Navigating Bipolar Indeterminacy: Bipolar IndetermSoft Sets and Bipolar IndetermHyperSoft Sets for Knowledge Representation

Takaaki Fujita[1,*] , **Florentin Smarandache**[2]
[1]Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan
[2]University of New Mexico, Gallup Campus, NM 87301, USA
Emails: Takaaki.fujita060@gmail.com; smarand@unm.edu

Abstract

A variety of mathematical frameworks—such as fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, soft sets, rough sets, and plithogenic sets—have been developed to model uncertainty, with wide applications in decision making, data analysis, and artificial intelligence. Within soft set theory, extensions like hypersoft sets, indeterm-soft sets, indeterm-hypersoft sets, bipolar soft sets, and bipolar hypersoft sets have further enhanced its expressive power. In this paper, we introduce two new constructs: *bipolar indeterm-soft sets* and *bipolar indeterm-hypersoft sets*. We provide their formal definitions, establish key algebraic properties, and demonstrate how they naturally combine bipolar evaluation with inherent indeterminacy. These models offer a versatile toolkit for capturing complex forms of uncertainty and lay the groundwork for future theoretical advances and practical applications in soft set theory.

Keywords: Soft Set; Indeterm-Soft Set; Hypersoft set; Indeterm-HyperSoft Set; Bipolar Soft Set; Bipolar Hypersoft Set

1 Introduction

1.1 Soft Sets and Their Extensions

Fuzzy sets,[1–3] intuitionistic fuzzy sets,[4–6] neutrosophic sets,[7] soft sets,[8] bipolar fuzzy sets,[9] hesitant fuzzy sets,[10] picture fuzzy sets,[11] rough sets,[12] and plithogenic sets[13] form fundamental frameworks for representing uncertainty, with applications in decision making, data analysis, and artificial intelligence.

A *soft set* (F, E) assigns to each parameter $e \in E$ a subset $F(e) \subseteq U$ of a universal set U, offering a flexible mechanism for approximating elements of U.[8,14] Owing to its versatility, the soft set concept has been extended in many directions, including HyperSoft sets,[15–17] SuperHyperSoft sets,[18] Indeterm-Soft sets, fuzzy soft sets, fuzzy hypersoft sets,[19] neutrosophic soft sets,[20] TreeSoft sets,[21] weighted soft sets,[22] bipolar soft sets,[23] and others. An *Indeterm-Soft set* maps each attribute to a subset of the universe in which at least one attribute or subset may remain inherently indeterminate.[24] An *Indeterm-HyperSoft set* further generalizes this idea to multiple attributes, mapping parameter tuples to subsets and permitting intrinsic indeterminacy at any tuple level.[25] These constructs have been applied to model phenomena that cannot be definitively classified as either true or false.

1.2 Our Contribution

Although Indeterm-Soft sets and Indeterm-HyperSoft sets play a crucial role, their bipolar counterparts remain largely unexplored. To bridge this gap, in this paper we introduce formal definitions of bipolar Indeterm-Soft sets and bipolar Indeterm-HyperSoft sets, and we investigate their algebraic structures. We establish fundamental properties of these new models and demonstrate how they seamlessly integrate bipolarity and indeterminacy within the soft set paradigm.

1.3 Structure of This Research Paper

This section outlines the structure of the present paper. Section 2 provides an overview of foundational concepts, including soft sets, hypersoft sets, Indeterm-Soft sets, and bipolar soft sets. Section 3 presents the main results of this paper, introducing and analyzing the newly proposed bipolar Indeterm-Soft sets and bipolar Indeterm-HyperSoft sets. Finally, Section 4 offers concluding remarks and discusses possible directions for future research.

2 Preliminaries

In this section, we recall the basic concepts and notation that will be used throughout the paper. Unless stated otherwise, all sets considered are finite and contain no duplicate elements.

2.1 Soft Sets

A *soft set* provides a parameterized collection of subsets of a universe of discourse.

Definition 2.1 (Soft Set[8])**.** Let U be a (finite) universal set and E a nonempty set of parameters. A *soft set* over U is a pair (F, E), where

$$F : E \longrightarrow \mathrm{POW}(U),$$

so that for each $e \in E$, the set $F(e) \subseteq U$ represents the collection of elements that approximately satisfy the parameter e.

Example 2.2 (Soft Set for Student Course Preferences)**.** Let

$$U = \{\mathrm{Haruka,\ Daiki,\ Yuki,\ Kenta}\}$$

be the set of students, and let

$$E = \{\text{Likes Math, Likes Literature, Likes Sports}\}$$

be the set of preference parameters. Define the mapping

$$F : E \longrightarrow \mathrm{POW}(U)$$

by

$$F(\text{Likes Math}) = \{\mathrm{Haruka,\ Daiki}\},$$

$$F(\text{Likes Literature}) = \{\mathrm{Daiki,\ Yuki,\ Kenta}\},$$

$$F(\text{Likes Sports}) = \{\mathrm{Haruka,\ Kenta}\}.$$

Then the pair

$$(F, E)$$

is a soft set over U, where each parameter $e \in E$ is associated with the subset $F(e) \subseteq U$ of students who exhibit that preference.

A bipolar soft set refines the soft set idea by distinguishing positive and negative information.

Definition 2.3 (Bipolar Soft Set[23])**.** Let U be a (finite) universe, E a (finite) set of parameters, and choose a (finite) subset $A \subseteq E$. Define its complement $\neg A = E \setminus A$. A *bipolar soft set* over U is a triple

$$(F,\, G,\, A),$$

where

- $F : A \longrightarrow \mathrm{POW}(U)$ is the *positive membership map*,
- $G : \neg A \longrightarrow \mathrm{POW}(U)$ is the *negative membership map*,
- for every $e \in A$, the *consistency* condition

  $$F(e) \;\cap\; G(\neg e) \;=\; \varnothing$$

  holds.

Equivalently, one can write

$$(F, G, A) \;=\; \bigl\{\, \bigl(e,\, F(e),\, G(\neg e)\bigr) \mid e \in A,\; F(e) \cap G(\neg e) = \varnothing \bigr\}.$$

Example 2.4 (Bipolar Soft Set in Staff Recruitment)**.** Let

$$U = \{\mathrm{Alice}, \mathrm{Bob}, \mathrm{Charlie}, \mathrm{Diana}\}, \quad E = \{\text{Leadership},\ \text{Teamwork},\ \text{Inexperience},\ \text{Poor Communication}\}.$$

We select the positive parameter set

$$A = \{\text{Leadership},\ \text{Teamwork}\}, \qquad \neg A = E \setminus A = \{\text{Inexperience},\ \text{Poor Communication}\}.$$

Define the positive and negative membership mappings

$$F(\text{Leadership}) = \{\mathrm{Alice},\ \mathrm{Bob}\}, \quad F(\text{Teamwork}) = \{\mathrm{Bob},\ \mathrm{Charlie},\ \mathrm{Diana}\},$$

$$G(\text{Inexperience}) = \{\mathrm{Charlie}\}, \quad G(\text{Poor Communication}) = \{\mathrm{Diana}\}.$$

Then the triple

$$(F,\, G,\, A) = \bigl\{(\text{Leadership}, \{\,\mathrm{Alice}, \mathrm{Bob}\}, \{\,\mathrm{Charlie}\}),\ (\text{Teamwork}, \{\,\mathrm{Bob}, \mathrm{Charlie}, \mathrm{Diana}\}, \{\,\mathrm{Diana}\})\bigr\}$$

is a bipolar soft set over U. One checks immediately that for each $e \in A$,

$$F(e) \;\cap\; G(\neg e) \;=\; \varnothing,$$

so the consistency condition is satisfied.

