# An Empirical Evaluation of the Stock Market Using Fuzzy Variant Black and Scholes Model Involving Central Fuzzy Measures

**K. Meenakshi[1,*], Pavithra S.[1,*], S. Sathish[1,*], Prabakaran N.[2,*]**
[1]School of Engineering, Presidency University, Bengaluru, Karnataka, India
[2]School of Computer Science and Engineering, Vellore Institute of Technology, Vellore, India
Emails: k.meenakshi@presidencyuniversity.in; pavithra.s@presidencyuniversity.in; s.sathish@presidencyuniversity.in; dhoni.praba@gmail.com

**Abstract**

This article defines the central tendency fuzzy measures, which include the weighted fuzzy possiblistic mean and the fuzzy probability mean involving octagonal fuzzy numbers. The same is supported by a fuzzy variant of the Black-Scholes option model, in which uncertain pricing parameters such as volatility, interest rate, and stock price are described using octagonal fuzzy numbers.

**Keywords:** Weighted fuzzy possiblistic mean; Interval-valued fuzzy expectation; Octagonal fuzzy numbers; Black-Scholes variant fuzzy option model

## 1 introduction

The concept of option pricing was first developed by Black and Scholes[6] in 1973. Since then, it has been the focus of in-depth study across various disciplines, including finance in a complex or uncertain environment. Cherubini[4] calculated the price of a corporate debt bond with a particular type of uncertain measure employed in the fuzzified version of the Black and Scholes model. A modified form of the formula was also suggested by Trenev.[27] Zmeskal applied the same technique[30] to evaluate the European equity call option.

Subsequently, Yoshida[29] examined a fuzzy analogue of the Black-Scholes formula to provide the acceptable range of expected prices, as well as the rational price of European options. In addition, Thavaneswaran et al.[23] and Thiagarajah et al.[26] have explored imprecise models. The first[26] investigated moment properties of fuzzy random coefficients, while the second[23] performed stochastic volatility of financial derivative contracts.

S.S. Appadoo et al.[25] discussed moment properties for quadratic adaptive fuzzy numbers, using which the European call option was estimated in his model. Further, Chrysafis et al.[5] computed fuzzy estimators to calculate the expected returns utilizing symmetric triangular fuzzy numbers. The author discussed both call/put options and obtained the fuzzy stock price and volatility with a possibilistic mean value using the proposed method. Furthermore, several authors worked on similar situations (to cite a few,[1],[2],[3],[7],[8],[9],[10],[11],[12],[13],[14],[15],[17],[18],[19],[20],[21],[22],[24],[28]).

## 2 Fuzzy Numbers - Basic

To describe the uncertainty associated with option pricing applications, only linear triangular and trapezoidal fuzzy numbers have been used in the literature. Among the different types of linear and non-linear membership functions, a special class of octagonal fuzzy numbers is considered in this study, using which a revised form of the Black-Scholes model is exemplified to evaluate stock options, as it provides a better optimal solution in decision-making scenarios because of its $k$ level.

**Definition 2.1.** [16] A fuzzy number $\widetilde{A}$ is a *linear octagonal fuzzy number* denoted $\widetilde{A} \approx (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k)$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8$ are real numbers with membership function $\mu_{\widetilde{A}}(x)$ given by (see Figure 1).

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & \text{for} \quad x < a_1 \\ \frac{k(x-a_1)}{(a_2-a_1)}, & \text{for} \quad a_1 \leq x \leq a_2 \\ k, & \text{for} \quad a_2 \leq x \leq a_3 \\ k + (1-k)\frac{(x-a_3)}{(a_4-a_3)}, & \text{for} \quad a_3 \leq x \leq a_4 \\ 1, & \text{for} \quad a_4 \leq x \leq a_5 \\ k + (1-k)\frac{(a_6-x)}{(a_6-a_5)}, & \text{for} \quad a_5 \leq x \leq a_6 \\ k, & \text{for} \quad a_6 \leq x \leq a_7 \\ k\frac{(a_8-x)}{(a_8-a_7)}, & \text{for} \quad a_7 \leq x \leq a_8 \\ 0, & \text{for} \quad x > a_8. \end{cases}$$

where $0 < k < 1$.

