Separation Axioms in Neutrosophic Bipolar Fuzzy Topological Space

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Abstract

The purpose of this research is to introduce the notion of neutrosophic bipolar T_i – spaces (i = 0, 1, 2, 3, 4) via neutrosophic bipolar fuzzy topological spaces, and investigate their different properties. By defining neutrosophic bipolar T_i – spaces (i = 0, 1, 2, 3, 4), some interesting results on neutrosophic bipolar separation axioms via neutrosophic bipolar fuzzy topological spaces are proved.

Keywords: Neutrosophic bipolar fuzzy point; Neutrosophic bipolar fuzzy topology; Neutrosophic bipolar fuzzy Ti-spaces (i = 0, 1, 2, 3, 4)

1. Introduction

Zadeh [17] has introduced fuzzy sets. Smarandache [11] introduced the concept of neutrosophic set theory, which is a generalization of fuzzy set and intuitionistic fuzzy set. Recently, many concepts of neutrosophic topological spaces have been extended to fuzzy neutrosophic topological spaces by the authors [5, 6]. The concept of fuzzy sets has been generalized to bipolar fuzzy (briefly bf) sets by Zhang [16]. Naveed et al. [14, 15] examined and thoroughly discussed the characteristics of bipolar fuzzy sets. Using several neutrosophic bipolar fuzzy transformation approaches, Raja Muhammad Hashim [9] established the concept of neutrosophic bipolar fuzzy sets and multi-attribute group decision making. AL-Nafee et al. [2] examined some separation axioms on neutrosophic crisp topological spaces. Suman Das and Surapati Pramanik [12] introduced the notion neutrosophic Separation Axioms.

(1) Neutrosophic sets are the more summed up class by which one can deal with uncertain informations in a more successful way when constrasted with fuzzy sets and all other versions of fuzzy sets. Neutrosophic sets have the greater adaptability, accuracy and similarity to the framework when contrasted with past existing fuzzy models.

(2) Bipolar fuzzy sets are proved very effective in uncertain problems which can characterized not only the positive characteristics but also the negative characteristics of certain problem.

By combining these two ideas, we initiate the study of neutrosophic bipolar fuzzy topological spaces, which are the generalization of bipolar fuzzy sets and neutrosophic sets. The concept of neutrosophic bipolar fuzzy Ti-spaces (i = 0, 1, 2, 3, 4) via neutrosophic bipolar fuzzy topological spaces is presented in this study, along with an examination of its characteristics. Using neutrosophic bipolar fuzzy topological spaces, some intriguing conclusions on neutrosophic bipolar fuzzy separation axioms are proved by defining neutrosophic bipolar fuzzy Ti-spaces (i = 0, 1, 2, 3, 4)

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- **Motivation**

The need for more expressive and adaptable models to handle imprecise, indeterminate, and bipolar information—particularly in complex, real-world problems like decision-making and image processing—is the driving force behind the extension of classical separation axioms (T_0 through T_4) to Neutrosophic Bipolar Fuzzy Topological Spaces (NBFTSs).

2. Preliminaries

Some basic definitions from the literature have been provided for subsequent use.

Definition 2.1 [1]

A neutrosophic set, is defined as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$ and X is a universe of discourse and A is characterized by a t-membership function $T_A : X \to]0^-, 1^+[$, an i-membership function $I_A : X \to]0^-, 1^+[$ and a f-membership function $F_A : X \to]0^-, 1^+[$. There is no condition on the sum of $T_A(x)$, $I_A(x)$, $F_A(x)$, so $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$

Definition 2.2 [5]

Let X be a non-empty fixed set. A fuzzy neutrosophic set (FNS for short) A is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions

$$\mu_A : X \to]^-0, 1^+[, \qquad \sigma_A : X \to]^-0, 1^+[, \qquad \nu_A : X \to]^-0, 1^+[$$

denote the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy function (namely $\sigma_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) respective of each element $x \in X$ to the set A and $^+0 \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 1^+$, for each $x \in X$.

Definition 2.3 [3]

Let X ba a nonempty set. Then A pair $A = (A^+, A^-)$ is called a bipolar-valued fuzzy set (or bipolar fuzzy set) in X, if $A^+ : X \to [0, 1]$ and $A^- : X \to [-1, 0]$ are mappings.

In particular, the bipolar fuzzy empty set [resp. the bipolar fuzzy whole set] denoted by $0_{bp} = (0^+_{bp}, 0^-_{bp})$ [resp. $1_{bp} = ((1^+_{bp}, 1^-_{bp}))$] is a bipolar fuzzy set in X denoted by: for each $x \in X$,

$$0^+_{bp}(x) = 0 = 0^-_{bp}(x) \text{ [resp. } 1^+_{bp}(x) = 1 \text{ and } 1^-_{bp}(x) = -1 \text{]}$$

We will denote the set of all bipolar fuzzy sets in X as BPF(X).

Definition 2.4 [7]

Let X be a non-empty set. Then a bipolar fuzzy set, is an object of the form $B = \langle x, \langle \mu^+(x), \mu^-(x)\rangle : x \in X\rangle$, where $\mu^+(x) : X \to [0, 1]$ and $\mu^-(x) : X \to [-1, 0]$, $\mu^+(x)$ is a positive material and $\mu^-(x)$ is a negative material of $x \in X$. For simplicity, we write the bipolar fuzzy set as $B = \langle \mu^+, \mu^-\rangle$ instead of $B = \langle x, \langle \mu^+(x), \mu^-(x)\rangle : x \in X\rangle$.

Definition 2.5 [9]

Let X be a non-vacuous set. Then a neutrosophic bipolar fuzzy set, is an object of the form $NB = (NB^+, NB^-)$ where $NB^+ = \langle x, \langle T_{NB^+}, I_{NB^+}, F_{NB^+}\rangle : x \in X\rangle$, $NB^- = \langle x, \langle T_{NB^-}, I_{NB^-}, F_{NB^-}\rangle : x \in X\rangle$ such that $T_{NB^+}, I_{NB^+}, F_{NB^+} : X \to [0, 1]$ and $T_{NB^-}, I_{NB^-}, F_{NB^-} : X \to [-1, 0]$.

Definition 2.6 [9]

Let X be a non-vacuous set. Then complement neutrosophic bipolar fuzzy set, is an object of the form

$$NB^c = \{\langle 1 - T_{NB^+}, 1 - I_{NB^+}, -1 - F_{NB^+} \; and \; 1 - T_{NB^-}, 1 - I_{NB^-}, \; 1 - F_{NB^-} \rangle\};$$

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Definition 2.7 [11]

A non-empty collection τ of NSs over a fixed set X is called a neutrosophic topology (NT) on X if the following three axioms hold :

(i) 0_N and 1_N are the members of τ ;
(ii) $R_1, R_2 \in \tau \Rightarrow R_1 \cap R_2 \in \tau$;
(iii) $\bigcup \{R_i : i \in \Delta\} \in \tau$, for every $\{R_i : i \in \Delta\} \in \tau$.

If τ is a NT on X, then the structure (X, τ) is called a neutrosophic topological space (NTS) [4]. Every member of τ is said to be a neutrosophic open set (NOS). If $R \in \tau$, then R^c is called a neutrosophic closed set (NCS).

Definition 2.8 [10]

A function $\xi : (X, \tau_1) \to (Y, \tau_2)$ is called a neutrosophic open mapping [5] if $\xi(K)$ is an NOS in Y, whenever K is an NOS in X.

Definition 2.9 [12]

Let $f : X \to Y$ be a function. Also let $A, A_i \in N(X)$, $i \in I$ and $B, B_j \in N(Y)$, $j \in J$. Then the following hold.

(i) $A_1 \subseteq A_2 \Leftrightarrow f(A_1) \subseteq f(A_2)$, $B_1 \subseteq B_2 \Leftrightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$.
(ii) $A \subseteq f^{-1}(f(A))$ and if f is injective then $A = f^{-1}(f(A))$.
(iii) $f^{-1}(f(B)) \subseteq B$ and if f is surjective then $f^{-1}(f(B)) = B$.
(iv) $f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)$ and $f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)$
(v) $f(\bigcup A_i) = \bigcup f(A_i)$, $f(\bigcap A_i) \subseteq \bigcap f(A_i)$ and if f is injective then $f(\bigcap A_i) = \bigcap f(A_i)$.
(vi) $f^{-1}(\tilde{\phi}_Y) = \tilde{\phi}_X, f^{-1}(\tilde{Y}) = \tilde{X}$.
(vii) $f^{-1}(\tilde{\phi}_X) = \tilde{\phi}_Y, f(\tilde{X}) = \tilde{Y}$ if f is surjective.