2.2 Hypersoft Sets and Bipolar Hypersoft Sets

As extensions of the above concepts, Hypersoft Sets[26] and Bipolar Hypersoft Sets[27] are known. Their definitions are provided below.

Definition 2.5 (Hypersoft Set[17])**.** Let U be a (finite) universal set and let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be nonempty attribute-value domains. Set

$$\mathrm{ATSET} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

A *hypersoft set* over U is a pair (G, ATSET) where

$$G : \mathrm{ATSET} \longrightarrow \mathcal{P}(U).$$

Equivalently,

$$(G, \mathrm{ATSET}) = \bigl\{\, (\gamma,\, G(\gamma)) \mid \gamma \in \mathrm{ATSET} \bigr\}.$$

For each tuple $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathrm{ATSET}$, the set $G(\gamma) \subseteq U$ collects all elements of U that match the attribute combination $\gamma_1, \ldots, \gamma_m$.

Example 2.6 (Hypersoft Set in a Recruitment Scenario)**.** Let $U = \{\text{Ayako}, \text{Hiroko}, \text{Shinya}, \text{Ichiro}\}$ be a set of job applicants. Consider two attribute domains:

$$\mathcal{A}_1 = \{\text{Bachelor, Master, PhD}\}, \quad \mathcal{A}_2 = \{\text{Junior, Mid, Senior}\}.$$

Form the Cartesian product $\text{ATSET} = \mathcal{A}_1 \times \mathcal{A}_2$. Define

$$G : \text{ATSET} \longrightarrow \text{POW}(U)$$

by the following table:

(γ_1, γ_2)	$G(\gamma_1, \gamma_2)$
(Bachelor, Junior)	{Hiroko, Shinya}
(Bachelor, Mid)	{Ayako}
(Bachelor, Senior)	{Ichiro}
(Master, Junior)	{Shinya}
(Master, Mid)	{Ayako, Hiroko}
(Master, Senior)	{Ichiro}
(PhD, Junior)	{Hiroko}
(PhD, Mid)	{Shinya, Ichiro}
(PhD, Senior)	{Ayako}

Then (G, ATSET) is a hypersoft set over U, assigning each education–experience combination the subset of applicants matching it.

Definition 2.7 (Bipolar Hypersoft Set)**.** [27] A (finite) *Bipolar Hypersoft Set (BHS-Set)* is a triple (F, G, A) over a universe of discourse U, where:

- $F : A \to \text{POW}(U)$ and $G : \neg A \to \text{POW}(U)$, with $\text{POW}(U)$ denoting the power set of U.
- The mappings satisfy the *consistency constraint*:

$$F(\alpha) \cap G(\neg\alpha) = \emptyset, \quad \forall \alpha \in A.$$

- $A = A_1 \times A_2 \times \cdots \times A_n$, where $A_i \subseteq E_i$ and $E = E_1 \times E_2 \times \cdots \times E_n$.
- $\neg A = \neg A_1 \times \neg A_2 \times \cdots \times \neg A_n$, where $\neg A_i = E_i \setminus A_i$.

The (finite) Bipolar Hypersoft Set (BHS-Set) (F, G, A) is represented as:

$$(F, G, A) = \{(\alpha, F(\alpha), G(\neg\alpha)) \mid \alpha \in A \text{ and } F(\alpha) \cap G(\neg\alpha) = \emptyset\}.$$

Example 2.8 (Bipolar Hypersoft Set in Customer Feedback)**.** Let $U = \{\text{ProductA}, \text{ProductB}, \text{ProductC}\}$ be items, and parameters

$$E_1 = \{\text{QualityHigh}, \text{QualityLow}\}, \quad E_2 = \{\text{PriceCheap}, \text{PriceExpensive}\}.$$

Set $A = E_1 \times \{\text{PriceCheap}\}$, so $\neg A = E_1 \times \{\text{PriceExpensive}\}$. Define

$$F : A \to \text{POW}(U), \quad G : \neg A \to \text{POW}(U)$$

by

$$\begin{aligned} F(\text{QualityHigh}, \text{PriceCheap}) &= \{\text{ProductA}, \text{ProductB}\}, \\ F(\text{QualityLow}, \text{PriceCheap}) &= \{\text{ProductC}\}, \\ G(\text{QualityHigh}, \text{PriceExpensive}) &= \{\text{ProductC}\}, \\ G(\text{QualityLow}, \text{PriceExpensive}) &= \{\text{ProductA}, \text{ProductB}\}. \end{aligned}$$

Since $F(\alpha) \cap G(\neg\alpha) = \emptyset$ for each $\alpha \in A$, the triple (F, G, A) is a bipolar hypersoft set over U. It captures positive (cheapandhighquality) and negative (expensiveand/lowquality) customer evaluations of the products.

2.3 Indeterm-Soft Sets

An Indeterm-Soft Set assigns each attribute value to a subset of a reference set while allowing inherent ambiguity—either in the attribute domain, in the power-set codomain, or in specific images.[28] Its multi-attribute generalization, the Indeterm-HyperSoft Set, maps tuples of attribute values to subsets with possible indeterminacy at any tuple level.[25]

Definition 2.9 (Indeterm-Soft Set[29])**.** Let U be a universe, $H \subseteq U$ a nonempty subset, and $\mathcal{P}(H)$ its powerset. For an attribute a with value set A, a function

$$F : A \longrightarrow \mathcal{P}(H)$$

is called an *Indeterm-Soft Set* if at least one of the following conditions holds:

(i) The domain A contains indeterminate elements.

(ii) The codomain $\mathcal{P}(H)$ itself exhibits indeterminacy.

(iii) There exists $v \in A$ such that $F(v)$ cannot be uniquely determined.

(iv) Any combination of the above.

Equivalently, one may view

$$F : A \longrightarrow H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ denotes an algebraic structure closed under indeterminate extensions of the usual set operations.

Example 2.10 (Indeterm-Soft Set in Restaurant Cuisine Classification)**.** Let $U = \{\mathrm{R1}, \mathrm{R2}, \mathrm{R3}, \mathrm{R4}\}$ be four restaurants:

$$\mathrm{R1} = \text{Pasta Palace}, \quad \mathrm{R2} = \text{Sushi World}, \quad \mathrm{R3} = \text{Fusion Delight}, \quad \mathrm{R4} = \text{Spice Garden}.$$

Set $H = U$. Let the attribute $a =$ "cuisine type" take values

$$A = \{\text{Italian, Japanese, Fusion}\}.$$

Define

$$F : A \longrightarrow \mathrm{POW}(H)$$

by

$$F(\text{Italian}) = \{\text{R1, R3}\},$$
$$F(\text{Japanese}) = \{\text{R2, R3}\},$$
$$F(\text{Fusion}) = \{\text{R3}\}.$$

Here restaurant R3 serves both Italian and Japanese fusion, so its membership in F(Italian) and F(Japanese) is *indeterminate*. Hence F is an *Indeterm-Soft Set* because for $v =$ Italian (and similarly for Japanese), $F(v)$ contains an element R3 whose classification is unclear.

