# On Graded S-semiprime Submodules of Graded Modules Over Graded Commutative Rings

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**Abstract**

Let $G$ be a group with identity $e$. Let $\mathfrak{T}$ be a commutative $G$-graded ring with non-zero identity, $\mathfrak{W}$ be a graded $\mathfrak{T}$-module and $S \subseteq h(\mathfrak{T})$ a multiplicatively closed subset of $\mathfrak{T}$. In this article, we introduce and study the concept of graded $S$-semiprime submodules. A graded submodule $K$ of $\mathfrak{W}$ with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$ is said to be graded $S$-semiprime, if there exists a fixed $s_t \in S$ such that whenever $r_i^n m_j \in K$ for some $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $t, i, j \in G$, and $n \in \mathbb{N}$, then $s_t r_i m_j \in K$. Some characterizations and properties of graded $S$-semiprime submodules are given.

**Keywords:** Graded S-semiprime submodule; Graded S-semiprime ideal; Graded semiprime submodule

## 1 Introduction and Preliminaries

Throughout this paper, we work under the assumption that $\mathfrak{T}$ is a commutative $G$-graded ring with unity, and $\mathfrak{W}$ denotes a unitary graded $\mathfrak{T}$-module.

The study of graded semiprime submodules within the context of graded modules over commutative graded rings has been developed in several works, including.[1,6,9,13] In,[7] P. Ghiasvand and F. Farzalipour extended this concept to modules over noncommutative graded rings. Additionally, the notion of graded semiprime ideals was initially introduced by Lee and Varmazyar in[9] and was further investigated in.[5] More recently, K. Al-Zoubi and S. Alghueiri proposed the idea of graded $J_{gr}$-semiprime submodules in,[2] offering a generalization of previously established graded semiprime submodule structures.

In this work, we introduce a new generalization of this framework: the concept of graded $S$-semiprime submodules of graded modules over commutative graded rings. Our aim is to explore foundational properties of this class of submodules and investigate their behavior and structure.

We begin by recalling several fundamental concepts related to graded rings and graded modules that are instrumental throughout this paper. For a more thorough treatment, the reader is referred to.[8,10–12]

Let $G$ be a group with identity element $e$. A ring $\mathfrak{T}$ is said to be $G$-graded if there exists a direct sum decomposition $\mathfrak{T} = \bigoplus_{h \in G} \mathfrak{T}_h$, where each $\mathfrak{T}_h$ is an additive subgroup of $\mathfrak{T}$ satisfying $\mathfrak{T}_g \mathfrak{T}_h \subseteq \mathfrak{T}_{gh}$ for all $g, h \in G$. Nonzero elements of the components $\mathfrak{T}_h$ are referred to as *homogeneous elements of degree* $h$, and the set of all such elements is denoted by $h(\mathfrak{T}) = \bigcup_{h \in G} \mathfrak{T}_h$. Any $a \in \mathfrak{T}$ can be uniquely expressed as a finite sum $a = \sum_{h \in G} a_h$ with $a_h \in \mathfrak{T}_h$, known as the homogeneous components of $a$. The component $\mathfrak{T}_e$ forms a subring of $\mathfrak{T}$ and contains the identity 1 (see[11]).

An ideal $J$ of $\mathfrak{T}$ is called a *graded ideal* (briefly, gr-Id)if $J = \bigoplus_{h\in G}(J \cap \mathfrak{T}_h) := \bigoplus_{h\in G} J_h$ (see[11]).

Similarly, a left $\mathfrak{T}$-module $\mathfrak{W}$ is said to be a *graded module* (or $G$-graded $\mathfrak{T}$-module) if there exists a direct sum decomposition $\mathfrak{W} = \bigoplus_{h\in G} \mathfrak{W}_h$ such that $\mathfrak{T}_g\mathfrak{W}_h \subseteq \mathfrak{W}_{gh}$ for all $g, h \in G$. An element in $\bigcup_{h\in G} \mathfrak{W}_h = h(\mathfrak{W})$ is termed *homogeneous*. For each $h \in G$, the component $\mathfrak{W}_h$ is an $\mathfrak{T}_e$-module.

A submodule $N$ of $\mathfrak{W}$ is said to be a *graded submodule* (briefly, gr-Sub) if $N = \bigoplus_{h\in G}(N \cap \mathfrak{W}_h) := \bigoplus_{h\in G} N_h$, where $N_h$ denotes the $h$-component of $N$. In this case, the quotient module $\mathfrak{W}/N$ inherits a natural $G$-grading, where the $h$-component is given by $(\mathfrak{W}/N)_h := (\mathfrak{W}_h + N)/N$ for each $h \in G$ (see[11]).