Definition 2.10 [13]

Let f be a function from an NTS (X, τ) to another NTS (Y, σ). Then

(i) f is called a neutrosophic continuous function if $f^{-1}(G) \in \tau$ for all $G \in \sigma$ then
(ii) f is called a neutrosophic open function if $f(G) \in \sigma$ for all $G \in \tau$
(iii) f is called a neutrosophic closed function if f(G) is a neutrosophic closed set in Y for every neutrosophic closed set G in X.
(iv) f is called a neutrosophic homeomorphism if the following three conditions hold :

a. f is a bijective function.
b. f is a neutrosophic continuous function
c. f^{-1} is a neutrosophic continuous function.

Definition 2.11 [12]

Let (X, τ) be a neutrosophic topological space and $A \in N(X)$. A collection $C = \{G_\lambda : \lambda \in \Delta\}$ of neutrosophic open sets of X is called a neutrosophic open cover (NOC, in short) of A if $A \subseteq \bigcup_{\lambda\in\Delta} G_\lambda$. We then say C covers A. In particular, C is said to be a NOC of X iff $\tilde{X} = \bigcup_{\lambda\in\Delta} G_\lambda$.

Let C be a NOC of the NS A and $C' \subseteq C$. Then C' is called a neutrosophic open subcover (NOSC, in short) of C if C' covers A.

Definition 2.12 [8]:

Let (X, τ_1), (Y, τ_2) be two bipolar fuzzy topological spaces. Then a mapping $f: (X, \tau_1) \to (Y, \tau_2)$ is said to be continuous, if $f^{-1}(V) \in \tau_1$, for each $V \in \tau_2$.

3. Neutrosophic Bipolar Fuzzy T_i-spaces

In this section, neutrosophic bipolar fuzzy separation axioms are introduced through neutrosophic bipolar fuzzy topological spaces, and their different connections are explored.

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Definition 3.1 Neutrosophic bipolar Fuzzy Point

Let x ∈ X neutrosophic bipolar fuzzy set NB = (NB^+, NB^-) is called a neutrosophic bipolar fuzzy point (NBP) iff for any y ∈ X and α = (α_1, α_2), β = (β_1, β_2), γ = (γ_1, γ_2) in (0, 1] x [-1, 0) where α_1, β_1, γ_1 ∈ (0, 1] and α_2, β_2, γ_2 ∈ [-1, 0).

$T^+_{NB}(y) = \alpha_1$ $\quad I^+_{NB}(y) = \beta_1$ $\quad F^+_{NB}(y) = \gamma_1$ for y = x

$T^+_{NB}(y) = 0$ $\quad I^+_{NB}(y) = 0$ $\quad F^+_{NB}(y) = 0$ for y ≠ x

$T^-_{NB}(y) = \alpha_2$ $\quad I^-_{NB}(y) = \beta_2$ $\quad F^-_{NB}(y) = \gamma_2$ for y = x

$T^-_{NB}(y) = 0$ $\quad I^-_{NB}(y) = 0$ $\quad F^-_{NB}(y) = 0$ for y ≠ x

A neutrosophic bipolar fuzzy point is denoted by $x_{\alpha,\beta,\gamma} = (x_{\alpha_1,\beta_1,\gamma_1},\ x_{\alpha_2,\beta_2,\gamma_2})$. For neutrosophic bipolar fuzzy point $x_{\alpha,\beta,\gamma}$, x will be called its support.

$x_{\alpha,\beta,\gamma}$ is said to belonging to NB = (NB^+, NB^-), denoted by $x_{\alpha,\beta,\gamma}$ ∈ NB if $T^+_{NB}(x) \geq \alpha_1$, $I^+_{NB}(x) \geq \beta_1$, $F^+_{NB}(x) \geq \gamma_1$ and $T^-_{NB}(x) \leq \alpha_2$, $I^-_{NB}(x) \leq \beta_2$, $F^-_{NB}(x) \leq \gamma_2$. The set of all neutrosophic bipolar fuzzy points in x is denoted as NBFP(X).

Definition 3.2:

A neutrosophic bipolar fuzzy topological space (X, τ_{NB}) is called a neutrosophic bipolar fuzzy T_0-space (NBF-T_0-S) if for any pair of neutrosophic bipolar fuzzy points (NBFPS) $x_{\alpha,\beta,\gamma} = (x_{\alpha_1,\beta_1,\gamma_1}, x_{\alpha_2,\beta_2,\gamma_2})$ and $y_{\lambda,\mu,\eta} = (y_{\lambda_1,\mu_1,\eta_1}, y_{\lambda_2,\mu_2,\eta_2})$ (x ≠ y) in X, there exists an neutrosophic bipolar fuzzy open set (NBFO) NB = (NB^+, NB^-) such that $x_{\alpha,\beta,\gamma}$ ∈ NB, $y_{\lambda,\mu,\eta}$ ∉ NB (or) $x_{\alpha,\beta,\gamma}$ ∉ NB, $y_{\lambda,\mu,\eta}$ ∈ NB.

Example 3.3

Let X = {x, y} and τ_{NB} = {0_{NB}, 1_{NB}, {⟨X, 0.5, 0.2, 0.3, –0.2, –0.1, –0.4⟩ ⟨y, 0.2, 0.1, 0.6, –0.3, –0.1, –0.4⟩}, {⟨x, 0.5, 0.2, 0.3, –0.2, –0.1, –0.4⟩}}. (X, τ_{NB}) is neutrosophic bipolar fuzzy T_0-space.

Example 3.4

Assume that X = {x, y} and τ_{NB} = {0_{NB}, 1_{NB}}. Clearly (X, τ_{NB}) is an NBFTS but it is not a NBF-T_0-space.

Theorem 3.5

Let τ_{NB} and τ^*_{NB} be two neutrosophic bipolar fuzzy topologies on a set X such that τ^*_{NB} is finer than τ_{NB}. If (X, τ_{NB}) is a NBF-T_0-space then (X, τ^*_{NB}) is also a NBF-T_0-space.

Proof

Assume that $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$, x ≠ y, be two NBFPS in X. Since (X, τ_{NB}) is a NBF-T_0-space, so there exists a NB ∈ τ_{NB} such that $x_{\alpha,\beta,\gamma}$ $x_{\alpha,\beta,\gamma}$ $y_{\lambda,\mu,\eta}$ ∉ NB or $x_{\alpha,\beta,\gamma}$ ∉ NB, $y_{\lambda,\mu,\eta}$ ∈ NB. Since τ^*_{NB} is finer than τ_{NB}, so NB ∈ τ_{NB} ⇒ NB ∈ τ^*_{NB}. Thus for any two NBFPS $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$, x ≠ y, there exists a NB ∈ τ^*_{NB} such that $x_{\alpha,\beta,\gamma}$ ∈ NB, $y_{\lambda,\mu,\eta}$ ∉ NB or $x_{\alpha,\beta,\gamma}$ ∉ NB, $y_{\lambda,\mu,\eta}$ ∈ NB. Hence (X, τ^*_{NB}) is also an NBF-T_0-space.

Example 3.6

Let X = {x, y}

Let $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$ be the points in x.

x = {α = (α_1, α_2) = (0.1, –0.2), β = (β_1, β_2) = (0.3, –0.4), γ = (γ_1, γ_2) = (0.1, –0.1)}

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and

$y = \{\lambda = (\lambda_1, \lambda_2) = (0.1, -0.1), \mu = (\mu_1, \mu_2) = (0.2, -0.3), \eta = (\eta_1, \eta_2) = (0.3, -0.5)\}$

Let (X, τ_{NB}) be a neutrosophic bipolar fuzzy NBF – T_0 space in X. Where τ_{NB} = {ϕ, X, NB_1}.

A neutrosophic bipolar fuzzy open (NBFO) set in X is defined as

NB_1 = {<x, 0.3, 0.4, 0.2, –0.3, –0.5, –0.5> <y, 0.2, 0.2, 0.4, –0.1, –0.3, –0.2>}

Since (X, τ_{NB}} is a NBF-T_0 space, for any two distinct NBF points $x_{\alpha,\beta,\gamma}$, $y_{\lambda,\mu,\eta}$. There exists a NBF open set NB_1 that contains one but not other in terms of membership grades.

⇒ The set NB_1

$T^{+}_{AB_1}(x) = 0.3 > T^{+}_{AB_1}(y) = 0.2$

Let τ^*_{NB} = {ϕ, X, NB_1, NB_2} is a NBF topology on X which is finer that τ_{NB}.

Therefore $\tau^*_{NB} \supseteq \tau_{NB}$.

Since τ^*_{NB} is finer than τ_{NB}, $NB_1 \in \tau^*_{NB}$.

Therefore (X, τ^*_{NB}) is also NBF-T_0-space.

Theorem 3.7

Let $\theta : (X, \tau_{NB_1}) \to (Y, \tau_{NB_2})$ be both one-one and neutrosophic bipolar fuzzy continuous function from a NBFTS (X, τ_{NB_1}) to another NBFTS (Y, τ_{NB_2}). Additionally, If (Y, τ_{NB_2}) is a NBF-T_0-space, then (X, τ_{NB_1}) is also a NBF-T_0-space.