Definition 2.11 (Indeterm-HyperSoft Set)**.** [29] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $POW(H)$ the powerset of H. Let $a_1, a_2, \ldots, a_n$ $(n \geq 1)$ be n distinct attributes, with attribute values $A_1, A_2, \ldots, A_n$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$. The pair $(F, A_1 \times A_2 \times \cdots \times A_n)$, where

$$F : A_1 \times A_2 \times \cdots \times A_n \to POW(H),$$

is called an *Indeterm-HyperSoft Set* if:

1. Any A_i or $POW(H)$ exhibits indeterminacy.
2. For $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$, $F(a_1, a_2, \ldots, a_n)$ is indeterminate.

Example 2.12 (Indeterm-HyperSoft Set in Multi-Attribute Restaurant Evaluation)**.** A food critic evaluates four restaurants on two attributes—cuisine type and price level—where the attribute categories intentionally overlap to introduce indeterminacy.

Restaurants and Data:

- R1: *Pasta Palace*—Italian cuisine, average entrée price $18
- R2: *Sushi World*—Japanese cuisine, average entrée price $28
- R3: *Fusion Delight*—offers both Italian and Japanese dishes, average entrée price $26
- R4: *Spice Garden*—Indian/Thai fusion (irrelevant to these attributes), average entrée price $16

Set

$$U = \{\mathrm{R1}, \mathrm{R2}, \mathrm{R3}, \mathrm{R4}\}, \quad H = U.$$

Attribute Definitions:

$$a_1 = \text{cuisine type}, \quad A_1 = \{\text{Italian, Japanese}\},$$
$$a_2 = \text{price level}, \quad A_2 = \{\text{Cheap, Expensive}\},$$

where

$$\text{Cheap : price} \le 20, \quad \text{Expensive : price} > 20.$$

Note that R1 and R4 are in "Cheap" (prices $18, $16) and R2, R3 in "Expensive" ($28, $26).

Mapping $F : A_1 \times A_2 \to \mathcal{P}OW(H)$**:**

(a_1, a_2)	$F(a_1, a_2)$
(Italian, Cheap)	$\{\mathrm{R1}\}$
(Italian, Expensive)	$\{\mathrm{R3}\}$
(Japanese, Cheap)	$\{\mathrm{R4}\}$
(Japanese, Expensive)	$\{\mathrm{R2, R3}\}$

Here:

- R1 clearly falls under (Italian, Cheap).
- R4 is assigned to (Japanese, Cheap) even though its cuisine is neither strictly Japanese nor Italian—this models a placeholder for out-of-scope items.
- R2 falls under (Japanese, Expensive).
- R3 appears in both (Italian, Expensive) and (Japanese, Expensive), because it serves both cuisines at a high price point.

Indeterminacy Analysis:

- The overlap in A_1 for R3—being both Italian and Japanese—makes its membership in the Expensive category *indeterminate*.
- A crisp set could only place R3 in one cuisine group; here F preserves the genuine ambiguity by including R3 in both relevant cells.
- Similarly, R4's placement under (Japanese, Cheap) despite its fusion background highlights the flexibility of an Indeterm-HyperSoft Set to handle "unknown" or "other" cases in a unified framework.

Since F maps some attribute-tuples to sets containing an element (R3 or R4) whose classification cannot be decided uniquely, $(F,\ A_1 \times A_2)$ satisfies the definition of an *Indeterm-HyperSoft Set*. This example demonstrates how overlapping attribute criteria and multi-attribute evaluations introduce real indeterminacy in practical decision making.

3 Main Results in This Paper

The results derived in this paper are presented below.

3.1 Bipolar Indeterm-Soft Set

A Bipolar IndetermSoft Set is a triple of positive and negative mappings over parameters, featuring inherent indeterminacy and disjoint images. The definition of a Bipolar Indeterm-Soft Set is provided below.

Definition 3.1 (Bipolar Indeterm-Soft Set)**.** Let U be a universe, $H \subseteq U$, E a set of parameters, and $A \subseteq E$. A *Bipolar Indeterm-Soft Set* over H is a triple

$$(F, G, A),$$

where

- $F : A \to \mathrm{POW}(H)$ and $G : E \setminus A \to \mathrm{POW}(H)$ are maps,
- at least one of A, $\mathrm{POW}(H)$, or some $F(a)$ or $G(b)$ is indeterminate,
- $F(a) \cap G(\neg a) = \emptyset$ for all $a \in A$.

Example 3.2 (University Admission with Bipolar Indeterm-Soft Set)**.** An admissions committee evaluates three applicants using three criteria:

- **High GPA (Academic)**: GPA $\geq$ 3.7
- **Extracurricular Involvement**: at least two leadership roles
- **Strong Recommendation**: letter score $\geq$ 4/5

Set

$$U = \{\text{Ayako}, \text{Hiroko}, \text{Shinya}\}, \quad H = U,$$

and parameter sets

$$E = \{\text{HighGPA, Extracurricular, Recommendation}\}, \quad A = \{\text{HighGPA, Extracurricular}\}.$$

Based on records:

$$\begin{aligned} &\text{GPA: Ayako} = 3.8,\ \text{Hiroko} = 3.9,\ \text{Shinya} = 3.2, \\ &\text{Clubs: Ayako} = 1,\ \text{Hiroko} = 5,\ \text{Shinya} = 3, \\ &\text{Rec. Score: Ayako} = 5,\ \text{Hiroko} = 3,\ \text{Shinya} = 2. \end{aligned}$$

We define

$$F(\text{HighGPA}) = \{\text{Ayako, Hiroko}\}, \quad F(\text{Extracurricular}) = \{\text{Hiroko, Shinya}\},$$

$$G(\text{Recommendation}) = \{\text{Ayako}\}.$$

Applicant	GPA	# Clubs	Rec. Score
Ayako	3.8 (HighGPA)	1 (none)	5 (strong)
Hiroko	3.9 (HighGPA)	5 (lead roles)	3 (moderate)
Shinya	3.2 (below cutoff)	3 (lead roles)	2 (weak)

Because Hiroko appears in both $F(\text{HighGPA})$ and $F(\text{Extracurricular})$, it is *indeterminate* which primary strength best represents him. Hence F is an *Indeterm-Soft Set* on A. Together with the negative map G on the leftover parameter, the triple

$$(F, G, A)$$

forms a *Bipolar Indeterm-Soft Set*, capturing both positive (academic and extracurricular) and negative (recommendation) aspects of the admission decision.

Example 3.3 (Employee Performance Evaluation with a Bipolar Indeterm-Soft Set)**.** Let

$$U = \{\text{Alice, Bob, Carol, Dave}\}, \quad H = U,$$

and let the set of evaluation criteria be

$$E = \{\text{Timeliness, Innovation, Conflict}\}.$$

Choose the positive-aspect parameters

$$A = \{\text{Timeliness, Innovation}\}, \qquad E \setminus A = \{\text{Conflict}\}.$$

Suppose the HR records show:

Employee	Deadlines Met	Ideas Proposed	Conflict Incidents
Alice	Yes	No	1 (minor)
Bob	Yes	Yes	0
Carol	No	Yes	2 (moderate)
Dave	No	No	0

Define the positive map F and the negative map G by

$$F(\text{Timeliness}) = \{\text{Alice, Bob}\}, \quad F(\text{Innovation}) = \{\text{Bob, Carol}\},$$

$$G(\text{Conflict}) = \{\text{Carol}\}.$$

Because Bob appears in both $F(\text{Timeliness})$ and $F(\text{Innovation})$, his principal strength is *indeterminate*. Hence F constitutes an *Indeterm-Soft Set* on A. Together with the negative map G on the remaining parameter, the triple

$$(F, G, A)$$

forms a *Bipolar Indeterm-Soft Set*, capturing both positive aspects (timeliness and innovation) and the negative aspect (conflict) of the employee evaluation.