It was shown in[3] that if $N$ is a graded submodule of $\mathfrak{W}$, then the annihilator $(N :_{\mathfrak{T}} \mathfrak{W}) = \{r \in \mathfrak{T} : r\mathfrak{W} \subseteq N\}$ is a graded ideal of $\mathfrak{T}$. For any homogeneous element $r_g \in h(\mathfrak{T})$, we define the graded submodule $(N :_{\mathfrak{W}} r_g) = \{m \in \mathfrak{W} : r_g m \in N\}$.

Finally, a nonempty subset $S$ of a $G$-graded ring $\mathfrak{T}$ is called a *multiplicatively closed subset* (abbreviated as m.c.s.) if it satisfies the following conditions: $0 \notin S$, $1 \in S$, and for all $r, t \in S$, we have $rt \in S$.

## 2 Graded S-semiprime submodules

**Definition 2.1.** Let $S \subseteq h(\mathfrak{T})$ be a m.c.s. of a graded ring $\mathfrak{T}$ and $K$ be a gr-Sub of $\mathfrak{W}$ with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Then $K$ is said to be a graded $S$-semiprime submodule (briefly, gr-S-SP-Sub) of $\mathfrak{W}$ if there exists a fixed $s_t \in S$ such that whenever $r_i^n m_j \in K$ for some $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $t, i, j \in G$, and $n \in \mathbb{N}$, then $s_t r_i m_j \in K$. In particular, a graded ideal $I$ of $\mathfrak{T}$ is said to be a graded $S$-semiprime ideal (briefly, gr-S-SP-Id) if $I$ is a gr-S-SP-Sub of the graded $\mathfrak{T}$-module $\mathfrak{T}$.

Recall from[6] that a proper gr-Sub $N$ of $\mathfrak{W}$ is said to be a graded semiprime submodule (briefly, gr-SP-Sub), if whenever $r_g^k m_h \in S$ for some $r_g \in h(\mathfrak{T})$, $m_h \in h(\mathfrak{W})$, $g, h \in G$ and $k \in \mathbb{Z}^+$, then $r_g m_h \in S$.

The following result can be easily obtain. (We omit the proof)

**Theorem 2.2.** *Let $S_1 \subseteq S_2 \subseteq h(\mathfrak{T})$ be two multiplicatively closed subsets of $\mathfrak{T}$. If $K$ is a gr-$S_1$-SP-Sub of $\mathfrak{W}$, then $K$ is a gr-$S_2$-SP-Sub of $\mathfrak{W}$ in case of $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S_2 = \emptyset$. In particular, every gr-SP-Sub $K$ with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$ is a gr-S-SP-Sub.*

The converse of Theorem 2.2 is not true in general, the following example illustrates that.

**Example 2.3.** Let $G = (\mathbb{Z}, +)$ and $\mathfrak{T} = (\mathbb{Z}, +, .)$. Define $\mathfrak{T}_g = \left\{ \begin{array}{cc} \mathbb{Z} & \text{if } g = 0 \\ 0 & \text{otherwise} \end{array} \right\}$. Then $\mathfrak{T}$ is a $G$-graded ring. Let $\mathfrak{W} = \mathbb{Z} \times \mathbb{Z}_4$. Then $\mathfrak{W}$ is a $G$-graded $\mathfrak{T}$-module with $\mathfrak{W}_g = \left\{ \begin{array}{cc} \mathbb{Z} \times \{\bar{0}\} & \text{if } g = 0 \\ \{0\} \times \mathbb{Z}_4 & \text{if } g = 1 \\ \{0\} \times \{\bar{0}\} & \text{otherwise} \end{array} \right\}$. Now, consider the zero gr-Sub $N = \{0\} \times \{\bar{0}\}$ of $\mathfrak{W}$. Note that $(N :_{\mathfrak{T}} \mathfrak{W}) = \{0\}$ and $2^2(0, \bar{1}) = (0, \bar{0}) \in N$ where $2 \in \mathfrak{T}_0$ and $(0, \bar{1}) \in \mathfrak{W}_1$. Since $2(0, \bar{1}) \notin N$, $N$ is not gr-SP-Sub of $\mathfrak{W}$. Now take $S = \mathbb{Z} - \{0\} \subseteq \mathbb{Z}_0 \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$ and put $s_g = 4 \in S$. An easy computation shows that $N$ is a gr-$S$-SP-Sub of $\mathfrak{W}$.

**Theorem 2.4.** *Let $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. If $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ and $S \subseteq u(\mathfrak{T}) \cap h(\mathfrak{T})$, then $K$ is a gr-SP-Sub of $\mathfrak{W}$.*

*Proof.* Assume that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ and $S \subseteq u(\mathfrak{T}) \cap h(\mathfrak{T})$. Let $r_i^n m_j \in K$ for some $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j \in G$, and $n \in \mathbb{N}$. Since $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, there exists a fixed $s_t \in S$, where $t \in G$ such that $s_t r_i m_j \in K$, but $s_t$ is unit, so it has an inverse $s_t^{-1} \in h(\mathfrak{T})$. Then $r_i m_j \in K$. Thus $K$ is a gr-SP-Sub of $\mathfrak{W}$. □

Let $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. The saturation of $S$ is defined by $S^* = \{s_g \in h(\mathfrak{T}) : \frac{s_g}{1}$ is unit in $h(S^{-1}\mathfrak{T})\}$. It is easy to show that $S^*$ is m.c.s. of $\mathfrak{T}$ containing $S$. Also, a m.c.s. $S \subseteq h(\mathfrak{T})$ is saturated set if $S^* = S$.