Proof

Assume that (Y, τ_{NB_2}) is an NBF-T_0-space. Also let $x_{\alpha,\beta,\gamma}$, $y_{\lambda,\mu,\eta}$, ($x \neq y$) be any two distinct NBFPS in (X, τ_{NB_1}). Since $\theta : (X, \tau_{NB_1}) \to (Y, \tau_{NB_2})$ is a one-one function, $\theta(x_{\alpha,\beta,\gamma})$, $\theta(y_{\lambda,\mu,\eta})$ are also distinct NBFPS in (Y, τ_{NB_2}). Since (Y, τ_{NB_2}) is a NBF-T_0-space, there exists a NBFOS NB in Y such that $\theta(x_{\alpha,\beta,\gamma}) \in NB$, $\theta(y_{\lambda,\mu,\eta}) \notin NB$ or $\theta(x_{\alpha,\beta,\gamma}) \notin NB$, $\theta(y_{\lambda,\mu,\eta}) \in NB$. Therefore, $x_{\alpha,\beta,\gamma} \in \theta^{-1}(NB)$, $y_{\lambda,\mu,\eta} \notin \theta^{-1}(NB)$ (or) $x_{\alpha,\beta,\gamma} \notin \theta^{-1}(NB)$, $y_{\lambda,\mu,\eta} \in \theta^{-1}(NB)$. Since, θ is a neutrosophic bipolar fuzzy continuous function, $\theta^{-1}(NB)$ is an NBFOS in (X, τ_{NB_1}). Therefore, for any pair of distinct NBFPS $x_{\alpha,\beta,\gamma}$, $y_{\lambda,\mu,\eta}$ in (X, τ_{NB_1}), there exists a NBPOS $\theta^{-1}(NB)$ such that $x_{\alpha,\beta,\gamma} \in \theta^{-1}(NB)$, $y_{\lambda,\mu,\eta} \notin \theta^{-1}(NB)$ or $x_{\alpha,\beta,\gamma} \notin \theta^{-1}(NB)$, $y_{\lambda,\mu,\eta} \in \theta^{-1}(NB)$. Therefore (X, τ_{NB_1}) is a NBF-T_0-space.

Theorem 3.8

Let $\theta : (X, \tau_{NB_1}) \to (Y, \tau_{NB_2})$ be a bijective neutrosophic bipolar fuzzy open function. If (X, τ_{NB_1}) is a NBF-T_0-space then (Y, τ_{NB_2}) is also a NBF-T_0-space.

Proof

Let (X, τ_{NB_1}) and (Y, τ_{NB_2}) be two NBF-T_0-spaces. Let (X, τ_{NB_1}) be a NBF-T_0-space and $\theta : X \to Y$ be a bijective neutrosophic bipolar fuzzy open function. Let $y_{\lambda,\mu,\eta}$ and $y'_{\lambda_1,\mu_1,\eta_1}$ be two NBFPS in Y such that $y \neq y'$. Since θ is bijective, so there exists two NBFPs $x_{\alpha,\beta,\gamma}$ and $x'_{\alpha_1,\beta_1,\gamma_1}$, $x \neq x'$ in X such that $\theta(x'_{\alpha_1,\beta_1,\gamma_1})$ = $y_{\lambda,\mu,\eta}$ and θ $(x'_{\alpha_1,\beta_1,\gamma_1})$ = $y'_{\lambda_1,\mu_1,\eta_1}$. Since X is NBF-$T_0$-space, there exists a τ_{NB}-open NBFS NB such that $x_{\alpha,\beta,\gamma} \in NB$ and

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$x'_{\alpha_1,\beta_1,\gamma_1} \notin$ NB (or) $x_{\alpha,\beta,\gamma} \notin$ NB and $x'_{\alpha_1,\beta_1,\gamma_1} \in$ NB. Suppose NB exists such that $x_{\alpha,\beta,\gamma} \in$ NB and $x'_{\alpha_1,\beta_1,\gamma_1} \notin$ NB. Since θ is a neutrosophic bipolar fuzzy open function, so θ(NB) is a τ_{NB_2} -open NBFS such that $y_{\lambda,\mu,\eta} = \theta(x_{\alpha,\beta,\gamma})$ $\in \theta$(NB) and $y'_{\lambda_1,\mu_1,\eta_1} = x'_{\alpha_1,\beta_1,\gamma_1} \notin \theta$(NB). Similarly, $y_{\lambda,\mu,\eta} = \theta(x_{\alpha,\beta,\gamma}) \notin \theta$(NB) and $y'_{\lambda_1,\mu_1,\eta_1} = x'_{\alpha_1,\beta_1,\gamma_1} \in \theta$(NB). Thus for any two NBFPs $y_{\lambda,\mu,\eta}$ and $y'_{\lambda_1,\mu_1,\eta_1}$ in Y such that $y \neq y'$ there exists a τ_{NB_2} -open NBFS θ(NB) such that $y_{\lambda,\mu,\eta} \in \theta$(NB), $y'_{\lambda_1,\mu_1,\eta_1} \notin \theta$(NB) or $y_{\lambda,\mu,\eta} \notin \theta$(NB), $y'_{\lambda_1,\mu_1,\eta_1} \in \theta$(NB). Therefore (Y, τ_{NB_2}) is a NBF-T_0-space. Hence proved.

Proposition 3.9

Let $\theta : (X, \tau_{NB_1}) \to (Y, \tau_{NB_2})$ be a bijective and neutrosophic bipolar fuzzy homomorphism. If (X, τ_{NB_1}) is a NBF-T_0-space then (Y, τ_{NB_2}) is also a NBF-T_0-space.

Proof

Let (X, τ_{NB_1}) and (Y, τ_{NB_2}) be two NBFTSS. Also let (X, τ_{NB_1}) be a NBF-T_0-space and $\theta : X \to Y$ be a bijective neutrosophic bipolar fuzzy homeomorphism. Therefore by the theorem 3.6, (Y, τ_{NB_2}) is a NBF-T_0-space.

Definition 3.10

A NBFTS (X, τ_{NB}) is called neutrosophic bipolar fuzzy T_1-space (NBF-T_1-space) if for any pair of NBFPS $x_{\alpha,\beta,\gamma}$, $y_{\lambda,\mu,\eta}$ ($x \neq y$) in X, there exists two NBFOS NB_1 and NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $x_{\alpha,\beta,\gamma} \notin NB_2$ and $y_{\lambda,\mu,\eta} \notin NB_1$, $y_{\lambda,\mu,\eta} \in NB_2$. Every neutrosophic bipolar fuzzy T_1-space is a neutrosophic bipolar fuzzy T_0-space.

Example 3.11

Suppose that X = {x, y}. Let τ_{NB} = {0_{NB}, 1_{NB}, {⟨x, 0.1, 0.2, 0.3, –0.2, –0.4, –0.1⟩, ⟨y, 0.3, 0.4, 0.1, –0.2, –0.1, –0.5⟩}, {⟨x, 0.1, 0.2, 0.3, –0.2, –0.4, –0.1⟩, ⟨y, 0.3, 0.4, 0.1, –0.2, –0.1, –0.5⟩}} be an NBFT on X. Therefore (X, τ_{NB}) is a neutrosophic bipolar fuzzy T_1-space.

Example 3.12

Let X = {a, b} and τ_{NB} = {ϕ, X}. It is clear that (X, τ_{NB}) is a NBFTS but it is not a NBF-T_1-space.

Proposition 3.13

Let τ_{NB} and τ^*_{NB} be two neutrosophic bipolar fuzzy topologies on a set X such that τ^*_{NB} is finer than τ_{NB}. If (X, τ_{NB}) is a NBF-T_1-space then (X, τ^*_{NB}) is also a NBF-T_1-space.

Proof

Let $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$, $x \neq y$, be two NBFPS in X. Then (X, τ_{NB}) is a NBF-T_1-space, so there exists a two NBFPS NB_1 and NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $x_{\alpha,\beta,\gamma} \notin NB_2$ and $y_{\lambda,\mu,\eta} \notin NB_1$, $y_{\lambda,\mu,\eta} \in NB_2$. Since τ^*_{NB} is a finer than τ_{NB}, so NB_1, $NB_2 \in \tau_{NB} \Rightarrow NB_1, NB_2 \in \tau^*_{NB}$. Thus for any two NBFPS $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$ in X such that $x \neq y$ there exists a two NBFPS NB_1 and NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $x_{\alpha,\beta,\gamma} \notin NB_2$ and $y_{\lambda,\mu,\eta} \notin NB_1$, $y_{\lambda,\mu,\eta} \in NB_2$. Hence (X, τ^*_{NB}) is a NBF-T_1-space.

Proposition 3.14

Let (X, τ_{NB}) be a NBFTS. If (X, τ_{NB}) is a NBF-T_1-space then it is a NBF-T_2-space.