Example 3.4 (Customer Satisfaction Survey with "MaaMaa" Ambiguity)**.** A small café asked four regulars to rate their experience as "Manzoku" (Satisfied), "MaaMaa" (So-so), or "Fuman" (Dissatisfied). The responses were:

Customer	Rating
Taro	Manzoku
Jiro	MaaMaa
Saburo	Manzoku and MaaMaa
Shiro	Fuman

Here the category "MaaMaa" is inherently vague, and Saburo's response reflects that ambiguity.

Let

$$U = \{\text{Taro, Jiro, Saburo, Shiro}\}, \quad H = U,$$

$$E = \{\text{Manzoku, MaaMaa, Fuman}\}, \quad A = \{\text{Manzoku, MaaMaa}\}, \quad E \setminus A = \{\text{Fuman}\}.$$

Define the positive and negative maps by

$$F(\text{Manzoku}) = \{\text{Taro, Saburo}\}, \quad F(\text{MaaMaa}) = \{\text{Jiro, Saburo}\},$$

$$G(\text{Fuman}) = \{\text{Shiro}\}.$$

Since Saburo's feedback appears in both F(Manzoku) and F(MaaMaa), it is *indeterminate* which category best fits. Moreover, the parameter "MaaMaa" itself is ambiguous, so the map F exhibits genuine indeterminacy. One checks that

$$F(a) \cap G(\neg a) = \varnothing \quad (\forall\, a \in A),$$

hence

$$(F,\, G,\, A)$$

constitutes a *Bipolar Indeterm-Soft Set*, capturing both positive ("Manzoku"/"MaaMaa") and negative ("Fuman") customer responses under linguistic uncertainty.

Theorem 3.5. *Every Indeterm-Soft Set $F : A \to \mathrm{POW}(H)$ can be viewed as a Bipolar Indeterm-Soft Set with an empty negative part, and every Bipolar Soft Set (F, G, A) is a special (crisp) case of a Bipolar Indeterm-Soft Set.*

Proof. We verify each direction by checking the defining conditions of a Bipolar Indeterm-Soft Set.

(1) From Indeterm-Soft Set to Bipolar Indeterm-Soft Set.

Let $F : A \to \mathrm{POW}(H)$ be an Indeterm-Soft Set. Define

$$E \;=\; A, \qquad G : E \setminus A \;=\; \varnothing \;\longrightarrow\; \mathrm{POW}(H)$$

to be the unique map from the empty set. We check the three requirements:

1. *Positive part:* F is the given map $A \to \mathrm{POW}(H)$.
2. *Negative part:* G is a (vacuous) map $\varnothing \to \mathrm{POW}(H)$.
3. *Indeterminacy condition:* Since F is an Indeterm-Soft Set, by hypothesis at least one of the following holds:
   - A has indeterminacy,
   - $\mathrm{POW}(H)$ has indeterminacy,
   - there exists $a \in A$ such that $F(a)$ is indeterminate.

   Thus the Bipolar Indeterm-Soft requirement "at least one of A, $\mathrm{POW}(H)$, or some $F(a)$, $G(b)$ is indeterminate" is satisfied.
4. *Consistency condition:* For every $a \in A$, we require

   $$F(a) \cap G(\neg a) \;=\; F(a) \cap \varnothing \;=\; \varnothing.$$

   Hence the disjointness holds trivially.

Therefore (F, G, A) is a Bipolar Indeterm-Soft Set with empty negative part.

(2) From Bipolar Soft Set to Bipolar Indeterm-Soft Set.

Let (F, G, A) be a Bipolar Soft Set over H. By definition:

- $F : A \to \mathrm{POW}(H)$ and $G : E \setminus A \to \mathrm{POW}(H)$ are maps.
- $F(a) \cap G(\neg a) = \varnothing$ for all $a \in A$.

We regard "no indeterminacy" as the special case in which none of A, $\mathrm{POW}(H)$, $F(a)$, or $G(b)$ exhibits indeterminacy. The Bipolar Indeterm-Soft Set definition does not exclude this degenerate case. Hence:

1. *Positive and negative parts:* the same maps F and G.
2. *(Degenerate) Indeterminacy condition:* zero indeterminate elements is permissible, so the "at least one" clause can be interpreted to allow none.
3. *Consistency condition:* $F(a) \cap G(\neg a) = \varnothing$ holds by the Bipolar Soft Set assumption.

Thus (F, G, A) qualifies as a Bipolar Indeterm-Soft Set with no indeterminacy. □

Let (F_1, G_1, A) and (F_2, G_2, A) be two Bipolar Indeterm-Soft Sets on the same (U, H, E).

Definition 3.6 (Union and Intersection)**.** Define their *union* and *intersection* by

$$(F_1, G_1, A) \vee (F_2, G_2, A) = \big(F_1 \cup F_2,\ G_1 \cap G_2,\ A\big),$$
$$(F_1, G_1, A) \wedge (F_2, G_2, A) = \big(F_1 \cap F_2,\ G_1 \cup G_2,\ A\big),$$

where $(F_1 \cup F_2)(a) = F_1(a) \cup F_2(a)$ and $(G_1 \cap G_2)(\neg a) = G_1(\neg a) \cap G_2(\neg a)$, etc.

Theorem 3.7 (Closure under Union)**.** *The union $(F_1 \cup F_2,\ G_1 \cap G_2,\ A)$ is again a Bipolar Indeterm-Soft Set.*

Proof. We must check the consistency and indeterminacy conditions. For each $a \in A$,

$$(F_1 \cup F_2)(a) \ \cap\ (G_1 \cap G_2)(\neg a) = \big(F_1(a) \cup F_2(a)\big) \cap \big(G_1(\neg a) \cap G_2(\neg a)\big).$$

Since $F_i(a) \cap G_i(\neg a) = \varnothing$ for $i = 1, 2$, each of $F_1(a) \cap G_1(\neg a)$ and $F_2(a) \cap G_2(\neg a)$ is empty. Hence $(F_1 \cup F_2)(a) \cap (G_1 \cap G_2)(\neg a) = \varnothing$. Moreover, if either (F_1, G_1, A) or (F_2, G_2, A) carries indeterminacy, then their union does as well. Thus the union is a valid Bipolar Indeterm-Soft Set. □

Theorem 3.8 (Closure under Intersection)**.** *The intersection $(F_1 \cap F_2,\ G_1 \cup G_2,\ A)$ is again a Bipolar Indeterm-Soft Set.*

Proof. Analogous to the union case: for each $a \in A$,

$$(F_1 \cap F_2)(a) \ \cap\ (G_1 \cup G_2)(\neg a) = \big(F_1(a) \cap F_2(a)\big) \cap \big(G_1(\neg a) \cup G_2(\neg a)\big),$$

which is contained in either $F_1(a) \cap G_1(\neg a)$ or $F_2(a) \cap G_2(\neg a)$, both of which are empty. Indeterminacy also persists under intersection. □

Theorem 3.9 (Idempotence)**.** *For any Bipolar Indeterm-Soft Set (F, G, A),*

$$(F, G, A) \vee (F, G, A) = (F, G, A), \qquad (F, G, A) \wedge (F, G, A) = (F, G, A).$$

Proof. Recall that

$$(F, G, A) \vee (F, G, A) = \big(F \cup F,\ G \cap G,\ A\big), \quad (F, G, A) \wedge (F, G, A) = \big(F \cap F,\ G \cup G,\ A\big).$$

We verify component-wise:

- For every $a \in A$,

  $$(F \cup F)(a) = F(a) \cup F(a) = F(a) \quad \text{(by idempotence of set-union)}.$$

- For every $\neg a \in E \setminus A$,

  $$(G \cap G)(\neg a) = G(\neg a) \cap G(\neg a) = G(\neg a) \quad \text{(by idempotence of set-intersection)},$$

  and similarly $(G \cup G)(\neg a) = G(\neg a)$.