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[scale=0.50] [thin,color=gray,step=0.9cm, dashed] (0, -1.9) grid (8,8); [thick,-¿] (-2.5,0) – (9,0) node[anchor=north west] x; [thick,-¿] (9,0) – (-2,0) node[anchor=north west] -x; [thick,-¿] (0,0) – (0,6) node[anchor=south east] $\mu_{\widetilde{A}}(x)$; (0,2.4) node[left]0.5; (0,1.2) node[left]0.25; (0,3.4) node[left]0.75; (0,0.1) node[left]0; (0,4.5) node[left]1; (a) at (0.5,0) [below] $a_1$; (b) at (1.6,0) [below] $a_2$; (c) at (2.4,0) [below] $a_3$; (d) at (3.4,0) [below] $a_4$; (e) at (4.8,0) [below] $a_5$; (f) at (5.8,0) [below] $a_6$; (g) at (6.6,0) [below] $a_7$; (h) at (7.7,0) [below] $a_8$; [blue,ultra thick, smooth] (-2.5,0) – (0.5,0); [blue,ultra thick, smooth] (0.5,0) – (1.8,1.8); [blue,ultra thick, smooth] (1.8,1.8) – (2.3,1.8); [blue,ultra thick, smooth] (2.3,1.8) – (2.6,1.8); [blue,ultra thick, smooth] (2.6,1.8) – (3.4,4.5); [blue,ultra thick, smooth] (3.4,4.5) – (4.9,4.5); [blue,ultra thick, smooth] (4.9,4.5) – (5.4,1.8); [blue,ultra thick, smooth] (5.4,1.8) – (6.3,1.8); [blue,ultra thick, smooth] (6.3,1.8) – (7.5,0); [blue,ultra thick, smooth] (7.5,0) – (9,0); [scale=0.50] [thin,color=gray,step=0.9cm, dashed] (0, -1.9) grid (8,8); [thick,-¿] (-2.5,0) – (9,0) node[anchor=north west] x; [thick,-¿] (9,0) – (-2,0) node[anchor=north west] -x; [thin,-¿] (0,0) – (0,7) node[anchor=south east] $\mu_{\widetilde{A}}(x)$; (0,4.6) node[left]0.75; (0,2.6) node[left]0.5; (0,0.8) node[left]0.25; (0,0.1) node[left]0; (0,6.3) node[left]1; (a) at (0.5,0) [below] $a_1$; (b) at (1.8,0) [below] $a_2$; (c) at (2.6,0) [below] $a_3$; (d) at (3.6,0) [below] $a_4$; (e) at (5,0) [below] $a_5$; (f) at (5.9,0) [below] $a_6$; (g) at (6.7,0) [below] $a_7$; (h) at (7.7,0) [below] $a_8$; [blue,ultra thick, smooth] (-2.5,0) – (0.5,0); [blue,ultra thick, smooth] (0.5,0) – (1.8,1.8); [blue,ultra thick, smooth] (1.8,1.8) – (2.3,1.8); [blue,ultra thick, smooth] (2.3,1.8) – (2.6,1.8); [blue,ultra thick, smooth] (2.6,1.8) – (3.6,6.3); [blue,ultra thick, smooth] (3.6,6.3) – (5,6.3); [blue,ultra thick, smooth] (5,6.3) – (5.6,1.8); [blue,ultra thick, smooth] (5.6,1.8) – (6.5,1.8); [blue,ultra thick, smooth] (6.5,1.8) – (7.5,0); [blue,ultra thick, smooth] (7.5,0) – (9,0);

Figure 1: Linear octagonal fuzzy number $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k)$ for different values of $k$

**Remark 2.2.** [16] In particular, if $k = 1$, the linear octagonal fuzzy number reduces to a trapezoidal fuzzy number given by $\widetilde{A} \approx (a_1, a_4, a_5, a_8)$. However, the use of trapezoidal fuzzy numbers may not fully convey the uncertainty associated with the fuzzy option pricing parameters mentioned above. Due to the frequent volatility in the stock market, the uncertainty in the option parameters would be greater. To deal with such a situation, we must consider octagonal fuzzy numbers, which can completely foresee ambiguities.

**Definition 2.3.** [16] The $\alpha - cut$ of a linear octagonal fuzzy number $(\widetilde{A})$ for $\alpha \in [0, 1]$ is given by

$$[\widetilde{A}]_\alpha = \begin{cases} [a_2 + \left(\frac{k-\alpha}{k}\right)(a_2 - a_1), a_7 + \left(\frac{k-\alpha}{k}\right)(a_8 - a_7)], \text{for} \quad \alpha \in [0, k] \\ [a_4 - \left(\frac{1-\alpha}{1-k}\right)(a_4 - a_3), a_5 + \left(\frac{1-\alpha}{1-k}\right)(a_6 - a_5)], \text{for} \quad \alpha \in (k, 1] \end{cases}$$

## 3 Highlights of the work

1. To compute central fuzzy measures

2. To apply the variant Black-Scholes model in a fuzzy setup
3. To obtain fuzzy call and put option values using octagonal fuzzy numbers

### 3.1 Sequence of the paper

The rest of this paper is constructed as follows: Section 4 defines fuzzy possibilistic measures involving octagonal fuzzy numbers. Furthermore, a computational approach to the variant fuzzy Black-Scholes model is proposed in Section 5 to estimate option fuzzy prices. This is illustrated with an empirical, real-time example in Section 6. Finally, the main conclusions are recorded in Section 7.