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Proof

Let $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$, such that x ≠ y, be two NBFPS in X. since X is NBF-T_1-space, so there exists a $NB_1 \in \tau_{NB}$ such that $x_{\alpha,\beta,\gamma} \in NB_1$, $y_{\lambda,\mu,\eta} \notin NB_1$ and there exists a $NB_2 \in \tau_{NB}$ such that $x_{\alpha,\beta,\gamma} \notin NB_2$, $y_{\lambda,\mu,\eta} \in NB_2$. Hence (X, τ_{NB}) is a NBF-T_0-space.

Remark 3.15

Converse of the proposition 3.14 is not true. The following counter examples are proved.

Let X = {a, b} and τ_{NB} = {ϕ, X, {⟨0.3, 0.4, 0.5, –0.2, –0.1, –0.3⟩}}. Clearly (X, τ_{NB}) is a NBF-T_0-space but not a NBF-T_1-space.

Theorem 3.16

Let $\theta : (X, \tau_{NB_1}) \rightarrow (Y, \tau_{NB_2})$ be both one-one and neutrosophic bipolar fuzzy continuous function from an NBFTS (X, τ_{NB_1}) to another NBFTS (Y, τ_{NB_2}). If (Y, τ_{NB_2}) be a NBF-T_1-space, then (X, τ_{NB_1}) is also a NBF-T_1-space.

Proof

Let (Y, τ_{NB_2}) be an NBF-T_1-space. Let $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$ (x ≠ y) be any two distinct NBFPS in X. Since θ : $(X, \tau_{NB_1}) \rightarrow (Y, \tau_{NB_2})$ is a one-one function, so $\theta(X_{\alpha,\beta,\gamma})$, $\theta(Y_{\lambda,\mu,\eta})$ are also distinct $NBFP_s$ in Y. Since (Y, τ_{NB_2}) is an NBF-T_1-S, there exist two NBFOSs, NB_1, NB_2 in Y such that $\theta(x_{\alpha,\beta,\gamma}) \in NB_1$, $\theta(x_{\alpha,\beta,\gamma}) \notin NB_2$ or $\theta(y_{\lambda,\mu,\eta}) \notin NB_1$, $\theta(y_{\lambda,\mu,\eta}) \in NB_2$,

Therefore $x_{\alpha,\beta,\gamma} \in \theta^{-1}(NB_1)$, $x_{\alpha,\beta,\gamma} \notin \theta^{-1}(NB_2)$ or $y_{\lambda,\mu,\eta} \notin \theta^{-1}(NB_1)$, $y_{\lambda,\mu,\eta} \in \theta^{-1}(NB_2)$. Since θ is a neutrosophic bipolar fuzzy continuous function, both $\theta^{-1}(NB_1)$, $\theta^{-1}(NB_2)$ are NBFDSs in X. Therefore, for any pair of distinct NBFPs $x_{\alpha,\beta,\gamma}$, $y_{\lambda,\mu,\eta}$ in X, there exist two NBFOSs $\theta^{-1}(NB_1)$, $\theta^{-1}(NB_2)$ such that $x_{\alpha,\beta,\gamma} \in \theta^{-1}(NB_1)$, $x_{\alpha,\beta,\gamma} \notin \theta^{-1}(NB_2)$ or $y_{\lambda,\mu,\eta} \notin \theta^{-1}(NB_1)$, $y_{\lambda,\mu,\eta} \in \theta^{-1}(NB_2)$. Therefore (X, τ_{NB}) is a NBF-T_1-space.

Theorem 3.17

If a NBFTS (X, τ_{NB}) is a NBF-T_1-space then every NBFP in X is a NBFCS.

Proof

Suppose that (X, τ_{NB}) is a NBF-T_1-Space. Assume that $x_{\alpha,\beta,\gamma}$ is an arbitrary NBFP in X. Lets now take a NBFP $y_{\lambda,\mu,\eta} \subseteq x^c_{\alpha,\beta,\gamma}$ (y ≠ x) in X. Since, (X, τ_{NB}) is a NBF-T_1-space, there exist two NBFOSs NB_1 and NB_2 such that $x^c_{\alpha,\beta,\gamma} \in NB_1$, $x^c_{\alpha,\beta,\gamma} \notin NB_2$ and $y_{\lambda,\mu,\eta} \notin NB_1$, $y_{\lambda,\mu,\eta} \in NB_2$. Therefore $x^c_{\alpha,\beta,\gamma} = \cup\, y_{\lambda,\mu,\eta} \subseteq x^c_{\alpha,\beta,\gamma}\{NB_1, NB_2 : x^c_{\alpha,\beta,\gamma} \in NB_1, x^c_{\alpha,\beta,\gamma} \notin NB_2$ and $y_{\lambda,\mu,\eta} \notin NB_1, y_{\lambda,\mu,\eta} \in NB_2\}$. Since $\cup\, y_{\lambda,\mu,\eta} \subseteq x^c_{\alpha,\beta,\gamma}\{NB_1, NB_2 : x^c_{\alpha,\beta,\gamma} \in NB_1, x^c_{\alpha,\beta,\gamma} \notin NB_2$ and $y_{\lambda,\mu,\eta} \notin NB_1, y_{\lambda,\mu,\eta} \in NB_2\}$ is a NBFOS in X, $x^c_{\alpha,\beta,\gamma}$ is a NBFOS in X. Therefore $X_{\alpha,\beta,\gamma}$ is an NBFCS in X.

Remark 3.18

Let (X, τ_{NB}) is an NBFTS, Then, X is an NBF-T_1-space iff $X_{\alpha,\beta,\gamma} = \cap\{NBF_{cl}(NB)$;

$X_{\alpha,\beta,\gamma} \in NBF_{cl}(NB)\}$.

Theorem 3.19

Let $\theta : (X, \tau_{NB_1}) \rightarrow (Y, \tau_{NB_2})$ be a bijective neutrosophic bipolar fuzzy open function. If (X, τ_{NB_1}) is a NBF-T_1-space then (Y, τ_{NB_2}) is also a NBF-T_1-space.

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Proof

Let (X, τ_{NB_1}) and (Y, τ_{NB_2}) be two NBFTSs. Also let (X, τ_{NB_1}) be a NBF-T_1-space and $\theta : X \to Y$ be a bijective neutrosophic bipolar fuzzy open function. Then (Y, τ_{NB_2}) is a NBF-T_1-space. Let $y_{\lambda,\mu,\eta}$ and $y'_{\lambda_1,\mu_1,\eta_1}$, $y \neq y'$, be two NBFPS in Y. Since θ is bijective, so there exists two NBFPS $x_{\alpha,\beta,\gamma}$ and $x'_{\alpha_1,\beta_1,\gamma_1}$, $x \neq x'$ in X such that θ $(x_{\alpha,\beta,\gamma}) = y_{\lambda,\mu,\eta}$ and $\theta(x'_{\alpha_1,\beta_1,\gamma_1}) = y'_{\lambda_1,\mu_1,\eta_1}$. Since X is NBF-$T_1$-space, so there exists a τ_{NB_1}-open NBFS NB_1 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $x'_{\alpha_1,\beta_1,\gamma_1} \notin NB_1$ and there exists τ_{NB_1}-open NBFS NB_2 such that $x_{\alpha,\beta,\gamma} \notin NB_2$, $x'_{\alpha_1,\beta_1,\gamma_1} \in NB_2$. Since θ is a neutrosophic bipolar fuzzy open function, so $\theta(NB_1)$ is a τ_{NB_2}-open NBFS such that $y_{\lambda,\mu,\eta} = \theta(x_{\alpha,\beta,\gamma}) \in \theta(NB_1)$ and $y'_{\lambda_1,\mu_1,\eta_1} = \theta(x'_{\alpha_1,\beta_1,\gamma_1}) \notin \theta(NB_1)$. Similarly $\theta(NB_2)$ is a τ_{NB_2}-open NBFS such that $y_{\lambda,\mu,\eta} = \theta(x_{\alpha,\beta,\gamma}) \in \theta(NB_2)$ and $y'_{\lambda_1,\mu_1,\eta_1} = \theta$ $(x'_{\alpha_1,\beta_1,\gamma_1}) \notin \theta(NB_2)$. Thus for any two NBFPS $y_{\lambda,\mu,\eta}$ and $y'_{\lambda_1,\mu_1,\eta_1}$ in Y such that $y \neq y'_1$, there exists a τ_{NB_2}-open NBFS $\theta(NB_1)$ such that $y_{\lambda,\mu,\eta} \in \theta(NB_1)$, $y'_{\lambda_1,\mu_1,\eta_1} \notin \theta(NB_1)$ and there exists a τ_{NB_2}-open NBFS, $\theta(NB_2)$ such that $y_{\lambda,\mu,\eta} \notin \theta(NB_2)$, $y'_{\lambda_1,\mu_1,\eta_1} \in \theta(NB_2)$. Therefore (Y, τ_{NB_2}) is a NBF-T_1-space. Hence proved.