Hence the union and intersection with itself leave both F and G unchanged, so the entire triple remains (F, G, A). □

Theorem 3.10 (Commutativity). *For any two Bipolar Indeterm-Soft Sets* (F_1, G_1, A) *and* (F_2, G_2, A)*,*

$$(F_1, G_1, A) \vee (F_2, G_2, A) = (F_2, G_2, A) \vee (F_1, G_1, A),$$

$$(F_1, G_1, A) \wedge (F_2, G_2, A) = (F_2, G_2, A) \wedge (F_1, G_1, A).$$

Proof. We check the union case; the intersection case is analogous.

Union:

$$(F_1, G_1, A) \vee (F_2, G_2, A) = \big(F_1 \cup F_2,\ G_1 \cap G_2,\ A\big).$$

For each $a \in A$,

$$(F_1 \cup F_2)(a) = F_1(a) \cup F_2(a) = F_2(a) \cup F_1(a) = (F_2 \cup F_1)(a).$$

For each $\neg a \in E \setminus A$,

$$(G_1 \cap G_2)(\neg a) = G_1(\neg a) \cap G_2(\neg a) = G_2(\neg a) \cap G_1(\neg a) = (G_2 \cap G_1)(\neg a).$$

Therefore $(F_1 \cup F_2,\ G_1 \cap G_2,\ A) = (F_2 \cup F_1,\ G_2 \cap G_1,\ A)$, establishing commutativity of $\vee$.

Intersection: Replace $\cup$ by $\cap$ and vice versa, and repeat the same argument using commutativity of set-intersection and set-union. □

Theorem 3.11 (Absorption). *For any Bipolar Indeterm-Soft Sets* (F_1, G_1, A) *and* (F_2, G_2, A)*,*

$$(F_1, G_1, A) \vee \big((F_1, G_1, A) \wedge (F_2, G_2, A)\big) = (F_1, G_1, A),$$

$$(F_1, G_1, A) \wedge \big((F_1, G_1, A) \vee (F_2, G_2, A)\big) = (F_1, G_1, A).$$

Proof. We show the first absorption law; the second is dual.

By definition,

$$(F_1, G_1, A) \wedge (F_2, G_2, A) = \big(F_1 \cap F_2,\ G_1 \cup G_2,\ A\big).$$

Hence

$$(F_1, G_1, A) \vee \big((F_1, G_1, A) \wedge (F_2, G_2, A)\big) = \big(F_1 \cup (F_1 \cap F_2),\ G_1 \cap (G_1 \cup G_2),\ A\big).$$

Now for each $a \in A$,

$$F_1(a) \cup \big(F_1(a) \cap F_2(a)\big) = F_1(a),$$

and for each $\neg a \in E \setminus A$,

$$G_1(\neg a) \cap \big(G_1(\neg a) \cup G_2(\neg a)\big) = G_1(\neg a).$$

Thus the resulting triple is (F_1, G_1, A), proving the absorption law. □

3.2 Bipolar Indeterm-HyperSoft Set

A Bipolar IndetermHyperSoft Set extends to parameter tuples, assigning positive and negative subset-values with inherent indeterminacy and enforcing disjointness. The definition of a Bipolar Indeterm-HyperSoft Set is provided below.

Definition 3.12 (Bipolar Indeterm-HyperSoft Set)**.** Let U, H and $\mathcal{E} = E_1 \times \cdots \times E_n$ as above, and choose $A \subseteq \mathcal{E}$. A *Bipolar Indeterm-HyperSoft Set* is a triple

$$(F, G, A),$$

with

$$F : A \to \mathrm{POW}(H), \quad G : \mathcal{E} \setminus A \to \mathrm{POW}(H),$$

satisfying:

(i) at least one of the E_i, $\mathrm{POW}(H)$, or some value $F(\alpha)$ or $G(\beta)$ is indeterminate,

(ii) $\forall \alpha \in A,\ F(\alpha) \cap G(\neg\alpha) = \varnothing$.

Example 3.13 (Detailed Supplier Evaluation with Indeterminacy)**.** A manufacturing firm must choose among three suppliers based on two features: product quality and delivery speed. The raw data are:

Supplier	Measured Quality Score	Typical Lead Time (days)
S1	92 (excellent)	4 (fast)
S2	88 (good, but variable)	6 (borderline fast/slow)
S3	76 (fair)	8 (slow)

We introduce categorical attributes that deliberately overlap to model indeterminacy:

- **QualityHigh:** "Quality $\geq$ 85"
- **QualityLow:** "Quality $\leq$ 90"
- **Fast:** "Lead time $\leq$ 6 days"
- **Slow:** "Lead time $\geq$ 6 days"

Here "QualityHigh" and "QualityLow" overlap for scores in $[85, 90]$, and "Fast" and "Slow" overlap at exactly 6 days. Thus, supplier S2 (score = 88, time = 6) belongs to both categories in each attribute, introducing intentional indeterminacy.

Let

$$U = \{\mathrm{S1}, \mathrm{S2}, \mathrm{S3}\}, \quad H = U, \quad E_1 = \{\text{QualityHigh}, \text{QualityLow}\}, \quad E_2 = \{\text{Fast}, \text{Slow}\},$$

so $\mathcal{E} = E_1 \times E_2$ has four possible tuples. We choose the "positive" set

$$A = \{(\text{QualityHigh}, \text{Fast}),\ (\text{QualityLow}, \text{Slow})\},$$

and define the Bipolar Indeterm-HyperSoft mappings:

$$\begin{aligned} F(\text{QualityHigh}, \text{Fast}) &= \{\mathrm{S1},\ \mathrm{S2}\}, \\ F(\text{QualityLow}, \text{Slow}) &= \{\mathrm{S2},\ \mathrm{S3}\}, \\ G(\text{QualityHigh}, \text{Slow}) &= \{\mathrm{S3}\}, \\ G(\text{QualityLow}, \text{Fast}) &= \{\mathrm{S1}\}. \end{aligned}$$

Interpretation:

- $(\text{QualityHigh}, \text{Fast})$: S1 clearly meets both high quality and fast delivery, while S2's quality is borderline and its lead time exactly at the overlap point, so it is indeterminately classified here.
- $(\text{QualityLow}, \text{Slow})$: S3 consistently performs poorly and slowly; S2 again falls into this category due to its borderline metrics.
- $(\text{QualityHigh}, \text{Slow})$ and $(\text{QualityLow}, \text{Fast})$ are treated negatively, flagging S3 for slow high-quality deliveries and S1 for fast but borderline-quality deliveries.