**Theorem 2.5.** *Let $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. Then $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ if and only if $K$ is a gr-$S^*$-SP-Sub of $\mathfrak{W}$.*

*Proof.* Suppose that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, so $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Firstly, we need to show that $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S^* = \emptyset$. If $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S^* \neq \varphi$, then we have $a_i \in (K :_{\mathfrak{T}} \mathfrak{W}) \cap S^*$. Hence $\frac{a_i}{1}$ is a unit in $h(S^{-1}\mathfrak{T})$. Thus there exists $\frac{x_j}{y_d} \in h(S^{-1}\mathfrak{T})$ where $j, d \in G$ such that $\frac{a_i}{1}\frac{x_j}{y_d} = 1$, so, $a_i x_j u_b = y_d u_b$ for some $u_b \in S$. Put $s' = y_d u_b \in S$, then $s' = a_i x_j u_b \in S$. Since $a_i \in (K :_{\mathfrak{T}} \mathfrak{W})$, then $s' \in (K :_{\mathfrak{T}} \mathfrak{W}) \cap S$ which is contradiction. Thus $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S^* = \emptyset$. Since $S \subseteq S^*$ then by Theorem 2.2, $K$ is a gr-$S^*$-SP-Sub of $\mathfrak{W}$. Conversely, suppose that $K$ is a gr-$S^*$-SP-Sub of $\mathfrak{W}$, so $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S^* = \emptyset$. Since $S \subseteq S^*$, then $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Now, let $r_i^n m_j \in K$ where $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j \in G$, and $n \in \mathbb{N}$. Then by the assumption there exists a fixed $s_g \in S^*$ where $g \in G$, such that $s_g r_i m_j \in K$. Since $s_g \in S^*$ then $\frac{s_g}{1}$ is a unit in $h(S^{-1}\mathfrak{T})$, it follows that there exists $x_y \in h(\mathfrak{T})$ and $t_v \in S$ with $\frac{s_g}{1} \cdot \frac{x_y}{t_v} = 1$, where $y, v \in G$. Hence $u_w t_v = s_g x_y u_w$ for some $u_w \in S$, $w \in G$. Put $s' = u_w t_v = s_g x_y u_w \in S$. Then $s' r_i m_j = x_y u_w s_g r_i m_j \in K$. Thus $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. □

**Theorem 2.6.** *Let $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. If $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then $S^{-1}K$ is a gr-SP-Sub of $S^{-1}\mathfrak{W}$.*

*Proof.* Suppose that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. Let $(\frac{r_i}{s_k})^n \frac{m_j}{t_l} = \frac{r_i^n m_j}{s_k^n t_l} \in S^{-1}K$, where $\frac{r_i}{s_k} \in h(S^{-1}\mathfrak{T})$, $\frac{m_j}{t_l} \in h(S^{-1}\mathfrak{W})$, and $n \in \mathbb{N}$. Then $v_b r_i^n m_j = r_i^n v_b m_j \in K$ for some $v_b \in S$, so there exists a fixed $s'_o \in S$, $o \in G$ such that $s'_o r_i v_b m_j \in K$ as $K$ is a gr-$S$-SP-Sub of $\mathfrak{W}$. So $\frac{r_i}{s_k}\frac{m_j}{t_l} = \frac{s'_o v_b r_i m_j}{s'_o v_b s_k t_l} \in S^{-1}K$. Thus $S^{-1}K$ is a gr-SP-Sub of $S^{-1}\mathfrak{W}$. □

**Theorem 2.7.** *Let $S = \{s_1, s_2, ......, s_k\} \subseteq h(\mathfrak{T})$ be a finite m.c.s. of $\mathfrak{T}$, and $K$ be a gr-Sub with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. If $S^{-1}K$ is a gr-SP-Sub of $S^{-1}\mathfrak{W}$, then $K$ is a gr-S-SP-Sub of $\mathfrak{W}$.*

*Proof.* Suppose that $S^{-1}K$ is a gr-SP-Sub of $S^{-1}\mathfrak{W}$. Let $r_i^n m_j \in K$ where $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j \in G$, and $n \in \mathbb{N}$. Then $\frac{r_i^n}{1}\frac{m_j}{1} = \frac{r_i^n m_j}{1} \in S^{-1}K$. Since $S^{-1}K$ is a gr-SP-Sub of $S^{-1}\mathfrak{W}$, we get $\frac{r_i}{1}\frac{m_j}{1} \in S^{-1}K$, so there exists $s_t \in S$ where $t \in G$ such that $s_t r_i m_j \in K$. Put $s' = s_1 s_2 ......s_t...s_k \in S$, then $s' r_i m_j \in K$. Thus $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. □