## 4 Main Results

**Definition 4.1.** Let $\tilde{\mathcal{P}}_*^1(\tilde{A}), \tilde{\mathcal{P}}_1^*(\tilde{A})$ and $\tilde{\mathcal{P}}_*^2(\tilde{A}), \tilde{\mathcal{P}}_2^*(\tilde{A})$ be the lower fuzzy possibilistic means and upper fuzzy possibilistic means of an octagonal fuzzy number $\widetilde{A}$ defined respectively in $\alpha \in [0, k]$ and in $\alpha \in (k, 1]$. Then we have,

$$\begin{aligned}
\tilde{\mathcal{P}}_*^1(\widetilde{A}) &= 2\int_0^k \alpha l_1(\alpha)d\alpha = 2\int_0^k \alpha\left[a_1 + \left(\frac{\alpha}{k}\right)(a_2 - a_1)\right]d\alpha \\
&= a_1 k^2 + \frac{2(a_2 - a_1)k^2}{3} \text{ and} \\
\tilde{\mathcal{P}}_1^*(\widetilde{A}) &= 2\int_0^k \alpha r_1(\alpha)d\alpha = 2\int_0^k \alpha\left[a_8 - \left(\frac{\alpha}{k}\right)(a_8 - a_7)\right]d\alpha \\
&= a_8 k^2 - \frac{2(a_8 - a_7)k^2}{3} \quad , \alpha \in [0, k]
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{P}}_*^2(\widetilde{A}) &= 2\int_k^1 \alpha l_2(\alpha)d\alpha = 2\int_k^1 \alpha\left[a_3 + \left(\frac{\alpha - k}{1 - k}\right)(a_4 - a_3)\right]d\alpha \\
&= a_3(1 - k^2) + \frac{2(a_4 - a_3)}{(1 - k)}\frac{(k^3 - 3k + 2)}{6} \text{ and} \\
\tilde{\mathcal{P}}_2^*(\widetilde{A}) &= 2\int_k^1 \alpha r_2(\alpha)d\alpha = 2\int_k^1 \alpha\left[a_6 - \left(\frac{\alpha - k}{1 - k)}\right)(a_6 - a_5)\right]d\alpha \\
&= a_6(1 - k^2) - \frac{2(a_6 - a_5)}{(1 - k)}\frac{(k^3 - 3k + 2)}{6} \quad , \alpha \in (k, 1]
\end{aligned}$$

**Definition 4.2.** The average of lower fuzzy possibilistic and upper fuzzy possibilistic means is defined for $\alpha \in [0, k]$ and $\alpha \in (k, 1]$ is denoted as $\underline{\mathrm{P}}(\widetilde{A})$ and $\overline{P}(\widetilde{A})$ respectively as follows:

$$\begin{aligned}
\underline{\mathrm{P}}(\widetilde{A}) &= (a_1 + a_8)\,k^2 + \frac{2k^2}{3}[a_2 - a_1 - a_8 + a_7] \quad \text{and} \\
\overline{P}(\widetilde{A}) &= (a_3 + a_6)\,(1 - k^2) + \frac{(k^3 - 3k + 2)}{3(1 - k)}\,(a_4 - a_3 - a_6 + a_5)
\end{aligned}$$

**Remark 4.3.** The interval-value fuzzy possibilistic mean of $\widetilde{A}$ is represented as $\tilde{\mathcal{P}}(\widetilde{A})$ and defined by

$$\tilde{\mathcal{P}}(\widetilde{A}) \quad = \quad \begin{cases} [a_1k^2 + \frac{2(a_2-a_1)k^2}{3}, a_8k^2 - \frac{2(a_8-a_7)k^2}{3}], \text{ for } & \alpha \in [0, k] \\ [a_3(1-k^2) + \frac{2(a_4-a_3)}{(1-k)}\frac{(k^3-3k+2)}{6}, a_6(1-k^2) - \frac{2(a_6-a_5)}{(1-k)}\frac{(k^3-3k+2)}{6}], \text{for} & \alpha \in (k, 1] \end{cases}$$

**Definition 4.4.** The weighted fuzzy possibilistic mean of an octagonal fuzzy number $\widetilde{A}$ in $[0, k]$ and in $(k, 1]$ are $\widetilde{E}^1(\widetilde{A})$ and $\widetilde{E}^2(\widetilde{A})$, respectively, given by,