Theorem 3.20

Let θ : $(X, \tau_{NB_1}) \to (Y, \tau_{NB_2})$ be a bijective and neutrosophic bipolar fuzzy homomorphism. If (X, τ_{NB_1}) is a NBF-T_1-space then (Y, τ_{NB_2}) is also a NBF-T_1-space.

Proof

Let (X, τ_{NB_1}) and (Y, τ_{NB_2}) be two NBFTSs. Also let (X, τ_{NB_1}) be a NBF-T_1-space and $\theta : X \to Y$ be a neutrosophic bipolar fuzzy homeomorphism. Since θ is a neutrosophic bipolar fuzzy homeomorphism, so θ is a bijective neutrosophic bipolar fuzzy open function. Therefore by the theorem 3.15, (Y, τ_{NB_2}) is a NBF-T_1-space. Hence proved.

Definition 3.21

A NBFTS (X, τ_{NB}) is said to be neutrosophic bipolar fuzzy T_2-space (NBF-T_2-S) or neutrosophic bipolar fuzzy Hausdroff space if for any pair of NBFPS $X_{\alpha,\beta,\gamma}$, $Y_{\lambda,\mu,\eta}$ ($X \neq y$) in X, there exists two NBFOS NB_1 and NB_2 such that $X_{\alpha,\beta,\gamma} \in NB_1$, $X_{\alpha,\beta,\gamma} \notin NB_2$ and $Y_{\lambda,\mu,\eta} \notin NB_1$, $Y_{\lambda,\mu,\eta} \in NB_2$ with $NB_1 \subseteq NB_2^c$. Obviously, every NBF-T_2-space is a NBF-T_1-space.

Example 3.22

Let X = {x, y, z} be a fixed set. Let τ_{NB} = {{0_{NB}, 1_{NB}, ⟨x, 0.3, 0.2, 0.1, –0.1, –0.2, –0.3 ⟩, ⟨y, 0.6, 0.4, 0.2, –0.3, –0.2, –0.1⟩}, {⟨x, 0.3, 0.2, 0.1, –0.1, –0.2, –0.3⟩, ⟨z, 0.2, 0.4, 0.5, –0.1, –0.3, –0.6⟩}, {⟨y, 0.6, 0.4, 0.2, –0.3, –0.2, –0.1⟩, ⟨z, 0.2, 0.4, 0.5, –0.1, –0.3, –0.6⟩}, {⟨x, 0.3, 0.2, 0.1, –0.1, –0.2, –0.3⟩}, {⟨y, 0.6, 0.4, 0.2, –0.3, –0.2, –0.1⟩}, {⟨z, 0.2, 0.4, 0.5, –0.1, –0.3, –0.6⟩}} be a NBFT on X. Clearly (X, τ_{NB}) is a NBF-T_2-space.

Example 3.23

Let x = {a, b} and $\tau = \{\varphi, X\}$. Clearly (X. τ_{NB}) is a NBFTS but it is not a NBF-T_2-space.

Theorem 3.24

Let τ_{NB} and τ^*_{NB} be two neutrosophic topologies on a set X such that τ^*_{NB} is finer than τ_{NB}. If (X, τ_{NB}) is a NBF-T_2-space then (X, τ^*_{NB}) is also a NBF-T_2-space.

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Proof

Let $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$, $x \neq y$, be two NBFPS in X. Here (X, τ_{NB}) is a NBF-T_2-space, so there exists $NB_1, NB_2 \in \tau_{NB}$ such that $x_{\alpha,\beta,\gamma} \in NB_1$, $y_{\lambda,\mu,\eta} \in NB_2$ and $NB_1 \cap NB_2 = \phi$. Since τ^*_{NB} is finer that τ_{NB}, so $NB_1, NB_2 \in \tau_{NB} \Rightarrow NB_1, NB_2 \in \tau^*_{NB}$. Thus for any two NBFPS $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$ in X such that $x \neq y$ there exists $NB_1, NB_2 \in \tau^*_{NB}$ such that $x_{\alpha,\beta,\gamma} \in NB_1$, $y_{\lambda,\mu,\eta} \in NB_2$ and $NB_1 \cap NB_2 = \phi$. Hence (X, τ^*_{NB}) is a NBF-T_2-space.

Proposition 3.25

Let (X, τ_{NB}) be a NBFTS. If (X, τ_{NB}) is a NBF-T_2-space then it is a NBF-T_1-space.

Proof

Let $x_{\alpha,\beta,\gamma}$ and $y_{\lambda,\mu,\eta}$ be any two NBFPS in X such that $x \neq y$. Since (X, τ_{NB}) is a NBF-T_2-space, so there exists τ_{NB}-open NBFSS NB_1 and NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$ and $y_{\lambda,\mu,\eta} \in NB_2$ and $NB_1 \cap NB_2 = \phi$. Since $x_{\alpha,\beta,\gamma} \in NB_1$ and $NB_1 \cap NB_2 = \phi$, so $x_{\alpha,\beta,\gamma} \notin NB_2$. Similarly $y_{\lambda,\mu,\eta} \notin NB_1$ that is there exists a $NB_1 \in \tau_{NB}$ such that $x_{\alpha,\beta,\gamma} \in NB_1$, $y_{\lambda,\mu,\eta} \notin NB_1$ and there exists a $NB_2 \in \tau_{NB}$ such that $x_{\alpha,\beta,\gamma} \notin NB_2$, $y_{\lambda,\mu,\eta} \in NB_2$. Hence (X, τ_{NB}) is a NBF-T_1-space.

Remark 3.26

Every neutrosophic bipolar fuzzy open sets ($NBFO_s$) in a NBF-T_2-space is a neutrosophic bipolar fuzzy closed set ($NBFC_s$).

Definition 3.27: Neutrosopbic Bipolar Fuzzy Continuity

Let X and Y be a non-empty sets, let $NB_1 \in NBF(X)$ and $NB_2 \in NBF(Y)$ and let $\theta : X \to Y$ be a mapping. Then

(i) The image of NB_1 under θ, denoted by $(\theta(NB_1)) = \left(\theta(NB_1^+),\ \theta(NB_1^-)\right)$ is a neutrosophic bipolar fuzzy set in Y defined as follows : for every $y \in Y$.

$$\left(\theta(NB_1^+)\right)(y) = \left\{\begin{matrix} V_{x\epsilon\theta^{-1}(y)} T_{NB_1^+(x)}, & V_{x\epsilon\theta^{-1}(y)} I_{NB_1^+(x)}, & \Lambda_{x\epsilon\theta^{-1}(y)} F_{NB_1^+(x)} \\ 0, & Otherwise & \end{matrix}\right\}$$

$$\left(\theta(NB_1^-)\right)(y) = \left\{\begin{matrix} \Lambda_{x\epsilon\theta^{-1}(y)} T_{NB_1^-(x)}, & \Lambda_{x\epsilon\theta^{-1}(y)} I_{NB_1^-(x)}, V_{x\epsilon\theta^{-1}(y)} F_{NB_1^-(x)} \\ 0, & Otherwise \end{matrix}\right\}$$

(ii) The pre-image of NB_2 under θ, denoted as $\left(\theta^{-1}(NB_2)\right) = \left(\theta^{-1}(NB_2^+)\right), \left(\theta^{-1}(NB_2^-)\right)$ is a neutrosophic bipolar fuzzy set in X defined as follows : for each $x \in X$.

$$\left(\theta^{-1}(NB_2^+)\right)(x) = \langle T_{NB_2^+} 0\theta(x), I_{NB_2^+} 0\theta(x), T_{NB_2^+}\left(\theta(x)\right), I_{NB_2^+}\left(\theta(x)\right), F_{NB_2^+}\left(\theta(x)\right) \rangle \ \&$$

$$\left(\theta^{-1}(NB_2^-)\right)(x) = \langle T_{NB_1^-}\left(\theta(x)\right), I_{NB_1^-}\left(\theta(x)\right), F_{NB_1^-}\left(\theta(x)\right) \rangle$$

Theorem 3.28

Assume that $\theta : (X, \tau_{NB_1}) \to (Y, \tau_{NB_2})$ be both one-one and neutrosophic bipolar fuzzy continuous function from a NBFTS (X, τ_{NB_1}) to another NBFTS (Y, τ_{NB_2}). If (Y, τ_{NB_2}) is an NBF-T_2-space, then (X, τ_{NB_1}) is also a NBF-T_2-space.

Proof

Since θ is a neutrosophic bipolar fuzzy continuous function, inverse image of an NBFOS in (Y, τ_{NB_2}) is also an NBFOS in (X, τ_{NB_1}). Also it is known that, the complement of NBFOS is NBFCS in NBFTS. Here, since (Y, τ_{NB_2}) is a NBF-T_2-space every NBFOS in (Y, τ_{NB_2}) is also a NBFCS in (Y, τ_{NB_2}). Now θ is a neutrosophic bipolar-fuzzy continuous function.