Because supplier S2 appears in both $F(\text{High}, \text{Fast})$ and $F(\text{Low}, \text{Slow})$, the map F exhibits genuine indeterminacy. Together with the negative map G, the triple

$$(F,\, G,\, A)$$

satisfies the Bipolar Indeterm-HyperSoft Set conditions: it captures overlapping categories (indeterminacy) and a bipolar structure (positive vs. negative assignments).

Example 3.14 (Smartphone Purchase Decision Highlighting Indeterminacy)**.** A shopper compares three phones under two intentionally overlapping criteria:

- **BatteryLong:** capacity ≥ 3000 mAh
- **BatteryShort:** capacity ≤ 3500 mAh
- **CameraGood:** resolution ≥ 12 MP
- **CameraAverage:** resolution ≤ 16 MP

Specifications:

Model	Battery (mAh)	Camera (MP)
PhoneA	3200	64
PhoneB	3300	12
PhoneC	2800	8

Define

$$U = \{\text{PhoneA}, \text{PhoneB}, \text{PhoneC}\},$$
$$E_1 = \{\text{BatteryLong}, \text{BatteryShort}\},$$
$$E_2 = \{\text{CameraGood}, \text{CameraAverage}\},$$

and choose the "positive" combinations

$$A = \{(\text{BatteryLong}, \text{CameraGood}),\ (\text{BatteryShort}, \text{CameraAverage})\}.$$

Set

$$F(\text{BatteryLong, CameraGood}) = \{\text{PhoneA, PhoneB}\},$$
$$F(\text{BatteryShort, CameraAverage}) = \{\text{PhoneA, PhoneB, PhoneC}\},$$

and for the remaining pairs define

$$G(\text{BatteryLong, CameraAverage}) = \{\text{PhoneC}\},$$
$$G(\text{BatteryShort, CameraGood}) = \{\text{PhoneC}\}.$$

Why indeterminacy matters:

- **PhoneB** lies in both "long" and "short" battery categories (3300 mAh), and in both "good" and "average" camera categories (12 MP). A crisp model would force a binary classification, but here F retains its ambiguous presence in both positive groups, preserving uncertainty.

- For $(\text{BatteryLong}, \text{CameraGood})$, PhoneB appears alongside PhoneA, yet its borderline specifications prevent a definitive classification—its status is indeterminate.
- For $(\text{BatteryShort}, \text{CameraAverage})$, all three phones are included. However, PhoneA and PhoneB have distinct strengths. Recognizing this ambiguity is essential to avoid over-simplifying the selection process.
- The negative map G flags PhoneC as undesirable in mixed categories, while F preserves potentially positive but uncertain memberships for PhoneA and PhoneB.

Thus (F, G, A) forms a *Bipolar Indeterm-HyperSoft Set* that explicitly tracks overlapping, indeterminate cases alongside clear negative assignments—enabling nuanced, uncertainty-aware decision support.

Example 3.15 (Retail Customer Satisfaction Survey)**.** A store asked four customers to rate price and quality on a 1–5 scale. The recorded scores were:

Customer	Price Score	Quality Score
Ayumi	5	5
Daichi	4	4
Emi	4	3.5
Fumio	3	2

We introduce overlapping categories to capture borderline cases:

$$\text{PriceSat} : \text{score} \geq 4, \quad \text{PriceUnsat} : \text{score} \leq 3,$$
$$\text{QualSat} : \text{score} \geq 4, \quad \text{QualUnsat} : \text{score} \leq 3.$$

Thus a rating of 3.5 (Emi's quality) lies between "satisfied" and "unsatisfied," creating genuine indeterminacy.

Let

$$U = \{\text{Ayumi}, \text{Daichi}, \text{Emi}, \text{Fumio}\}, \quad H = U,$$
$$E_1 = \{\text{PriceSat}, \text{PriceUnsat}\}, \quad E_2 = \{\text{QualSat}, \text{QualUnsat}\}, \quad \mathcal{E} = E_1 \times E_2.$$

Choose the "positive" parameter-tuples

$$A = \{(\text{PriceSat}, \text{QualSat}),\ (\text{PriceSat}, \text{QualUnsat})\}.$$

Define

$$F(\text{PriceSat}, \text{QualSat}) = \{\text{Ayumi, Daichi, Emi}\},$$
$$F(\text{PriceSat}, \text{QualUnsat}) = \{\text{Emi}\},$$
$$G(\text{PriceUnsat}, \text{QualSat}) = \varnothing,$$
$$G(\text{PriceUnsat}, \text{QualUnsat}) = \{\text{Fumio}\}.$$

Since Emi's quality rating is ambiguous, she appears in both

$$F(\text{PriceSat}, \text{QualSat})$$

and

$$F(\text{PriceSat}, \text{QualUnsat})$$

, satisfying the indeterminacy condition. One checks that for each $\alpha \in A$,

$$F(\alpha) \cap G(\neg\alpha) = \varnothing,$$

so $(F,\ G,\ A)$ indeed forms a Bipolar Indeterm-HyperSoft Set, simultaneously representing positive feedback, negative feedback, and uncertain responses.

Theorem 3.16 (Generalization)**.** *(a) Every Indeterm-HyperSoft Set $(F, \mathcal{E})$ can be viewed as a Bipolar Indeterm-HyperSoft Set by choosing $A = \mathcal{E}$ and taking G to be the empty mapping on $\mathcal{E} \setminus A = \varnothing$.*

(b) *Every Bipolar HyperSoft Set (F, G, A) is automatically a Bipolar Indeterm-HyperSoft Set (possibly with no actual indeterminacy).*

Proof. **(a) From Indeterm-HyperSoft to Bipolar Indeterm-HyperSoft.**

- *Starting data:* An Indeterm-HyperSoft Set consists of a pair

$$(F,\ \mathcal{E}), \quad F : \mathcal{E} \ =\ E_1 \times \cdots \times E_n \ \longrightarrow\ \mathrm{POW}(H),$$

  where at least one attribute domain E_i, or at least one image $F(\alpha)$, is indeterminate.

- *Construction of the bipolar version:* Set

$$A \ =\ \mathcal{E}, \qquad G : \mathcal{E} \setminus A = \varnothing \ \longrightarrow\ \mathrm{POW}(H) \quad \text{(empty map)}.$$

  Thus we obtain the triple

$$\big(F, G, A\big).$$

- *Verification of conditions:*
  1. *Domain and codomain:* By construction F and G have the required domains A and $\mathcal{E} \setminus A$.
  2. *Consistency:* For each $\alpha \in A$,

$$F(\alpha)\ \cap\ G(\neg\alpha) = F(\alpha)\ \cap\ G(\text{nothing}) = F(\alpha)\ \cap\ \varnothing = \varnothing,$$

     so the consistency constraint holds vacuously.
  3. *Indeterminacy:* Since the original Indeterm-HyperSoft Set had at least one indeterminate component (either some E_i or some $F(\alpha)$), the same indeterminacy persists in (F, G, A).

  Hence (F, G, A) satisfies the definition of a Bipolar Indeterm-HyperSoft Set.

(b) From Bipolar HyperSoft to Bipolar Indeterm-HyperSoft.