$$\begin{aligned}
\widetilde{E}^1(\widetilde{A}) &= \int_0^k \alpha\left(l_1(\alpha) + r_1(\alpha)\right) d\alpha \\
&= \int_0^k \alpha\left[a_1 + \left(\frac{\alpha}{k}\right)(a_2 - a_1) + a_8 - \left(\frac{\alpha}{k}\right)(a_8 - a_7)\right] d\alpha \\
&= \int_0^k \alpha\left[(a_1 + a_8) + \left(\frac{\alpha}{k}\right)(a_2 - a_1 - a_8 + a_7)\right] d\alpha \\
&= \int_0^k \left[\alpha(a_1 + a_8) + \left(\frac{\alpha^2}{k}\right)(a_2 - a_1 - a_8 + a_7)\right] d\alpha \\
&= \left(\frac{\alpha^2}{2}\right)(a_1 + a_8) + \left(\frac{\alpha^3}{3k}\right)(a_2 - a_1 - a_8 + a_7) \\
&= \left(\frac{k^2}{2}\right)(a_1 + a_8) + \left(\frac{\alpha^3}{3k}\right)(a_2 - a_1 - a_8 + a_7) \\
&= \left(\frac{k^2}{2}\right)(a_1 + a_8) + \left(\frac{k^2}{3}\right)(a_2 - a_1 - a_8 + a_7) \\
&= \frac{1}{6}\left[3k^2(a_1 + a_8) + 2k^2(a_2 - a_1 + a_7 - a_8)\right] \\
&= \frac{3k^2(a_1 + a_8) + 2k^2(a_2 - a_1 + a_7 - a_8)}{6}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{E}^2(\widetilde{A}) &= \int_k^1 \alpha\left(l_2(\alpha) + r_2(\alpha)\right) d\alpha \\
&= \int_k^1 \alpha\left[a_3 + \left(\frac{\alpha - k}{1 - k}\right)(a_4 - a_3) + a_6 - \left(\frac{\alpha - k}{1 - k}\right)(a_6 - a_5)\right] d\alpha \\
&= \int_k^1 \left[\alpha(a_3 + a_6) + \alpha\left(\frac{\alpha - k}{1 - k}\right)(a_4 - a_3 - a_6 + a_5)\right] d\alpha \\
&= (a_3 + a_6)\left(\frac{\alpha^2}{2}\right) + (a_4 - a_3 - a_6 + a_5)\left(\frac{1}{1 - k}\right)\left(\frac{\alpha^3}{3} - \frac{\alpha^2 k}{2}\right) \\
&= (a_3 + a_6)\left(\frac{1}{2} - \frac{k^2}{2}\right) + \left(\frac{a_4 - a_3 - a_6 + a_5}{1 - k}\right)\left\{\left(\frac{1}{3} - \frac{k}{2}\right) - \left(\frac{k^3}{3} - \frac{k^3}{2}\right)\right\} \\
&= (a_3 + a_6)\left(\frac{1 - k^2}{2}\right) + \frac{(a_4 - a_3 - a_6 + a_5)}{6(1 - k)}\left(2 - 3k + k^3\right)
\end{aligned}$$

**Interval-valued fuzzy expectation of an octagonal fuzzy number:**
The lower fuzzy probability means and upper fuzzy probability means of $\widetilde{A}$, respectively, in $[0, k]$ and in $(k, 1]$ are $\widetilde{E}^1_*(\widetilde{A})$, $\widetilde{E}^2_*(\widetilde{A})$ and $\widetilde{E}^*_1(\widetilde{A})$, $\widetilde{E}^*_2(\widetilde{A})$ expressed as given below:

$$\begin{aligned}
\widetilde{E}^1_*(\widetilde{A}) &= \int_0^k l_1(\alpha) d\alpha = \int_0^k \left[a_1 + \left(\frac{\alpha}{k}\right)(a_2 - a_1)\right] d\alpha \\
&= a_1 k + \frac{(a_2 - a_1)k}{2} = \frac{(a_1 + a_2)k}{2} \text{ and} \\
\widetilde{E}^2_*(\widetilde{A}) &= \int_k^1 l_2(\alpha) d\alpha = \int_k^1 \left[a_3 + \left(\frac{\alpha - k}{1 - k}\right)(a_4 - a_3)\right] d\alpha \\
&= a_3(1 - k) + \frac{(a_4 - a_3)}{(1 - k)}\frac{(k^2 - 2k + 1)}{2}
\end{aligned}$$

and

$$\widetilde{E}_1^*(\widetilde{A}) = \int_0^k r_1(\alpha)d\alpha = \int_0^k \left[a_8 - \left(\frac{\alpha}{k}\right)(a_8 - a_7)\right] d\alpha$$
$$= a_8 k + \frac{(a_7 - a_8)k}{2} = \frac{(a_7 + a_8)k}{2}$$
$$\widetilde{E}_2^*(\widetilde{A}) = \int_k^1 r_2(\alpha)d\alpha = \int_k^1 \left[a_6 - \left(\frac{\alpha - k}{1-k}\right)(a_6 - a_5)\right] d\alpha$$
$$= a_6(1-k) - \frac{(a_6 - a_5)}{(1-k)}\frac{(k^2 - 2k + 1)}{2}$$

**Definition 4.5.** The interval-valued fuzzy probability mean of $\widetilde{A}$ is $\tilde{\mathbb{E}}(\widetilde{A})$ given by

$$\tilde{\mathbb{E}}(\widetilde{A}) = \begin{cases} \left[\frac{(a_1+a_2)k}{2}, \frac{(a_7+a_8)k}{2}\right], \text{ for } \alpha \in [0,k] \\ \left[a_3(1-k) + \frac{(a_4-a_3)}{(1-k)}\frac{(k^2-2k+1)}{2}, a_6(1-k) - \frac{(a_6-a_5)}{(1-k)}\frac{(k^2-2k+1)}{2}\right], \text{for } \alpha \in (k,1] \end{cases}$$