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$\Rightarrow$ $\theta(X) = Y$ is a NBFOS in τ_{NB_2}

$\Rightarrow$ $\theta^{-1}(Y) = X$ is a NBFOS in τ_{NB_1}

Therefore, if (Y, τ_{NB_2}) is a NBF-T_2-space, then $(\theta^{-1}(Y), \tau_{NB_1})$ is a NBF-T_2-space. Hence (X, τ_{NB_2}) is an NBF-T_2-space.

Example 3.29

Let X = {a, b}

Let NB_1 be a neurosophic bipolar fuzzy set centered at a and NB_2 be a neutrosophic bipolar fuzzy set centered at b and are defined as

NB_1 = {<a, 1.0, 0.0, 0.0, 0, 0, –1>, <b, 0, 0.2, 0.8, –0.6, –0.2, –0.2>}

and NB_2 = {<a, 0.2, 0.1, 0.7, –0.8, –0.1, –0.1>, <b, 1.0, 0, 0, 0, 0, –1, 0>}

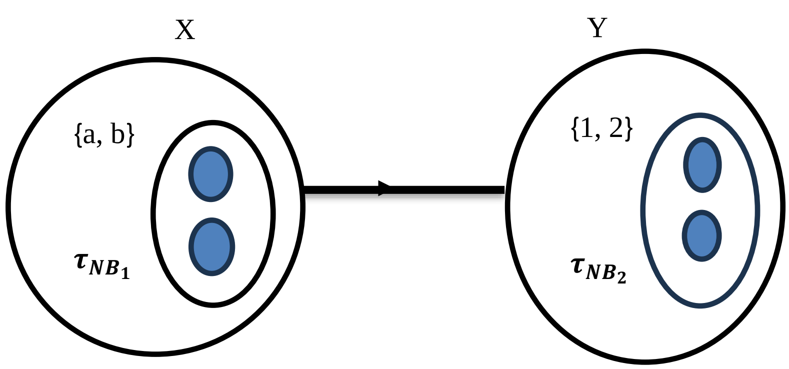
Then τ_{NB_1} = {ϕ, X, NB_1, NB_2} is a neutrosophic bipolar fuzzy topology on X. Then (X, τ_{NB_1}) is a NBFTS.

Let Y = {1, 2}

Define a one-one NBFC function θ : X – Y, such that θ(a) = 1 and θ(b) = 2. This is clearly injective (one-one).

Define neutrosophic bipolar fuzzy topological spaces (Y, τ_{NB_2}).

Let $NB_3 = \theta(NB_1)$ and $NB_4 = \theta(NB_2)$



Let NB_3 be a NBFS centered at 1 and NB_4 be a NBFS centered at 2 and are

NB_3 = {<1, 1, 0, 0, 0, 0, –1>, <2, 0, 0.2, 0.8, –0.6, –0.2, –0.2>}

and NB_4 = {<1, 0.2, 0.1, 0.7, –0.8, –0.1, –0.1>, <2, 1.0, 0, 0, 0,0, –1.0>}.

Then τ_{NB_2} = {ϕ, Y, NB_3, NB_4} is a neutrosophic bipolar fuzzy topology on Y.

Then (Y, τ_{NB_2}) becomes a NBFTS.

Verify θ is NBFC function.

A function θ is neutrosopbic bipolar fuzzy continuous if, for all $V \in \tau_{NB_2}$, $\theta^{-1}(V) \in \tau_{NB_1}$

$\theta^{-1}(NB_3) = NB_1 \in \tau_{NB_1}$

$\theta^{-1}(NB_4) = NB_2 \in \tau_{NB_1}$

So θ is NBFC function.

Let $C_{\alpha\beta\gamma}$, $d_{\lambda\mu\eta}$ be two distinct NBF points in Y.

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Since (Y, $\boldsymbol{\tau_{NB_2}}$) is NBF T_2 space, for any two distinct neutrosophic bipolar fuzzy points there exists disjoint neutrosophic bipolar fuzzy open sets NB_3 centered around c and NB_4 centered around d in Y.

As θ : (X, $\boldsymbol{\tau_{NB_1}}$) → (Y, $\boldsymbol{\tau_{NB_2}}$) is a NBFC function for NB_3 and NB_4 in Y there will exist $\theta^{-1}(NB_3) = NB_1$ and $\theta^{-1}(NB_4) = NB_2$ in X centered around x and y (x ≠ y) respectively.

Therefore (X, $\boldsymbol{\tau_{NB_1}}$) is also NBF-T_2-space.

Theorem 3.30

Assume that (X, τ_{NB}) is a NBF-T_1-space with the condition that the complement of each NBFOS is a NBFOS then (X, τ_{NB}) is a NBF-T_2-space.

Proof

Assume that (X, τ_{NB}) is an NBF-T_1-space with the condition that the complement of each NBFOS is a NBFOS that is if $\tilde{N}$ is an NBFOS in (X, τ_{NB}) then $\tilde{N}^c = \tilde{N}$ ………….. (1)

Suppose that $\tilde{N}$ is an NBFOS in (X, τ_{NB}). Then $\tilde{N}^c$ is a NBFCS in (X, τ_{NB}). Again by equation (1), $\tilde{N}^c$ is an NBFOS in (X, τ_{NB}). So $\tilde{N}^c = \tilde{N}$. Again $(\tilde{N}^c)^c = \tilde{N}^c = \tilde{N}$. Therefore, every NBFCS in (X, τ_{NB}) is both NBFOS and NBFCS in (X, τ_{NB}). Hence by the Remark 3.22, (X, τ_{NB}) is an NBF-T_2-space.

Theorem 3.31

Let θ : $(X, \tau_{NB_1}) \to (Y, \tau_{NB_2})$ be a bijective neutrosophic bipolar fuzzy open function. If (X, τ_{NB_1}) is a NBF-T_2-space then (Y, τ_{NB_2}) is also a NBF-T_2-space.

Proof

Let (X, τ_{NB_1}) and (Y, τ_{NB_2}) be two NBFTSs. Also let (X, τ_{NB_1}) be a NBF-T_2-space and $\theta : X \to Y$ be a bijective neutrosophic bipolar fuzzy open function. Then (Y, τ_{NB_2}) is a NBF-T_2-space. Let $y_{\lambda,\mu,\eta}$ and $y'_{\lambda_1,\mu_1,\eta_1}$, y ≠ y′, be two NBFPS in Y. Since θ is bijective, so there exists two NBFPS $x_{\alpha,\beta,\gamma}$ and $x'_{\alpha_1,\beta_1,\gamma_1}$, x ≠ x′$_1$ in X such that $\theta(x_{\alpha,\beta,\gamma}) = y_{\lambda,\mu,\eta}$ and $\theta(x'_{\alpha_1,\beta_1,\gamma_1}) = y'_{\lambda_1,\mu_1,\eta_1}$. Since X is NBF-$T_2$-space, so there exists a τ_{NB_1}-open NBFS NB_1, NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $x'_{\alpha_1,\beta_1,\gamma_1} \in NB_2$ and $NB_1 \cap NB_2 = \phi$. Since θ is a neutrosophic bipolar fuzzy open function, so $\theta(G)$, $\theta(H)$ are τ_{NB_1}-open NBFSS such that $y_{\lambda,\mu,\eta} = \theta(x_{\alpha,\beta,\gamma}) \in \theta(NB_1)$, $y'_{\lambda_1,\mu_1,\eta_1} = \theta(x'_{\alpha_1,\beta_1,\gamma_1}) \notin \theta(NB_2)$. Again since θ is bijective, so $\theta(NB_1) \cap \theta(NB_2) = \theta(NB_1 \cap NB_2) = \theta(\phi) = \phi$. Thus for any two NBFPS $y_{\lambda,\mu,\eta}$ and $y'_{\lambda_1,\mu_1,\eta_1}$ is Y such that y ≠ y'_1, there exists τ_{NB_2}-open NBFSS $\theta(NB_1)$, $\theta(NB_2)$ such that $y_{\lambda,\mu,\eta} \in \theta(NB_1)$, $y'_{\lambda_1,\mu_1,\eta_1} \in \theta(NB_2)$ and $\theta(NB_1) \cap \theta(NB_2) = \phi$. Therefore (Y, τ_{NB_2}) is a NBF-T_2-space. Hence proved.

Theorem 3.32

Let θ : $(X, \tau_{NB_1}) \to (Y, \tau_{NB_2})$ be a bijective and neutrosophic bipolar fuzzy homomorphism. If (X, τ_{NB_1}) is a NBF-T_1-space then (Y, τ_{NB_2}) is also a NBF-T_1-space.

Proof

Let (X, τ_{NB_1}) and (Y, τ_{NB_2}) be two NBFTSs. Also let (X, τ_{NB_1}) be a NBF-T_2-space and $\theta : X \to Y$ be a bijective neutrosophic bipolar fuzzy homeomorphism. Therefore by the theorem 3.25, (Y, τ_{NB_2}) is a NBF-T_2-space. Hence proved.

