilon)$ and $x' \in h(\mathcal{K}')$, since ψ is a Gr-epimorphism, then there exists $x \in h(\mathcal{K})$ such that $\psi(x) = x'$, thus $0 \neq \psi(abx) \in \psi(V)$ so there exists $v \in V \cap h(\mathcal{K})$ such that $\psi(abx) = \psi(V)$. This yields that $abx - v \in Ker\psi \subseteq V$. Thus $0 \neq abx \in V$, since V is a Gr-WJCP-Sub of $\mathcal{K}$, implies either $ax \in V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$. It follows that either $\psi(ax) \in \psi(V) + \psi(J_{gr}(\mathcal{K})) \subseteq \psi(V) + J_{gr}(\mathcal{K}')$ or $\psi(bx) \in \psi(V) + \psi(J_{gr}(\mathcal{K})) \subseteq \psi(V) + J_{gr}(\mathcal{K}')$, by [10, Theorem 2.12 (i)], we get either $ax' \in \psi(V) + J_{gr}(\mathcal{K}')$ or $bx' \in \psi(V) + J_{gr}(\mathcal{K}')$. Therefore, $\psi(V)$ is a Gr-WJCP-Sub of $\mathcal{K}'$.

(ii) $\psi^{-1}(V') <^{sub}_{\mho} \mathcal{K}$, (suppose $\psi^{-1}(V') = \mathcal{K}$, then $\psi(\psi^{-1}(V')) = \psi(\mathcal{K}) = \mathcal{K}'$ since ψ is a onto, hence $V' = \mathcal{K}'$, a contradiction). Now, let $0 \neq abx \in \psi^{-1}(V')$ where a, $b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$, then $0 \neq \psi(abx) \in V'$(if $\psi(abx) = 0$, then $abx \in Ker\psi$, but ψ is one to one, so we get $Ker\psi = \{0\}$, thus $abx = 0$, which is a contradiction). It follows that, $0 \neq ab\psi(x) \in V'$, but V' is a Gr-W-J_{gr}-C-prime submodule of $\mathcal{K}'$. So either $a\psi(x) \in V' + J_{gr}(\mathcal{K}')$ or $b\psi(x) \in V' + J_{gr}(\mathcal{K}')$. Since ψ is a Gr-epimorphism and $Ker\psi$ is a Gr-small of $\mathcal{K}$, then $\psi(J_{gr}(\mathcal{K})) = J_{gr}(\mathcal{K}')$, by [10, theorem 2.12 (ii)], implies either $\psi(ax) \in V' + \psi(J_{gr}(\mathcal{K}))$ or $\psi(bx) \in V' + \psi(J_{gr}(\mathcal{K}))$. Hence either $ax \in \psi^{-1}(V') + J_{gr}(\mathcal{K})$ or $bx \in \psi^{-1}(V') + J_{gr}(\mathcal{K})$. Therefore, $\psi^{-1}(V')$ is a Gr-WJCP-Sub of $\mathcal{K}$. □

We call that a graded Υ-module $\mathcal{K}$ is a graded hollow (Gr-hollow) if every $<^{sub}_{\mho} \mathcal{K}$ is a Gr-small submodule of $\mathcal{K}$, see.[9]

Theorem 2.21. *Let $\mathcal{K}$ and $\mathcal{K}'$ be two Gr-hollow Υ-module and have no Gr-maximal submodule. If V, V' are Gr-WJCP-Subs of $\mathcal{K}$, $\mathcal{K}'$ (respectively), then $V \times V'$ is a Gr-WJCP-Sub of $\mathcal{K} \times \mathcal{K}'$.*

Proof. Let $(0,0) \neq ab(x,x') \in V \times V'$ where $a, b \in h(\Upsilon)$ and $(x,x') \in h(\mathcal{K} \times \mathcal{K}')$, then either $abx \neq 0$ or $abx' \neq 0$. If $abx \neq 0$, then $0 \neq abx \in V$, since V is a Gr-WJCP-Sub of $\mathcal{K}$, then either $ax \in V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$. But $\mathcal{K}$ is a Gr-hollow, then $V \subseteq J_{gr}(\mathcal{K})$ implies either $ax \in J_{gr}(\mathcal{K})$ or $bx \in J_{gr}(\mathcal{K})$. Since $\mathcal{K}'$ has no Gr-maximal submodule, then $J_{gr}(\mathcal{K}') = \mathcal{K}'$. So either $a(x,x') \in J_{gr}(\mathcal{K}) \times J_{gr}(\mathcal{K}') = J_{gr}(\mathcal{K} \times \mathcal{K}') \subseteq (V \times V') + J_{gr}(\mathcal{K} \times \mathcal{K}')$ or $b(m,m') \in J_{gr}(\mathcal{K}) \times J_{gr}(\mathcal{K}') = J_{gr}(\mathcal{K} \times \mathcal{K}') \subseteq (V \times V') + J_{gr}(\mathcal{K} \times \mathcal{K}')$. Similarily, for case if $abx' \neq 0$. Therefore, $V \times V'$ is a Gr-WJCP-Sub of $\mathcal{K} \times \mathcal{K}'$. □

Theorem 2.22. *Let $\mathcal{K}$ and $\mathcal{K}'$ have no Gr-maximal submodule and let V be a Gr-small submodule of $\mathcal{K}$. Then V is a Gr-WJCP-Sub of $\mathcal{K}$ if and only if $V \times \mathcal{K}^{'}$ is a Gr-WJCP-Sub of $\mathcal{K} \times \mathcal{K}^{'}$.*