**Remark 4.6.** The average of the lower fuzzy probability mean and the upper fuzzy probability mean are defined in $\alpha \in [0,k]$ and $\alpha \in (k,1]$ are $\underline{\mathrm{E}}(\widetilde{A})$ and $\overline{E}(\widetilde{A})$ respectively, as follows:

$$\underline{\mathrm{E}}(\widetilde{A}) = (a_1 + a_8)\,k + \frac{k}{2}[a_2 - a_1 - a_8 + a_7] \quad \text{and}$$
$$\overline{E}(\widetilde{A}) = (a_3 + a_6)\,(1-k) + \frac{(k^2 - 2k + 1)}{2(1-k)}\,(a_4 - a_3 - a_6 + a_5)$$

## 5 Methodology and Implementation

The variant fuzzy Black-Scholes methodology is validated by considering its application to the underlying asset of the listed stock of the Walmart (WMT) company on the NASDAQ Stock Exchange. Options always have a precisely known expiration date and strike price, except for the interest rate, stock price, and volatility, as these parameters may not always occur precisely in the real world due to the continuous fluctuations in the stock market. This present study employs octagonal fuzzy numbers to model various characteristics, including volatility $\tilde{\delta}$, interest rate $\tilde{\mathbb{I}}_r$, initial stock price $\tilde{S}_I$ and constant strike price $\tilde{S}_K$. The aforementioned uncertain option pricing parameters are represented using a possibilistic mean of octagonal fuzzy numbers in the proposed fuzzy Black-Scholes model as follows:
$\tilde{\delta} \approx (\tilde{\delta}_1, \tilde{\delta}_2, \tilde{\delta}_3, \tilde{\delta}_4, \tilde{\delta}_5, \tilde{\delta}_6, \tilde{\delta}_7, \tilde{\delta}_8)$;
$\tilde{\mathbb{I}}_r \approx (\tilde{\mathbb{I}}_{r_1}, \tilde{\mathbb{I}}_{r_2}, \tilde{\mathbb{I}}_{r_3}, \tilde{\mathbb{I}}_{r_4}, \tilde{\mathbb{I}}_{r_5}, \tilde{\mathbb{I}}_{r_6}, \tilde{\mathbb{I}}_{r_7}, \tilde{\mathbb{I}}_{r_8})$;
$\tilde{S}_I \approx (\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4, \tilde{S}_5, \tilde{S}_6, \tilde{S}_7, \tilde{S}_8)$;
$\tilde{S}_K \approx (\tilde{S}_K, \tilde{S}_K, \tilde{S}_K, \tilde{S}_K, \tilde{S}_K, \tilde{S}_K, \tilde{S}_K, \tilde{S}_K)$

**Computational procedure to estimate fuzzy call/put option values using extended fuzzy Black-Scholes:**

1. Compute the weighted possibilistic mean value of the stock price $\tilde{E}(\tilde{S}_I)$; the weighted possibilistic mean value of expected volatility, $\tilde{E}(\tilde{\delta})$ and the weighted possibilistic mean value of the interest rate $\tilde{E}(\tilde{\mathbb{I}}_r)$ (see definition 4.4). Assume the stock consistently yields dividends at a predetermined rate $\gamma$.

2. Determine

$$\tilde{C} \approx \tilde{S}_I e^{-\gamma\mathcal{T}}\mathcal{N}(b_1) - \tilde{S}_K e^{\tilde{\mathbb{I}}_r\mathcal{T}}\mathcal{N}(b_2), \text{and}$$

$$\tilde{P} \approx \tilde{S}_K e^{\tilde{\mathbb{I}}_r\mathcal{T}}\mathcal{N}(-b_2) - \tilde{S}_I e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1)\text{where}$$

$$b_1 \approx \left[\frac{ln\left(\frac{\tilde{E}(\tilde{S}_I)}{\tilde{S}_K}\right) + \left(\tilde{E}(\tilde{\mathbb{I}}_r) - \gamma + \frac{\tilde{E}(\tilde{\delta})^2}{2}\right)\mathcal{T}}{\tilde{E}(\tilde{\delta})\sqrt{\mathcal{T}}}\right] \text{and} \quad (1)$$

$$b_2 \approx b_1 - \tilde{E}(\tilde{\delta})\sqrt{\mathcal{T}} \quad (2)$$

3. Compute the $\alpha-$ cuts of the fuzzy call/put option values as follows:
For $\alpha \in [0,k]$, $\left[\widetilde{F}C_1\right]_\alpha = \left[(\widetilde{F}C_L)^1_\alpha, (\widetilde{F}C_R)^1_\alpha\right]$ and $\left[\widetilde{F}P_1\right]_\alpha = \left[(\widetilde{F}P_L)^1_\alpha, (\widetilde{F}P_R)^1_\alpha\right]$, where