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Definition 3.33

Let (X, τ_{NB}) is a NBFTS. Then X is called a neutrosophic bipolar fuzzy regular-space if for any NBFP $x_{\alpha,\beta,\gamma}$ in X, and NBFCS Q with $x_{\alpha,\beta,\gamma} = Q^c$, there exists two NBFOSs NB_1 and NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $Q \subseteq NB_2$ and $NB_1 \subseteq NB_2^c$.

Definition 3.34

A NBFTS (X, τ_{NB}) is said to be a neutrosophic bipolar fuzzy T_3-space (NBF-T_3-space) if it is a NBF-T_1-space and a neutrosophic bipolar fuzzy regular space. Obviously, every NBF-T_2-space is a NBF-T_3-Space.

Example 3.35

The neutrosophic bipolar fuzzy discrete topological space (X, τ_{NB}) is a neutrosophic bipolar fuzzy regular space as well as NBF-T_1-space. Therefore (X, τ_{NB}) is a NBF-T_3-space.

Definition 3.36

Let (X, τ_{NB}) be a neutrosophic bipolar fuzzy topological space and let A ∈ NBF(X). Then the neutrosophic bipolar fuzzy closure of A, denoted by $NBF\tilde{c}l(A)$ is a neutrosophic bipolar fuzzy set in X defined by,

$$NBF\tilde{c}l(A) = \cap \{NB \in NBF(X) : NB^c \in \tau_{NB}, A \in NB\}.$$

Theorem 3.37

For any NBFTS (X, τ_{NB}), the following results are equivalent :

(i) X is a neutrosophic bipolar fuzzy regular-space

(ii) For any NBFP $X_{\alpha,\beta,\gamma}$ and any NBFOS NB_2 containing $X_{\alpha,\beta,\gamma}$ there exists anNBFOS NB_1 such that $X_{\alpha,\beta,\gamma} \in NB_1 \subseteq NBF\tilde{c}l(NB_1) \subseteq NB_2$.

Proof

(i) $\Rightarrow$ (ii)

Suppose that (X, τ_{NB}) is a neutrosophic bipolar fuzzy regular space. Then, for any NBFP $X_{\alpha,\beta,\gamma}$ in X, and a NBFCS Q with $X_{\alpha,\beta,\gamma} \in Q^c$, there exists two NBFOSs NB_1 and NB_2 such that $X_{\alpha,\beta,\gamma} \in NB_1$, $Q \subseteq NB_2$ and $NB_1 \subseteq NB_2^c$.

Again, since NB_2^c is a NBFCS, there exists a NBFCS, H such that $NB_1 \subseteq H$ and so $NBF\tilde{c}l(NB_1) \subseteq H$.

Again, for NBFCS, H there exists an NBFOS NB_2 such that $H \subseteq NB_2$. Therefore, $X_{\alpha,\beta,\gamma} \in NB_1 \subseteq NBF\tilde{c}l(NB_1) \subseteq H \subseteq NB_2$.

$\Rightarrow$ $X_{\alpha,\beta,\gamma} \in NB_1 \subseteq NBF\tilde{c}l(NB_1) \subseteq NB_2$.

(ii) $\Rightarrow$ (i)

The result is obvious for the neutrosophic regular space.

Definition 3.38

A NBFTS (X, τ_{NB}) is said to be a neutrsophic bipolar fuzzy normal-space, if for any pair of NBFCSs $\tilde{NB}_1$ and $\tilde{NB}_2$ with $\tilde{NB}_1 \subseteq \tilde{NB}_2^c$ in X, there exists two NBFOS, NB_1 and NB_2 in X such that $\tilde{NB}_1 \subseteq NB_1$, $\tilde{NB}_2 \subseteq NB_2$ and $NB_1 \subseteq \tilde{NB}_2^c$.

Example 3.39

Let X = {x, y} be a fixed set.

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Let τ_{NB} = {0_{NB}, 1_{NB}, {⟨x, 0.3, 0.2, 0.1, –0.2, –0.1, –0.2⟩, ⟨y, 0.5, 0.4, 0.3, –0.2, –0.1, –0.3⟩}, {⟨x, 0.3, 0.2, 0.1, –0.3, –0.1, –0.2⟩, ⟨y, 0.5, 0.4, 0.3, –0.2, –0.1, –0.3⟩}}.

Consider the closed set = {0_{NB}, 1_{NB}, {⟨x, 0.7, 0.8, 0.9, –0.7, –0.9, –0.8⟩, ⟨y, 0.5, 0.6, 0.7, –0.8, –0.9, –0.7⟩}, {⟨x, 0.7, 0.8, 0.9, –0.7, –0.9, –0.8⟩, ⟨y, 0.5, 0.6, 0.7, –0.8, –0.9, –0.7⟩}}.

Therefore (X, τ_{NB}) is a neutrosophic bipolar fuzzy normal-space.

Definition 3.40

A NBFTS (X, τ_{NB}) is said to be a neutrosophic bipolar fuzzy T_4-space (NBF-T_4-space) if it is both NBF-T_1-space and neutrosophic bipolar fuzzy normal-space. Obviously, every NBF-T_4-space is also a NBF-T_1-space.

Theorem 3.41

For any NBFTS (X, τ_{NB}), the following results are equivalent.

(i) X is a neutrosophic bipolar fuzzy normal-space.

(ii) For every NBFCS $\tilde{NB}_1$ and NBFOS NB_1 with $\tilde{NB}_1 \subseteq NB_1$, there exists an NBFOS NB_2 such that $\tilde{NB}_1 \subseteq NB_2 \subseteq NBF\tilde{c}l(NB_2) \subseteq NB_1$.

Proof

(i) ⇒ (ii)

Assume that X is a neutrosophic bipolar fuzzy normal space. Then for any two NBFCSs $\tilde{NB}_1$, $\tilde{NB}_2$ with $\tilde{NB}_1 \subseteq \tilde{NB}_2^c$ in X, there exists two NBFOSs NB_1 and NB_2 in X such that $NB_1 \subseteq NB_2^c$.

For every NBFCS $\tilde{NB}_1$ and $\tilde{NB}_2$ with $\tilde{NB}_1 \subseteq \tilde{NB}_2^c$. It is clear that $\tilde{NB}_1 \subseteq \tilde{NB}_2^c \Rightarrow \tilde{NB}_1 \cap \tilde{NB}_2 = \phi$.

Consider the NBFOS NB_1 that contains $\tilde{NB}_1$ and NB_2 that contains $\tilde{NB}_2$ i.e., $\tilde{NB}_1 \subseteq NB_1$ and $\tilde{NB}_2 \subseteq NB_2$. There are two NBFOS NB_3 and NB_4 that are $NB_1 \cap NB_3$ and $NB_2 \cap NB_4$, where NB_3 and NB_4 may or may not be disjoint NBFOS.

Here, in terms of first part, $\tilde{NB}_1 \subseteq NB_1$ and $NBF\tilde{c}l(NB_1) \subseteq NB_3$

$$\Rightarrow \quad \tilde{NB}_1 \subseteq NB_1 \subseteq NBF\tilde{c}l(NB_1) \subseteq NB_3$$

Assuming that $NB_1 = NB_2$ and $NB_3 = NB_1$ are both $NBFO_s$, we obtain the following result.

(ii) ⇒ (i)

There exists a NBFOS NB_2 such that $\tilde{NB}_1 \subseteq NB_2 \subseteq NBF\tilde{c}l(NB_2) \subseteq NB_1$ for any NBFCS $\tilde{NB}_1$ and NBFOS NB_1 with $\tilde{NB}_1 \subseteq NB_1$. Now, it is necessary to demonstrate that X is a neutrosophic bipolar fuzzy normal space.

There are two NBFOSs NB_1 and NB_2 for any two NBFCSs $\tilde{NB}_1$ and $\tilde{NB}_2$ with $\tilde{NB}_1 \subseteq \tilde{NB}_2^c$ such that $\tilde{NB}_1 \subseteq NB_1$ and $\tilde{NB}_2 \subseteq NB_2$ as $\tilde{NB}_1 \cap \tilde{NB}_2 = \phi$.

As a result, any two NBFCSs $\tilde{NB}_1$ and $\tilde{NB}_2$ with $\tilde{NB}_1 \subseteq \tilde{NB}_2^c$ in X have two NBFOSs NB_1 and NB_2, and we have, $\tilde{NB}_1 \subseteq NB_1$, $\tilde{NB}_2 \subseteq NB_2$ and $NB_1 \subseteq NB_2^c$. Therefore, X is a neutrosophic bipolar fuzzy normal-space.

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To compare the neutrosophic bipolar fuzzy separation axiom T0 – T4 with classical and fuzzy separation axioms, the following comparison table has been given to highlight the distinctions between them.