**Theorem 2.8.** *Let $S \subseteq h(\mathfrak{T})$ be a multiplicatively closed subset of $\mathfrak{T}$, and $K$ be a gr-Sub of $\mathfrak{W}$ with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Then the following statements are equivalent:*

1. *$K$ is a gr-S-SP-Sub of $\mathfrak{W}$.*
2. *There is a fixed $s_t \in S$ such that whenever $J^n A \subseteq K$ where $J$ is a gr-Id of $\mathfrak{T}$, $A$ is a gr-Sub of $\mathfrak{W}$, and $n \in \mathbb{N}$, then $s_t JA \subseteq K$.*

*Proof.* (1) $\Rightarrow$ (2) Suppose that $K$ is a gr-$S$-SP-Sub of $\mathfrak{W}$ and $J^n A \subseteq K$ where $J$ is a graded ideal of $\mathfrak{T}$, $A$ is a gr-Sub of $\mathfrak{W}$, and $n \in \mathbb{N}$. Let $a = \sum_{j \in G} a_j \in J$ and $m = \sum_{r \in G} m_r \in A$. It follows that, $\forall i, r \in G$, $a_i^n m_r \in J^n A \subseteq K$. Since $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then there exists a fixed $s_t \in S$ such that $s_t a_i m_r \in K$, for all $i, r \in G$. It follows that, $s_t am \in K$. Thus $s_t JA \subseteq K$.

(2) $\Rightarrow$ (1) Suppose that there is a fixed $s_t \in S$ where $t \in G$ such that whenever $J^n A \subseteq K$ where $J$ is a gr-Id of $\mathfrak{T}$, $A$ is a gr-Sub of $\mathfrak{W}$, and $n \in \mathbb{N}$, then $s_t JA \subseteq K$. Let $r_i^n m_j \in K$ where $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j \in G$, and $n \in \mathbb{N}$. Put $J = Rr_i$ and $A = Rm_j$. Then $J$ is a gr-Id of $\mathfrak{T}$ and $A$ is a gr-Sub of $\mathfrak{W}$. Now $J^n A = Rr_i^n m_j \subseteq K$ so by the assumption $s_t JA = Rs_t r_i m_j \subseteq K$, so $s_t r_i m_j \in K$. Thus $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. □

By the previous theorem we have the following result.

**Corollary 2.9.** *Let $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$, and $K$ be a gr-Id of $\mathfrak{T}$ with $K \cap S = \emptyset$. Then $K$ is agr-S-SP-Id of $\mathfrak{T}$ if and only if there is a fixed $s_t \in S$ such that whenever $I^n J \subseteq K$ for some gr-Ids $I, J$ of $\mathfrak{T}$, and $n \in \mathbb{N}$, then $s_t IJ \subseteq K$.*

**Theorem 2.10.** *Let $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$, and $K$ be a gr-Sub of $\mathfrak{W}$. If $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then $(K :_{\mathfrak{T}} \mathfrak{W})$ is a gr-S-SP-Id of $\mathfrak{T}$.*

*Proof.* Suppose that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. Then $(K :_{\mathfrak{T}} \mathfrak{W})$ is a gr-Id of $\mathfrak{T}$ with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Let $r_i^n \in (K :_{\mathfrak{T}} \mathfrak{W})$ for some $r_i \in h(\mathfrak{T})$, $i \in G$, and $n \in \mathbb{N}$, so $r_i^n \mathfrak{W} \subseteq K$. Put $J = Rr_i$. Then $J$ is a gr-Id of $\mathfrak{T}$. Hence $J^n \mathfrak{W} = Rr_i^n \mathfrak{W} \subseteq K$. Now, by Theorem 2.8, there exists a fixed $s_t \in S$ where $t \in G$ such that $s_t JM = s_t Rr_i \mathfrak{W} \subseteq K$, so $s_t r_i \mathfrak{W} \subseteq K$ and hence $s_t r_i \in (K :_{\mathfrak{T}} \mathfrak{W})$. Thus $(K :_{\mathfrak{T}} \mathfrak{W})$ is a gr-$S$-SP-Id of $\mathfrak{T}$. □

Recall from[4] that a graded $\mathfrak{T}$-module $\mathfrak{W}$ is called a graded multiplication if for each gr-Sub $N$ of $\mathfrak{W}$, we have $N = IM$ for some gr-Id $I$ of $\mathfrak{T}$. If $N$ is gr-Sub of a graded multiplication module $\mathfrak{W}$, then $N = (N :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W}$.