$$(\widetilde{F}C_L)^1_\alpha = \tilde{S}_1 e^{-\gamma\mathcal{T}}\mathcal{N}(b_1) - \tilde{S}_K e^{-\tilde{\mathbb{I}}_{r_1}\mathcal{T}}\mathcal{N}(b_2) + \left(\tilde{S}_K\mathcal{N}(b_2)(e^{-\tilde{\mathbb{I}}_{r_1}\mathcal{T}} - e^{-\tilde{\mathbb{I}}_{r_2}\mathcal{T}}) + (\tilde{S}_2 - \tilde{S}_1)e^{-\gamma\mathcal{T}}\mathcal{N}(b_1)\right)\left(\frac{\alpha}{k}\right)$$

$$(\widetilde{F}C_R)^1_\alpha = \tilde{S}_8 e^{-\gamma\mathcal{T}}\mathcal{N}(b_1) - \tilde{S}_K e^{-\tilde{\mathbb{I}}_{r_8}\mathcal{T}}\mathcal{N}(b_2) + \left(\tilde{S}_K\mathcal{N}(b_2)(e^{-\tilde{\mathbb{I}}_{r_7}\mathcal{T}} - e^{-\tilde{\mathbb{I}}_{r_8}\mathcal{T}}) + (\tilde{S}_8 - \tilde{S}_7)e^{-\gamma\mathcal{T}}\mathcal{N}(b_1)\right)\left(\frac{\alpha}{k}\right) \text{ and}$$

$$(\widetilde{F}P_L)^1_\alpha = \tilde{S}_K e^{-\tilde{\mathbb{I}}_{r_1}\mathcal{T}}\mathcal{N}(-b_2) - \tilde{S}_1 e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1) + \left(\tilde{S}_K\mathcal{N}(-b_2)(e^{-\tilde{\mathbb{I}}_{r_2}\mathcal{T}} - e^{-\tilde{\mathbb{I}}_{r_1}\mathcal{T}}) - (\tilde{S}_2 - \tilde{S}_1)e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1)\right)\left(\frac{\alpha}{k}\right)$$

$$(\widetilde{F}P_R)^1_\alpha = \tilde{S}_K e^{-\tilde{\mathbb{I}}_{r_8}\mathcal{T}}\mathcal{N}(-b_2) - \tilde{S}_8 e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1) \left((\tilde{S}_8 - \tilde{S}_7)e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1) - \tilde{S}_K\mathcal{N}(-b_2)(e^{-\tilde{\mathbb{I}}_{r_8}\mathcal{T}} - e^{-\tilde{\mathbb{I}}_{r_7}\mathcal{T}})\right)\left(\frac{\alpha}{k}\right)$$

For $\alpha \in [k,1]$, we have $\left[\widetilde{F}C_2\right]_\alpha = \left[(\widetilde{F}C_L)^2_\alpha, (\widetilde{F}C_R)^2_\alpha\right]$ and $\left[\widetilde{F}P_2\right]_\alpha = \left[(\widetilde{F}P_L)^2_\alpha, (\widetilde{F}P_R)^2_\alpha\right]$.

$$[\widetilde{F}C_L]^2_\alpha = \tilde{S}_3 e^{-\gamma\mathcal{T}}\mathcal{N}(b_1) - \tilde{S}_K e^{-\tilde{\mathbb{I}}_{r_3}\mathcal{T}}\mathcal{N}(b_2) + \left(\tilde{S}_K\mathcal{N}(b_2)(e^{-\tilde{\mathbb{I}}_{r_3}\mathcal{T}} - e^{-\tilde{\mathbb{I}}_{r_4}\mathcal{T}}) + (\tilde{S}_4 - \tilde{S}_3)e^{-\gamma\mathcal{T}}\mathcal{N}(b_1)\right)\left(\frac{\alpha-k}{1-k}\right)$$

$$[\widetilde{F}C_R]^2_\alpha = \tilde{S}_6 e^{-\gamma\mathcal{T}}\mathcal{N}(b_1) - \tilde{S}_K e^{-\tilde{\mathbb{I}}_{r_6}\mathcal{T}}\mathcal{N}(b_2) + \left(\tilde{S}_K\mathcal{N}(b_2)(e^{-\tilde{\mathbb{I}}_{r_5}\mathcal{T}} - e^{-\tilde{\mathcal{I}}_{r_6}\mathbb{T}}) + (\tilde{S}_6 - \tilde{S}_5)e^{-\gamma\mathcal{T}}\mathcal{N}(b_1)\right)\left(\frac{\alpha-k}{1-k}\right)$$

$$[\widetilde{F}P_L]^2_\alpha = \tilde{S}_K e^{-\tilde{\mathbb{I}}_{r_3}\mathcal{T}}\mathcal{N}(-b_2) - \tilde{S}_3 e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1) + \left(\tilde{S}_K(e^{-\tilde{\mathbb{I}}_{r_4}\mathcal{T}} - e^{-\tilde{\mathbb{I}}_{r_3}\mathcal{T}})\mathcal{N}(-b_2) - (\tilde{S}_4 - \tilde{S}_3)e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1)\right)\left(\frac{\alpha-k}{1-k}\right)$$