- *Starting data:* A Bipolar HyperSoft Set is given by

$$(F, G, A), \quad F : A \to \mathrm{POW}(H), \quad G : \mathcal{E} \setminus A \to \mathrm{POW}(H),$$

  with the consistency condition $F(\alpha) \cap G(\neg\alpha) = \varnothing$ and no requirement of indeterminacy.

- *Verification:*
  1. *Domain and codomain:* The pair (F, G) already has the correct domains A and $\mathcal{E} \setminus A$.
  2. *Consistency:* By hypothesis of the Bipolar HyperSoft Set, $\forall\alpha \in A,\ F(\alpha) \cap G(\neg\alpha) = \varnothing$.
  3. *Indeterminacy clause:* The definition of a Bipolar Indeterm-HyperSoft Set only requires *at least one* of the parameter domains, codomain, or image-sets to be indeterminate. We may simply note that 'zero indeterminacy' is allowed, and so this condition is vacuously satisfied.

  Therefore, every Bipolar HyperSoft Set is also a Bipolar Indeterm-HyperSoft Set.

This completes the proof of both parts. □

Definition 3.17 (Union and Intersection)**.** Let $\mathrm{BIP}_i = (F_i, G_i, A)$ $(i = 1, 2)$ be two Bipolar Indeterm-HyperSoft Sets with the same A. Define

$$\mathrm{BIP}_1 \vee \mathrm{BIP}_2 = \big(F_1 \cup F_2,\ G_1 \cap G_2,\ A\big),$$
$$\mathrm{BIP}_1 \wedge \mathrm{BIP}_2 = \big(F_1 \cap F_2,\ G_1 \cup G_2,\ A\big),$$

where $(F_1 \cup F_2)(\alpha) = F_1(\alpha) \cup F_2(\alpha)$ and similarly for the other operations.

Theorem 3.18 (Closure under Union). *If* BIP_1 *and* BIP_2 *are Bipolar Indeterm-HyperSoft Sets, then* $\mathrm{BIP}_1 \vee \mathrm{BIP}_2$ *is also one.*

Proof. Let $\mathrm{BIP}_1 \vee \mathrm{BIP}_2 = (F, G, A)$. For each $\alpha \in A$,

$$F(\alpha) \;=\; F_1(\alpha) \cup F_2(\alpha), \quad G(\neg\alpha) \;=\; G_1(\neg\alpha) \cap G_2(\neg\alpha).$$

Consistency:

$$F(\alpha) \cap G(\neg\alpha) = \big(F_1(\alpha) \cup F_2(\alpha)\big) \cap \big(G_1(\neg\alpha) \cap G_2(\neg\alpha)\big) \subseteq \big(F_1(\alpha) \cap G_1(\neg\alpha)\big) \;\cup\; \big(F_2(\alpha) \cap G_2(\neg\alpha)\big) = \varnothing.$$

Indeterminacy: If either BIP_1 or BIP_2 had indeterminacy in some E_i, $\mathrm{POW}(H)$, or a value-set, then so does $\mathrm{BIP}_1 \vee \mathrm{BIP}_2$. Thus $\mathrm{BIP}_1 \vee \mathrm{BIP}_2$ satisfies the definition. □

Theorem 3.19 (Closure under Intersection). *If* BIP_1 *and* BIP_2 *are Bipolar Indeterm-HyperSoft Sets, then* $\mathrm{BIP}_1 \wedge \mathrm{BIP}_2$ *is also one.*

Proof. Analogous to union. For each $\alpha \in A$,

$$F(\alpha) = F_1(\alpha) \cap F_2(\alpha), \quad G(\neg\alpha) = G_1(\neg\alpha) \cup G_2(\neg\alpha).$$

Consistency:

$$F(\alpha) \cap G(\neg\alpha) \subseteq \big(F_1(\alpha) \cap G_1(\neg\alpha)\big) \;\cup\; \big(F_2(\alpha) \cap G_2(\neg\alpha)\big) = \varnothing.$$

Indeterminacy likewise persists under intersection. □

Theorem 3.20 (Idempotence, Commutativity, Associativity). *Let* $\mathrm{BIP}_i = (F_i, G_i, A)$ *be Bipolar Indeterm-HyperSoft Sets, and write* $\mathrm{BIP} = (F, G, A)$. *Then:*

$$\mathrm{BIP} \vee \mathrm{BIP} = \mathrm{BIP}, \quad \mathrm{BIP} \wedge \mathrm{BIP} = \mathrm{BIP},$$

$$\mathrm{BIP}_1 \vee \mathrm{BIP}_2 = \mathrm{BIP}_2 \vee \mathrm{BIP}_1, \quad \mathrm{BIP}_1 \wedge \mathrm{BIP}_2 = \mathrm{BIP}_2 \wedge \mathrm{BIP}_1,$$

$$(\mathrm{BIP}_1 \vee \mathrm{BIP}_2) \vee \mathrm{BIP}_3 = \mathrm{BIP}_1 \vee (\mathrm{BIP}_2 \vee \mathrm{BIP}_3), \quad (\mathrm{BIP}_1 \wedge \mathrm{BIP}_2) \wedge \mathrm{BIP}_3 = \mathrm{BIP}_1 \wedge (\mathrm{BIP}_2 \wedge \mathrm{BIP}_3).$$

Proof. We prove each claim by checking the defining component-maps F and G pointwise on their domains.

1. Idempotence.

$$\mathrm{BIP} \vee \mathrm{BIP} = \big(F \cup F,\; G \cap G,\; A\big).$$

For every $\alpha \in A$,

$$(F \cup F)(\alpha) = F(\alpha) \;\cup\; F(\alpha) = F(\alpha),$$

by the idempotence of union in $\mathrm{POW}(H)$. Similarly,

$$(G \cap G)(\neg\alpha) = G(\neg\alpha) \;\cap\; G(\neg\alpha) = G(\neg\alpha).$$

Hence $\mathrm{BIP} \vee \mathrm{BIP} = (F, G, A) = \mathrm{BIP}$. The proof for $\mathrm{BIP} \wedge \mathrm{BIP}$ is identical, using $F \cap F = F$ and $G \cup G = G$.

2. Commutativity.

$$\mathrm{BIP}_1 \vee \mathrm{BIP}_2 = \big(F_1 \cup F_2,\; G_1 \cap G_2,\; A\big), \quad \mathrm{BIP}_2 \vee \mathrm{BIP}_1 = \big(F_2 \cup F_1,\; G_2 \cap G_1,\; A\big).$$

For each α:

$$F_1(\alpha) \cup F_2(\alpha) = F_2(\alpha) \cup F_1(\alpha),$$

and for each $\neg\alpha$:

$$G_1(\neg\alpha) \cap G_2(\neg\alpha) = G_2(\neg\alpha) \cap G_1(\neg\alpha).$$

Thus $\mathrm{BIP}_1 \vee \mathrm{BIP}_2 = \mathrm{BIP}_2 \vee \mathrm{BIP}_1$. The meet case is analogous with $\cap \leftrightarrow \cup$.

3. Associativity. First for union:

$$(\mathrm{BIP}_1 \vee \mathrm{BIP}_2) \vee \mathrm{BIP}_3 = \big((F_1 \cup F_2) \cup F_3,\ (G_1 \cap G_2) \cap G_3,\ A\big).$$

By associativity of union and intersection in $\mathrm{POW}(H)$,

$$(F_1 \cup F_2) \cup F_3 = F_1 \cup (F_2 \cup F_3), \quad (G_1 \cap G_2) \cap G_3 = G_1 \cap (G_2 \cap G_3).$$

Hence $(\mathrm{BIP}_1 \vee \mathrm{BIP}_2) \vee \mathrm{BIP}_3 = \mathrm{BIP}_1 \vee (\mathrm{BIP}_2 \vee \mathrm{BIP}_3)$. The intersection associativity is shown similarly.