**Theorem 2.11.** *Let $\mathfrak{W}$ be a graded multiplication $\mathfrak{T}$-module, $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$, and $K$ be a gr-Sub. Then $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ if and only if $(K :_{\mathfrak{T}} \mathfrak{W})$ is a gr-S-SP-Id of $\mathfrak{T}$.*

*Proof.* Suppose that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then $(K :_{\mathfrak{T}} \mathfrak{W})$ is a gr-S-SP-Id of $\mathfrak{T}$ by Theorem 2.10. Conversely, assume that $(K :_{\mathfrak{T}} \mathfrak{W})$ is a gr-S-SP-Id of $\mathfrak{T}$. Let $J^n A \subseteq K$ where $J$ is a gr-Id of $\mathfrak{T}$, $A$ is a graded submodule of $\mathfrak{W}$, and $n \in \mathbb{N}$. Since $J^n A \subseteq K$ , $J^n(A :_{\mathfrak{T}} \mathfrak{W}) \subseteq (J^n A :_{\mathfrak{T}} \mathfrak{W}) \subseteq (K :_{\mathfrak{T}} \mathfrak{W})$. As $A$ is a gr-Sub of $\mathfrak{W}$, we have $(A :_{\mathfrak{T}} \mathfrak{W})$ is a gr-Id of $\mathfrak{T}$. Since $(K :_{\mathfrak{T}} \mathfrak{W})$ is a gr-$S$-SP-Id of $\mathfrak{T}$, by Corollary 2.9, there exists a fixed $s_t \in S$ where $t \in G$ such that $s_t J(A :_{\mathfrak{T}} \mathfrak{W}) \subseteq (K :_{\mathfrak{T}} \mathfrak{W})$. Since $\mathfrak{W}$ is a graded multiplication module, $A = (A :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W}$ and $K = (K :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W}$. It follows that, $s_t JA = s_t J(A :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W} \subseteq (K :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W} = K$. Now, by Theorem 2.8, $K$ is a gr-$S$-SP-Sub of $\mathfrak{W}$. □

**Theorem 2.12.** *Let $\mathfrak{W}$ be a graded multiplication $\mathfrak{T}$-module, $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$, and $K$ be a gr-Sub with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Then the following statements are equivalent:*

1. *$K$ is a gr-S-SP-Sub of $\mathfrak{W}$.*
2. *There exists a fixed $s_t \in S$, where $t \in G$ such that whenever $L^n A \subseteq K$ for some gr-Subs $L, A$ of $\mathfrak{W}$ and $n \in \mathbb{N}$, then $s_t LA \subseteq K$.*

*Proof.* (1) ⇒ (2) Suppose that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. Let $L^n A \subseteq K$ for some gr-Subs $L, A$ of $\mathfrak{W}$ and $n \in \mathbb{N}$. Since $\mathfrak{W}$ is a graded multiplication module, then $L = I\mathfrak{W}$ and $A = J\mathfrak{W}$ for some gr-Ids $I, J$ of $\mathfrak{T}$, so $L^n A = (I^n \mathfrak{W})(J\mathfrak{W}) = I^n J\mathfrak{W} = I^n A \subseteq K$. By Theorem 2.8, there exists a fixed $s_t \in S$ where $t \in G$ such that $s_t IA \subseteq K$, so $s_t IJ\mathfrak{W} \subseteq K$. Thus $s_t(I\mathfrak{W})(J\mathfrak{W}) = s_t LA \subseteq K$.

(2) ⇒ (1 Let $I^n A \subseteq K$ for some gr-Id $I$ of $\mathfrak{T}$, gr-Sub $A$ of $\mathfrak{W}$, and $n \in \mathbb{N}$. Since $\mathfrak{W}$ is a graded multiplication module, $A = (A :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W}$. Then $I^n A = I^n(A :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W}$. Now, put $L = I\mathfrak{W}$ so, $L^n A = I^n \mathfrak{W}(A :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W} = I^n(A :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W} \subseteq K$. By the assumption, there exists a fixed $s_t \in S$ where $t \in G$ such that $s_t LA \subseteq K$, it follows that, $s_t LA = s_t I\mathfrak{W}(A :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W} = s_t I(A :_{\mathfrak{T}} \mathfrak{W})\mathfrak{W} = s_t IA \subseteq K$. By Theorem 2.8, $K$ is a gr-$S$-SP-Sub of $\mathfrak{W}$. □

Let $\mathfrak{T}$ be a $G$-graded ring and $\mathfrak{W}, \mathfrak{W}'$ graded $\mathfrak{T}$-modules. Let $\varphi : \mathfrak{W} \to \mathfrak{W}'$ be an $\mathfrak{T}$-module homomorphism. Then $\varphi$ is said to be a graded homomorphism if $\varphi(\mathfrak{W}_g) \subseteq \mathfrak{W}'_g$ for all $g \in G$ (see[11]).