$$[\widetilde{F}P_R]^2_\alpha = \tilde{S}_K e^{-\tilde{\mathbb{I}}_{r_6}\mathcal{T}}\mathcal{N}(-b_2) - \tilde{S}_6 e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1) + \left((\tilde{S}_6 - \tilde{S}_5)e^{-\gamma\mathcal{T}}\mathcal{N}(-b_1) - \tilde{S}_K(e^{-\tilde{\mathbb{I}}_{r_6}\mathcal{T}} - e^{-\tilde{\mathcal{I}}_{r_5}\mathbb{T}})\mathcal{N}(-b_2)\right)\left(\frac{\alpha-k}{1-k}\right)$$

## 6 Mathematical Illustration

Given the effective variant fuzzy Black-Scholes model, a hypothetical scenario of validating European call and put options on Walmart Inc. (WMT) is considered when evaluating options.
Following are the input parameters:
The current stock price is trading at 174, with a historical volatility of approximately 0.43, a 6-month risk-free interest rate of 0.05, a strike price of 73, a maturity date of the option is 2, and a dividend of 0.01.
**Parameters modelled using octagonal fuzzy numbers:**

1. Initial fuzzy stock price $\tilde{S}_t \approx (166, 168, 170, 172, 174, 176, 178, 180)$
2. Constant fuzzy strike price $\tilde{S}_K \approx (73, 73, 73, 73, 73, 73, 73, 73)$
3. Fuzzy interest rate $\tilde{\mathbb{I}}_r \approx (0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09)$
4. Fuzzy Voltality $\tilde{\delta} \approx (0.41, 0.43, 0.45, 0.47, 0.49, 0.51, 0.53, 0.55)$

Table 1: Sensitivity Analysis - I

| Fuzzy call option prices | | |
|---|---|---|
| $\alpha$ | $[\widetilde{F}C_1]_\alpha$ | $[\widetilde{F}C_2]_\alpha$ |
| 0 | 78.7796 | 95.2019 |
| 0.1 | 79.1221 | 95.5299 |
| 0.2 | 79.4647 | 95.8579 |
| 0.3 | 79.8072 | 96.1859 |
| 0.4 | 80.1498 | 96.5138 |
| 0.5 | 80.4923 | 96.8418 |
| 0.6 | 80.8349 | 97.1698 |
| 0.7 | 81.1774 | 97.4978 |
| 0.8 | 84.3449 | 91.3700 |
| 0.9 | 85.1324 | 92.1461 |
| 1.0 | 85.9199 | 92.9223 |

By choosing $k = 0.7$ for the choices $k$ $(0, 5 < k < 1)$ to maximize and minimize the profit and loss, respectively, in the fuzzy call and put options.
By step (i), $\tilde{E}(\tilde{S}_I) = 84.77$; $\tilde{E}(\tilde{\delta}) = 0.2352$ and $\tilde{E}(\tilde{\mathbb{I}}_r) = 0.027$.

From step (ii), $b_1 = 0.7179$ and $b_2 = 0.3853$, (see equations (1) and (2)) using which $\mathcal{N}(b_1) = 0.7637$ and $\mathcal{N}(b_2) = 0.6485$. Now, the fuzzy call option values of WMT are computed as given below:

$$[\widetilde{F}C_L]^1_\alpha = \left[78.7796 + 2.3978\left(\frac{\alpha}{k}\right)\right] \quad \alpha \in [0, k] \tag{3}$$

$$[\widetilde{F}C_R]^1_\alpha = \left[95.2019 + 2.2959\left(\frac{\alpha}{k}\right)\right], \quad \alpha \in [0, k] \tag{4}$$

$$[\widetilde{F}C_L]^2_\alpha = \left[83.5574 + 2.3625\left(\frac{\alpha - k}{1 - k}\right)\right], \alpha \in [k, 1] \tag{5}$$

$$[\widetilde{F}C_R]^2_\alpha = \left[90.5938 + 2.3285\left(\frac{\alpha - k}{1 - k}\right)\right], \alpha \in [k, 1] \tag{6}$$

Using $\mathcal{N}(-b_1) = 0.2353$ and $\mathcal{N}(-b_2) = 0.3525$, We have the following fuzzy put option values of WMT:

$$[\widetilde{F}P_L]^1_\alpha = \left[-13.56285 - 0.950839\left(\frac{\alpha}{k}\right)\right] \quad \alpha \in [0, k] \tag{7}$$

$$[\widetilde{F}P_R]^1_\alpha = \left[-20.02174 + 0.89548\left(\frac{\alpha}{k}\right)\right], \quad \alpha \in [0, k] \tag{8}$$

$$[\widetilde{F}P_L]^2_\alpha = \left[-15.454836 - 0.931643\left(\frac{\alpha - k}{1 - k}\right)\right], \alpha \in [k, 1] \tag{9}$$

$$[\widetilde{F}P_R]^2_\alpha = \left[-18.222011 + 0.009362\left(\frac{\alpha - k}{1 - k}\right)\right], \alpha \in [k, 1] \tag{10}$$

The various fuzzy call and put option prices are obtained for different values $\alpha$ in the range $[0, 1]$ using equations from (3) to (10), and the same is recorded in Tables (1 and 2).