□

Theorem 3.21 (Distributivity)**.** *For any* $\mathrm{BIP}_1, \mathrm{BIP}_2, \mathrm{BIP}_3$,

$$\mathrm{BIP}_1 \wedge (\mathrm{BIP}_2 \vee \mathrm{BIP}_3) = (\mathrm{BIP}_1 \wedge \mathrm{BIP}_2) \vee (\mathrm{BIP}_1 \wedge \mathrm{BIP}_3),$$

$$\mathrm{BIP}_1 \vee (\mathrm{BIP}_2 \wedge \mathrm{BIP}_3) = (\mathrm{BIP}_1 \vee \mathrm{BIP}_2) \wedge (\mathrm{BIP}_1 \vee \mathrm{BIP}_3).$$

Proof. We verify the first identity; the second is dual.

Left-hand side:

$$\mathrm{BIP}_1 \wedge (\mathrm{BIP}_2 \vee \mathrm{BIP}_3) = \big(F_1 \cap (F_2 \cup F_3),\ G_1 \cup (G_2 \cap G_3),\ A\big).$$

Right-hand side:

$$(\mathrm{BIP}_1 \wedge \mathrm{BIP}_2) \vee (\mathrm{BIP}_1 \wedge \mathrm{BIP}_3) = \big((F_1 \cap F_2) \cup (F_1 \cap F_3),\ (G_1 \cup G_2) \cap (G_1 \cup G_3),\ A\big).$$

By distributivity in $\mathrm{POW}(H)$,

$$F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3), \quad G_1 \cup (G_2 \cap G_3) = (G_1 \cup G_2) \cap (G_1 \cup G_3).$$

Therefore the two sides coincide pointwise, proving distributivity. □

4 Conclusion and Future Work

This paper introduced two new frameworks—*Bipolar Indeterm-Soft Sets* and *Bipolar Indeterm-HyperSoft Sets*—and analysed their mathematical foundations in depth. We also demonstrated their practical relevance through several concrete, real-life examples.

Looking ahead, we encourage further investigation into these models from both theoretical and applied perspectives. Promising directions include (i) developing and evaluating construction algorithms, (ii) designing computational experiments, and (iii) testing the frameworks in real-world decision-making and knowledge-representation tasks. Such efforts will strengthen the practical utility of Bipolar Indeterm-Soft Sets and Bipolar Indeterm-HyperSoft Sets and broaden their impact across diverse application domains.

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Data Availability

This manuscript is entirely theoretical and did not involve the generation or analysis of empirical data. We invite future researchers to undertake experimental or observational studies to test and extend the concepts presented here.

Ethical Approval

Because this work is purely conceptual and does not involve human participants or animal subjects, ethical approval was not required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this manuscript.

Disclaimer

The ideas and models introduced in this paper remain theoretical and have not yet been validated in real-world settings. While we have aimed for accuracy and proper citation, inadvertent errors or omissions may occur, and readers should verify sources independently. The opinions and conclusions expressed herein are those of the authors alone and do not necessarily represent the views of their affiliated institutions.

References

[1] YS Yun. Parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set. *Journal of Algebra & Applied Mathematics*, 21(2), 2023.

[2] LC Platil and GC Petalcorin. Fuzzy γ-semimodules over γ-semirings. *Journal of Analysis and Applications*, 15(1):71–83, 2017.

[3] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[4] Krittika Linesawat and Somsak Lekkoksung. A study on multi-intuitionistic fuzzy sets and their application in ordered semigroups. *International Journal of Analysis and Applications*, 23:63–63, 2025.

[5] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[6] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[7] Madeleine Al Tahan, Saba Al-Kaseasbeh, and Bijan Davvaz. Neutrosophic quadruple hv-modules and their fundamental module. *Neutrosophic Sets and Systems*, 72:304–325, 2024.

[8] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[9] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[10] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[11] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.

[12] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[13] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[14] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[15] Xingsi Xue, Himanshu Dhumras, Garima Thakur, Rakesh Kumar Bajaj, and Varun Shukla. Schweizer-sklar t-norm operators for picture fuzzy hypersoft sets: Advancing suistainable technology in social healthy environments. *Computers, Materials & Continua*, 84(1), 2025.

[16] Atiqe Ur Rahman, Muhammad Saeed, HAEW Khalifa, Walaa Abdullah Afifi, et al. Decision making algorithmic techniques based on aggregation operations and similarity measures of possibility intuitionistic fuzzy hypersoft sets. *AIMS Math*, 7(3):3866–3895, 2022.

[17] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[18] Pankaj Bhambri. Enhancing cybersecurity with superhypersoft computing approaches. In *Modern SuperHyperSoft Computing Trends in Science and Technology*, pages 127–148. IGI Global Scientific Publishing, 2025.

[19] Baravan A. Asaad, Sagvan Y. Musa, and Zanyar A. Ameen. Fuzzy bipolar hypersoft sets: A novel approach for decision-making applications. *Mathematical and Computational Applications*, 2024.

[20] Sumbal Ali, Asad Ali, Ahmad Bin Azim, Ahmad Aloqaily, and Nabil Mlaiki. Utilizing aggregation operators based on q-rung orthopair neutrosophic soft sets and their applications in multi-attributes decision making problems. *Heliyon*, 2024.

[21] Akbar Rezae, Karim Ghadimi, and Florentin Smarandache. Associated a nexus with a treesoft sets and vice versa. 2024.

[22] Holy-Heavy M Balami, Aliyu G Dzarma, and Mohammed A Mohammed. Weighted soft set and its application in parameterized decision making processes. *International Journal of Development Mathematics (IJDM)*, 2(1):131–144, 2025.

[23] Orhan Dalkılıç. A decision-making approach to reduce the margin of error of decision makers for bipolar soft set theory. *International Journal of Systems Science*, 53:265 – 274, 2021.

[24] Li Li. Indetermsoft set for talent training quality assessment in university engineering management under the background of" dual carbon". *Neutrosophic Sets and Systems*, 83:148–158, 2025.

[25] Adebisi Sunday Adesina. Further operations (complement, intersection, union) for indetermsoft set, indetermhypersoft set, and treesoft set and their applications. *Journal of Basic & Applied Sciences*, 21:47–52, 2025.

[26] Muhammad Saqlain, Poom Kumam, and Wiyada Kumam. Multi-criteria decision-making method based on weighted and geometric aggregate operators of linguistic fuzzy-valued hypersoft set with application. *Journal of Fuzzy Extension and Applications*, 6(2):344–370, 2025.

[27] Sagvan Y. Musa. N-bipolar hypersoft sets: Enhancing decision-making algorithms. *PLOS ONE*, 19, 2024.

[28] Tao Shen and Chunmei Mao. Indetermsoft set for digital marketing effectiveness evaluation driven by big data based on consumer behavior. *Neutrosophic Sets and Systems*, 82:352–369, 2025.

[29] Florentin Smarandache. *New types of soft sets "hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set": an improved version*. Infinite Study, 2023.