**Theorem 2.13.** *Let $\mathfrak{W}, \mathfrak{W}'$ be a graded $\mathfrak{T}$-modules, $f : \mathfrak{W} \to \mathfrak{W}'$ be a graded $\mathfrak{T}$-homomorphism, and $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. If $K'$ is a gr-S-SP-Sub of $\mathfrak{W}'$ such that $(f^{-1}(K') :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$, then $f^{-1}(K')$ is a gr-$S$-SP-Sub of $\mathfrak{W}$.*

*Proof.* Suppose that $K'$ is a gr-S-SP-Sub of $\mathfrak{W}'$ with $(f^{-1}(K') :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Let $r_i^n m_j \in f^{-1}(K')$ where $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j \in G$, and $n \in \mathbb{N}$. Then $f(r_i^n m_j) = r_i^n f(m_j) \in K'$. Since $f$ is a graded $\mathfrak{T}$-homomorphism, $f(\mathfrak{W}_j) \in \mathfrak{W}'_j$ so $f(m_j) \in h(\mathfrak{W}')$. Since $K'$ is a gr-S-SP-Sub of $\mathfrak{W}'$, there exists $s_t \in S$, where $t \in G$ such that $s_t r_i f(m_j) = f(s_t r_i m_j) \in K'$ so $s_t r_i m_j \in f^{-1}(K')$. Thus $f^{-1}(K')$ is a gr-$S$-SP-Sub of $\mathfrak{W}$. □

**Corollary 2.14.** *Let $L$ be a gr-Sub of $\mathfrak{W}$, and $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. If $K'$ is a gr-S-SP-Sub of $\mathfrak{W}$ with $(K' :_{\mathfrak{T}} L) \cap S = \emptyset$, then $L \cap K'$ is a gr-S-SP-Sub of $L$.*

*Proof.* Suppose that $K'$ is a gr-S-SP-Sub of $\mathfrak{W}$ with $(K' :_{\mathfrak{T}} L) \cap S = \emptyset$. Consider the graded injection $f : L \to \mathfrak{W}$ defind by $f(x) = x$ for all $x \in L$. So $f^{-1}(K') = L \cap K'$. By $(K' :_{\mathfrak{T}} L) \cap S = \emptyset$, we get $(f^{-1}(K') :_{\mathfrak{T}} L) \cap S = \emptyset$. Now, by Theorem 2.13, $L \cap K'$ is a gr-$S$-SP-Sub of $L$. □

**Theorem 2.15.** *Let $\mathfrak{W}, \mathfrak{W}'$ be a graded $\mathfrak{T}$-modules, $f : \mathfrak{W} \to \mathfrak{W}'$ be a graded $\mathfrak{T}$-epimorphism, and $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. If $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ such that $ker(f) \subseteq K$, then $f(K)$ is a gr-S-SP-Sub of $\mathfrak{W}'$.*

*Proof.* Suppose that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ such that $ker(f) \subseteq K$. If $\exists s_t \in (f(K) :_{\mathfrak{T}} \mathfrak{W}') \cap S$ where $t \in G$, then $s_t\mathfrak{W}' \subseteq f(K)$, it follows that $f(s_t\mathfrak{W}) = s_t f(\mathfrak{W}) \subseteq f(K)$, thus we have $s_t\mathfrak{W} \subseteq s_t\mathfrak{W} + ker(f) \subseteq K + ker(f) \subseteq K$, so $s_t \in (K :_{\mathfrak{T}} \mathfrak{W}) \cap S$ which is a contradiction. Thus $(f(K) :_{\mathfrak{T}} \mathfrak{W}') \cap S = \emptyset$ . Now, let $r_i^n m_j' \in f(K)$, where $r_i \in h(\mathfrak{T})$, $m_j' \in h(\mathfrak{W}')$, $i, j \in G$, and $n \in \mathbb{N}$. Since $m_j' \in h(\mathfrak{W}')$ and $f$ is a graded epimorphism, $\exists m_j \in h(\mathfrak{W})$ such that $f(m_j) = m_j'$. Then $r_i^n m_j' = r_i^n f(m_j) = f(r_i^n m_j) \in f(K)$, so there exists $b_d \in K \cap h(\mathfrak{W})$ such that $f(r_i^n m_j - b_d) = 0$, so $r_i^n m_j - b_d \in ker(f) \subseteq K$, which gives $r_i^n m_j \in K$. Since $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then $\exists s_t \in S$, $t \in G$ such that $s_t r_i m_j \in K$. It follows that, $s_t r_i m_j' \in f(K)$. Thus $f(K)$ is a gr-$S$-SP-Sub of $\mathfrak{W}'$. □

**Corollary 2.16.** *Let $L$, $K$ be two gr-Subs of $\mathfrak{W}$ with $L \subseteq K$, and $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. Then $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ if and only if $K/L$ is a gr-S-SP-Sub of $\mathfrak{W}/L$.*