**Remark 1** It is evident from the computational approach that octagonal fuzzy numbers are more effective in characterizing the intricacies of ambiguity in applications involving fuzzy option pricing. In addition, it provides the best rates at which financial analysts can buy call and put options with assurance and use them as a tool for decision-making. Hence, the proposed method is useful in many cases where the conditions in any real-world example can be described in terms of octagonal fuzzy numbers.

**Remark 2** Our present approach differs from other fuzzy Black-Scholes applications is that it uses linear octagonal fuzzy numbers, as it could capture the uncertainties involved in the option parameters completely. This helps the decision-maker to predict both the amount the stock will gain if it increases and the amount

Table 2: Sensitivity Analysis - II

| Fuzzy put option prices | | |
|---|---|---|
| $\alpha$ | $[\widetilde{FP}_1]_\alpha$ | $[\widetilde{FP}_2]_\alpha$ |
| 0 | -13.56285 | -20.02174 |
| 0.1 | -13.69868 | -19.89381 |
| 0.2 | -13.83452 | -19.76589 |
| 0.3 | -13.97035 | -19.63796 |
| 0.4 | -14.10619 | -19.51004 |
| 0.5 | -14.24202 | -19.38211 |
| 0.6 | -14.37785 | -19.25419 |
| 0.7 | -14.51369 | -19.12626 |
| 0.8 | -15.76538 | -18.21889 |
| 0.9 | -16.07593 | -18.21577 |
| 1.0 | -16.38648 | -18.21265 |

[scale=0.60] [ width=0.8height=0.6xlabel= Fuzzy call option prices, ylabel=$\alpha$, xmin=76, xmax=98, ymin=0, ymax=1, xticklabel style=/pgf/number format/1000 sep=] [ color=red, ultra thick, mark=square, ] coordinates (78.7796, 0)(79.1221, 0.1)(79.4647, 0.2)(79.8072, 0.3) (80.1498, 0.4)(80.4923, 0.5)(80.8349, 0.6)(81.1774, 0.7)(84.3449, 0.8)(85.1324, 0.9)(85.9199, 1.0) ; $[\widetilde{FC}_1]_\alpha$ [ color=red, ultra thick, mark=square, ] coordinates (95.2019, 0)(95.5299, 0.1)(95.8579, 0.2) (96.1859, 0.3)(96.5138, 0.4)(96.8418, 0.5)(97.1698, 0.6)(97.4978, 0.7)(91.3700, 0.8)(92.1461, 0.9)(92.9223, 1.0) ; $[\widetilde{FC}_2]_\alpha$ [scale=0.60] [ width=0.8height=0.6xlabel=Fuzzy put option prices, ylabel=$\alpha$, xmin=-21, xmax=-12, ymin=0, ymax=1, xticklabel style=/pgf/number format/1000 sep=, ] [ color=blue, ultra thick, mark=square, ] coordinates (-13.56285, 0)(-13.69868, 0.1)(-13.83452, 0.2) (-13.97035, 0.3)(-14.10619, 0.4)(-14.24202, 0.5)(-14.37785, 0.6)(-14.51369, 0.7)(-15.76538, 0.8)(-16.07593, 0.9)(-16.38648, 1.0) ; $[\widetilde{FP}_1]_\alpha$ [ color=blue, ultra thick, mark=square, ] coordinates (-20.02174, 0)(-19.89381, 0.1)(-19.76589 , 0.2) (-19.63796, 0.3)(-19.51004, 0.4)(-19.38211, 0.5)(-19.25419, 0.6)(-19.12626, 0.7)(-18.21889 , 0.8)(-18.21577, 0.9)(-18.21265, 1.0) ; $[\widetilde{FP}_2]_\alpha$

Figure 2: Fuzzy call and put option prices using variant fuzzy Black-Scholes Model involving octagonal fuzzy numbers

the stock will lose if it decreases. bag = [text width=0.5em, text centered] end = [] **Concluding Remark** The prices of fuzzy call and put options against different values $\alpha \in [0,1]$ are represented diagrammatically in the form of a figure (see Figure 2). The sensitive analysis made in this study helps option traders to make valuable insights into market trends and to choose the best option strategy in the stock market.

## 7 Conclusion and future work

In light of the conclusions above, the approach covered in this work has applications in all fields where it is necessary to estimate parameters from statistical data. The fuzzy-aided Black-Scholes paradigm using octagonal fuzzy numbers illustrated during this study provides a more precise approximation of the option price, enabling the financial analyst to make the best possible choice when pricing investment instruments and choosing the optimal one for their goal. The sensitivity analysis of both fuzzy call and put option prices will be analyzed in the future, involving octagonal fuzzy numbers.

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