Table 1: comparison between the neutrosophic bipolar fuzzy separation axiom T0 – T4 with classical and fuzzy separation axioms

Axioms	Classical Topology	Fuzzy Topology	NBF TOPOLOGY
T_0	For any two distinct points x, y ∈ x, there exist an open set that contains one of them but not the other.	For every pair of distinct points x, y ∈ x, there exists a fuzzy open set that contains one of them but not the other.	For any pair of neutrosophic bipolar fuzzy points (NBFPS) $x_{\alpha,\beta,\gamma} = \left(x_{a_1,\beta_1,\gamma_1}, x_{a_2,\beta_2,\gamma_2}\right)$ and $y_{\lambda,\mu,\eta} = \left(y_{\lambda_1,\mu_1,\eta_1}, x_{\lambda_2,\mu_2,\eta_2}\right)$ (x ≠ y) in X, there exist on neutrosophic bipolar fuzzy open set (NBFO) NB = $(NB^+, NB^-) \ni x_{\alpha,\beta,\gamma} \in NB$, $y_{\lambda,\mu,\eta} \notin NB$ (or) $x_{\alpha,\beta,\gamma} \notin NB$, $y_{\lambda,\mu,\eta} \in NB$. That is when there are two different NBFPs, at least one of them has an NBF open set where the truth degree of the other is strictly less
T_1	For every pair of distinct points x, y ∈ X, there exist two open set A, B ∈ τ such that x∈ A and x ∈ B or y ∉ A and y ∈ B	For every pair of distinct points x, y ∈ X, there exists fuzzy open sets A, B ∈ τ such that A(x) = 1, A(y) = 0 or B(y) = 1, B(x) = 0.	For any pair of NBFPS $x_{\alpha,\beta,\gamma}$, $y_{\lambda,\mu,\eta}$ (x ≠ y) in X, there exists two NBFOS NB_1 and NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $x_{\alpha,\beta,\gamma} \notin NB_2$ and $y_{\lambda,\mu,\eta} \notin NB_1$, $y_{\lambda,\mu,\eta} \in NB_2$.. Every neutrosophic bipolar fuzzy T_1-space is a neutrosopic bipolar fuzzy T_0-space. That is Every NBFP has an NBF open set that excludes the other on a higher truth/falsity value.
T_2	For any two distinct points x, y ∈ X, there exists disjoint open sets U and V such that x ∈ U, y ∈V.	For every pair of distinct points x, y ∈ X, there exists fuzzy open sets A, B ∈τ ∋ A(x) = 1, A(y) =0 or B(y) = 1, B(y) = 0 and A ∩ B = 0.	For any pair of NBFPS $x_{\alpha,\beta,\gamma}$, $y_{\lambda,\mu,\eta}$ (x ≠ y) in X, there exists two NBFOS NB_1, NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $x_{\alpha,\beta,\gamma} \notin NB_2$ and $y_{\lambda,\mu,\eta} \notin NB_1$, $y_{\lambda,\mu,\eta} \in NB_2$ with $NB_1 \subset NB_2$. That is two distinct NBFPs have disjoint NBF open sets based on bipolar and indeterminate values.
T_3	For any point x ∈ X and a closed set A ⊂ X with x ∉ A, there exists disjoint open sets U and V such that x ∈ U, A ⊂ Y.	If it is FT_1 and for every fuzzy point x_α and a fuzzy closed set F with x ∉ F, there exist fuzzy opne sets A, B ∈ τ such that x ∈ A, F ⊂ S and A ∩ B = 0.	For any NBFP $x_{\alpha,\beta,\gamma}$ in X, and NBFCS Q with $x_{\alpha,\beta,\gamma} = Q^c$, there exists two NBFOSs NB_1 and NB_2 such that $x_{\alpha,\beta,\gamma} \in NB_1$, $Q \subseteq NB_2$ and $NB_1 \subseteq NB_2^c$. If it is a NBF-T_1-space and a neutrosophic bipolar fuzzy regular space, obviously, every NBF-T_2 space is a NBF-T_3-space. That is NBF open sets can be used to separate each NBFP and NBF closed set that does not contain it.
T_4	For any two distinct closed sets A, B ⊂ X, there exists disjoint open sets U, V such that A ⊂ U, B ⊂ V.	If it is FT_1 and for every pair of disjoint fuzzy closed sets A, B, there exists disjoint fuzzy open sets U, V ∈ τ such that A ⊂ U, B ⊂ V and U ∩ V = 0.	For any pair of NBFCSs $\tilde{NB}_1$ and $\tilde{NB}_2$ with $\tilde{NB}_1 \subseteq \tilde{NB}_2^c$ in X, there exists two NBFOS, NB_1 and NB_2 in X such that $\tilde{NB}_1 \subseteq NB_1$, $\tilde{NB}_2 \subseteq NB_2$ and $NB_1 \subseteq \tilde{NB}_2^c$. If it is both NBF-$T_1$-space and neutrosophic bipolar fuzzy normal space, obviously, every NBF-T_4 space is also a NBF-T_1-space. NBF open sets can be used to divide any two disjoint NBF closed sets according to their triple-valued memberships.

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3. Conclusion

In this article, a powerful framework for handling problems involving uncertainty, indeterminacy and bipolarity is introduced by applying the fuzzy and classical separation axioms to the neutrosophic bipolar fuzzy environment (T_0–T_4). Each of the NBF separation axioms (T_0 to T_4) provides the theoretical support of this framework: T_0 guarantees basic distinguishability, T_1 improves pointwise separation, T_3 (Hausdorff) introduces disjoint neighborhoods, T_3 (Regularity) permits the separation of points from closed sets, and T_4 (Normality) offers complete neighborhood-based separation of disjoint closed sets. These characteristics provide a hierarchical interpretation of separation within the neutrosophic bipolar fuzzy environment, leading for useful applications in decision-making, medical diagnosis and image processing.

References

[1] M. Abdel-Basset et al., "An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number," *Appl. Soft Comput.*, vol. 77, pp. 438–452, 2019.

[2] B. Al-Nafee, R. K. Al-Hamido, and F. Smarandache, "Separation axioms in neutrosophic crisp topological spaces," *Neutrosophic Sets Syst.*, vol. 25, pp. 25–32, 2017.

[3] M. Azhagappan and M. Kamaraj, "Notes on bipolar valued fuzzy RW-closed and bipolar valued fuzzy RW-open sets in bipolar valued fuzzy topological spaces," *Int. J. Math. Archive*, vol. 7, no. 3, pp. 30–36, 2016.

[4] C. L. Chang, "Fuzzy Topological Spaces," *J. Math. Anal. Appl.*, vol. 24, pp. 182–190, 1968.

[5] M. Fatimah, A. Anas, and F. Shaymaa, "Fuzzy Neutrosophic Weakly-Generalised Closed set in Fuzzy Neutrosophic Topological Spaces," *Univ. Anbar Pure Sci.*, vol. 12, pp. 63–73, 2018.

[6] M. Fatimah and W. Sarah, "Generalized b Closed Sets and Generalized b Open Sets in Fuzzy Neutrosophic bi-Topological Spaces," *Neutrosophic Sets Syst.*, vol. 35, pp. 188–197, 2020.

[7] J. Kim, P. K. Lim, J. G. Lee, and K. Hur, "The category of bipolar fuzzy sets," to be published.

[8] J. Kim, P. K. Lim, J. G. Lee, and K. Hur, "Bipolar Fuzzy Topological Spaces," *Ann. Fuzzy Math. Inform*, vol. 17, no. 3, pp. 205–229, 2019.

[9] R. M. Hashim and M. Gulistan, "Neutrosophic Bipolar Fuzzy Set and its Application in Medicines Preparations," *Neutrosophic Sets Syst.*, vol. 31, 2020.

[10] M. S. A. Alkhateeb and A. A. D. Alsharif, “Neutrosophic Logic-Based Approach for Decision Making in Smart Cities,” *Journal of Computational and Theoretical Nanoscience*, vol. 18, no. 1, pp. 1-7, 2021.

[11] F. Smarandache, "Neutrosophic Set-A Generalization of the Intuitionistic Fuzzy Logic," vol. 4, pp. 403–409, 2010, ISBN 978-952-67349-2-7.

[12] S. Das, R. Das, and S. Pramanik, "Neutrosophic Separation Axioms," *Neutrosophic Sets Syst.*, vol. 49, 2022.

[13] S. Dey and G. C. Raj, "Separation Axioms in Neutrosophic Topological Spaces," *Neutrosophic Syst. Appl.*, vol. 2, 2023.

[14] N. Yaqoob and M. A. Ansari, "Bipolar–Fuzzy ideals in ternary semigroups," *Int. J. Math. Anal.*, vol. 7, pp. 1775–1782, 2013.

[15] N. Yaqoob et al., "Structures of Bipolar Fuzzy–Hyper ideals in semihyper groups," *J. Intell. Fuzzy Syst.*, vol. 21, pp. 3015–3032, 2014.

[16] W.-R. Zhang, "Bipolar Fuzzy Set," in *Proc. IEEE*, 1998, pp. 835–840.

[17] L. A. Zadeh, "Fuzzy Sets," *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965.

DOI: https://doi.org/10.54216/IJNS.270220