Proof. ($\Longrightarrow$) Let $(0,0) \neq ab(x,x^{'}) \in V \times \mathcal{K}^{'}$ where a, $b \in h(\Upsilon)$ and $(x,x^{'}) \in h(\mathcal{K} \times \mathcal{K}^{'})$, then either $abx \neq 0$ or $abx^{'} \neq 0$. If $0 \neq abx \in V$, since V is a Gr-WJCP-Sub of $\mathcal{K}$, this implies either $ax \in V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$. But V is a Gr-small of $\mathcal{K}$, then $V \subseteq J_{gr}(\mathcal{K})$, it follows that either $ax \in J_{gr}(\mathcal{K})$ or $bx \in J_{gr}(\mathcal{K})$. But $J_{gr}(\mathcal{K}^{'}) = \mathcal{K}^{'}$ as $\mathcal{K}'$ has no Gr-maximal submodule. Thus either $a(x,x^{'}) = (ax, ax^{'}) \in J_{gr}(\mathcal{K}) \times J_{gr}(\mathcal{K}^{'}) = J_{gr}(\mathcal{K} \times \mathcal{K}^{'}) \subseteq (V \times \mathcal{K}^{'}) + J_{gr}(\mathcal{K} \times \mathcal{K}^{'})$ or $b(x,x^{'}) = (bx, bx^{'}) \in J_{gr}(\mathcal{K}) \times J_{gr}(\mathcal{K}^{'}) = J_{gr}(\mathcal{K} \times \mathcal{K}^{'}) \subseteq (V \times \mathcal{K}^{'}) + J_{gr}(\mathcal{K} \times \mathcal{K}^{'})$. If $0 \neq abx' \in \mathcal{K}'$, since $\mathcal{K}'$ has no Gr-maximal submodule, thus $0 \neq abx' \in J_{gr}(\mathcal{K}')$. But $x \in h(\mathcal{K}) \subseteq \mathcal{K} = J_{gr}(\mathcal{K})$, so either $a(x,x') = (ax, ax') \in J_{gr}(\mathcal{K}) \times J_{gr}(\mathcal{K}') \subseteq J_{gr}(\mathcal{K} \times \mathcal{K}') \subseteq (V \times \mathcal{K}') + J_{gr}(\mathcal{K} \times \mathcal{K}')$ or $b(x,x') = (bx, bx') \in J_{gr}(\mathcal{K}) \times J_{gr}(\mathcal{K}') \subseteq J_{gr}(\mathcal{K} \times \mathcal{K}') \subseteq (V \times \mathcal{K}') + J_{gr}(\mathcal{K} \times \mathcal{K}')$. Therefore, $V \times \mathcal{K}^{'}$ is a Gr-WJCP-Sub of $\mathcal{K} \times \mathcal{K}^{'}$.

($\Longleftarrow$) Let $0 \neq abx \in V$ where a, $b \in h(\Upsilon)$ and $x \in h(\mathcal{K})$, then $(0,0) \neq ab(x,x^{'}) \in V \times \mathcal{K}^{'}$ for each $x^{'} \in h(\mathcal{K}^{'})$. It follows that either $a(x,x^{'}) \in (V \times \mathcal{K}^{'}) + J_{gr}(\mathcal{K} \times \mathcal{K}^{'}) = (V \times \mathcal{K}^{'}) + (J_{gr}(\mathcal{K}) \times J_{gr}(\mathcal{K}^{'}))$ or $b(x,x^{'}) \in (V \times \mathcal{K}^{'}) + J_{gr}(\mathcal{K} \times \mathcal{K}^{'}) = (V \times \mathcal{K}^{'}) + (J_{gr}(\mathcal{K}) \times J_{gr}(\mathcal{K}^{'}))$ as $V \times \mathcal{K}^{'}$ is a Gr-WJCP-Sub of $\mathcal{K} \times \mathcal{K}^{'}$. Since N is a Gr-small of $\mathcal{K}$ and $\mathcal{K}'$ has no Gr-maximal submodule then $V \subseteq J_{gr}(\mathcal{K})$ and $J_{gr}(\mathcal{K}^{'}) = \mathcal{K}^{'}$. Thus either $a(x,x^{'}) \in (V \times \mathcal{K}^{'}) + (((V + J_{gr}(\mathcal{K})) \times \mathcal{K}^{'})$ or $b(x,x^{'}) \in (V \times \mathcal{K}^{'}) + (((V + J_{gr}(\mathcal{K})) \times \mathcal{K}^{'})$. But $V \times \mathcal{K}^{'} \subseteq (V + J_{gr}(\mathcal{K})) \times \mathcal{K}^{'}$, so, implies either $a(x,x^{'}) \in (V \times \mathcal{K}^{'}) + (((V + J_{gr}(\mathcal{K})) \times \mathcal{K}^{'}) = ((V + J_{gr}(\mathcal{K})) \times \mathcal{K}^{'})$ or $b(x,x^{'}) \in (V \times \mathcal{K}^{'}) + ((V + J_{gr}(\mathcal{K})) \times \mathcal{K}^{'}) = ((V + J_{gr}(\mathcal{K})) \times \mathcal{K}^{'})$. Hence either $ax \in V + J_{gr}(\mathcal{K})$ or $bx \in V + J_{gr}(\mathcal{K})$. Therefore, V is a Gr-WJCP-Sub of $\mathcal{K}$. □

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