*Proof.* Suppose that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. Consider the canonical homomorphism $\pi : \mathfrak{W} \to \mathfrak{W}/L$ defined by $\pi(m) = m + L$, for all $m \in \mathfrak{W}$. Then $\pi$ $\pi$ is a graded epimorphism and $ker(\pi) = L \subseteq K$. So by Theorem 2.15, $\pi(K) = K/L$ is a gr-S-SP-Sub of $\mathfrak{W}/L$. Conversely, suppose that $K/L$ is a gr-S-SP-Sub of $\mathfrak{W}/L$. Let $r_i^n m_j \in K$ where $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j \in G$, and $n \in \mathbb{N}$. Then $r_i^n(m_j + L) \in K/L$ where $m_j + L \in h(\mathfrak{W}/L)$. Since $K/L$ is a gr-S-SP-Sub of $\mathfrak{W}/L$, then $\exists s_t \in S$ where $t \in G$ such that $(s_t r_i m_j) + L \in K/L$, so $s_t r_i m_j \in L \subseteq K$ or $s_t r_i m_j \in K$. Thus $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. □

**Lemma 2.17.** *Let $S \subseteq h(\mathfrak{T})$ be a m.c.s. of $\mathfrak{T}$. If $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then there exists $s_t \in S$ such that $(K :_{\mathfrak{W}} s_t^2) = (K :_{\mathfrak{W}} s_t^n)$, for all $n \geqslant 2$.*

*Proof.* Assume that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ and $s_t$ is the element that satisfies the condition of gr-S-SP-Sub. Let $n \geqslant 2 \in \mathbb{N}$, and $m = \sum_{j \in G} m_j \in (K :_{\mathfrak{W}} s_t^n)$. Since $(K :_{\mathfrak{W}} s_t^n)$ is a gr-Sub of $\mathfrak{W}$, we have $m_j \in (K :_{\mathfrak{W}} s_t^n)$ $\forall j \in G$. Then $s_t^n m_j \in K$ where $s_t \in S \subseteq h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, and $n \in \mathbb{N}$. Since $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then $s_t(s_t m_j) = s_t^2 m_j \in K$. Hence $m_j \in (K :_{\mathfrak{W}} s_t^2)$ $\forall j \in G$ which gives $m = \sum_{j \in G} m_j \in (K :_{\mathfrak{W}} s_t^2)$. Thus $(K :_{\mathfrak{W}} s_t^n) \subseteq (K :_{\mathfrak{W}} s_t^2)$. Let $m \in (K :_{\mathfrak{W}} s_t^2)$. Then $s_t^2 m \in K$, so $s_t^n m \in K$. Hence $m \in (K :_{\mathfrak{W}} s_t^n)$. Thus $(K :_{\mathfrak{W}} s_t^2) \subseteq (K :_{\mathfrak{W}} s_t^n)$. Thus $(K :_{\mathfrak{W}} s_t^n) = (K :_{\mathfrak{W}} s_t^2)$, $\forall n \geq 2$. □

**Theorem 2.18.** *Let $S \subseteq h(\mathfrak{T})$ be a multiplicatively closed subset of $\mathfrak{T}$ and $K$ be a gr-Sub of $\mathfrak{W}$ with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Then $K$ is a gr-S-SP-Sub of $\mathfrak{W}$ if and only if $(K :_{\mathfrak{W}} s_t)$ is a gr-SP-Sub for some $s_t \in S$.*

*Proof.* Suppose that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. Then there exists $s_t \in S$ where $t \in G$ satisfies the gr-S-SP-Sub condition. Let $r_i^n m_j \in (K :_{\mathfrak{W}} s_t^2)$ where $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j \in G$, and $n \in \mathbb{N}$. Then $r_i^n s_t^2 m_j \in K$. Since $K$ is a gr-$S$-SP-Sub of $\mathfrak{W}$ and $s_t^2 m_j \in h(\mathfrak{W})$, we get $s_t r_i s_t^2 m_j = s_t^3 r_i m_j \in K$, so $r_i m_j \in (K :_{\mathfrak{W}} s_t^3)$. By Lemma 2.17, we have $(K :_{\mathfrak{W}} s_t^3) = (K :_{\mathfrak{W}} s_t^2)$. Thus $(K :_{\mathfrak{W}} s_t^2)$ is a gr-SP-Sub of $\mathfrak{W}$. Conversely, suppose that $(K :_{\mathfrak{W}} s_t)$ is a gr-SP-Sub for some $s_t \in S$. Let $r_i^n m_j \in K \subseteq (K :_{\mathfrak{W}} s_t)$ where $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j \in G$, and $n \in \mathbb{N}$. Since $(K :_{\mathfrak{W}} s_t)$ is a gr-SP-Sub of $\mathfrak{W}$, $r_i m_j \in (K :_{\mathfrak{W}} s_t)$ and so $s_t r_i m_j \in K$. Thus $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. □

Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be a $G$-graded rings. Then $\mathfrak{T} = \mathfrak{T}_1 \times \mathfrak{T}_2$ is a $G$-graded ring with $\mathfrak{T}_g = (\mathfrak{T}_1)_g \times (\mathfrak{T}_2)_g$ for all $g \in G$. Let $\mathfrak{W}_1$ be a $G$-graded $\mathfrak{T}_1$-module, $\mathfrak{W}_2$ be a $G$-graded $\mathfrak{T}_2$-module and $\mathfrak{T} = \mathfrak{T}_1 \times \mathfrak{T}_2$. Then $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$ is a $G$-graded $\mathfrak{T}$-module with $(\mathfrak{W})_g = (\mathfrak{W}_1)_g \times (\mathfrak{W}_2)_g$ for all $g \in G$, (see[11]). Also if $S_1 \subseteq h(\mathfrak{T}_1)$ is a m.c.s. of $\mathfrak{T}_1$ and $S_2 \subseteq h(\mathfrak{T}_2)$ is a m.c.s. of $\mathfrak{T}_2$, then $S = S_1 \times S_2$ is a m.c.s. of $\mathfrak{T}$.

**Theorem 2.19.** *Let $\mathfrak{W}_i$ be a graded $\mathfrak{T}_i$-module, $S_i \subseteq h(\mathfrak{T}_i)$ be a m.c.s. of $\mathfrak{T}_i$, and $K_i$ is a gr-Sub of $\mathfrak{W}_i$ for each $i = 1, 2$. Let $\mathfrak{W} = \mathfrak{W}_1 \times \mathfrak{W}_2$ be a graded $\mathfrak{T} = \mathfrak{T}_1 \times \mathfrak{T}_2$-module, $S = S_1 \times S_2$ be a m.c.s. of $\mathfrak{T}$ and $K = K_1 \times K_2$. If $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then $K_1$ is a gr-$S_1$-SP-Sub of $\mathfrak{W}_1$ and $(K_2 :_{\mathfrak{T}_2} \mathfrak{W}_2) \cap S_2 \neq \emptyset$ or $(K_1 :_{\mathfrak{T}_1} \mathfrak{W}_1) \cap S_1 \neq \emptyset$, and $K_2$ is a gr-$S_2$-SP-Sub of $\mathfrak{W}_2$.*

*Proof.* Assume that $K$ is a gr-S-SP-Sub of $\mathfrak{W}$. Then we have $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$, it follows that, $(K_2 :_{\mathfrak{T}_2} \mathfrak{W}_2) \cap S_2 = \emptyset$ or $(K_1 :_{\mathfrak{T}_1} \mathfrak{W}_1) \cap S_1 = \emptyset$. Suppose that $(K_2 :_{\mathfrak{T}_2}) \cap S_2 \neq \emptyset$. We will show that $K_1$ is a gr-$S_1$-SP-Sub of $\mathfrak{W}_1$. Let $r_i^n m_j \in K_1$ where $r_i \in h(\mathfrak{T}_1)$, $m_j \in h(\mathfrak{W}_1)$, $i, j \in G$, and $n \in \mathbb{N}$. Then $(r_i^n m_j, 0) = (r_i, 0)^n (m_j, 0) \in K$. Since $K$ is a gr-S-SP-Sub of $\mathfrak{W}$, then there exists a fixed $s_g = (s'_g, s''_g) \in S$ such that $s_g(r_i, 0)(m_j, 0) = (s'_g r_i m_j, 0) \in K$, which gives that $s'_g r_i m_j \in K_1$. Thus $K_1$ is a gr-$S_1$-SP-Sub of $\mathfrak{W}_1$. If $(K_1 :_{\mathfrak{T}_1} \mathfrak{W}_1) \cap S_1 \neq \emptyset$, similarly $K_2$ is a gr-$S_2$-SP-Sub of $\mathfrak{W}_2$. □

A gr-Sub $N$ of $\mathfrak{W}$ is said to be *a graded maximal* if $N \neq \mathfrak{W}$ and if there is a gr-Sub $L$ of $\mathfrak{W}$ such that $N \subseteq L \subseteq \mathfrak{W}$, then $N = L$ or $L = \mathfrak{W}$ (see[11]). *The graded Jacobson radical of a graded module* $\mathfrak{W}$, denoted by $gr\text{-}Jac(\mathfrak{T})$, is defined to be the intersection of all graded maximal submodules of $\mathfrak{W}$ (if $\mathfrak{W}$ has no graded maximal submodule then we shall take, by definition, $gr\text{-}Jac(\mathfrak{T}) = \mathfrak{W}$. (see[11]).

**Theorem 2.20.** *Let $K$ be a gr-Sub of $\mathfrak{W}$ such that $(K :_{\mathfrak{T}} \mathfrak{W}) \subseteq gr\text{-}Jac(\mathfrak{T})$. Then $K$ is a gr-SP-Sub of $\mathfrak{W}$ if and only if $K$ is a gr-$S_I = (h(\mathfrak{T}) - I)$-SP-Sub of $\mathfrak{W}$ for each graded maximal ideal $I$ of $\mathfrak{T}$.*

*Proof.* It is clear that $S_I \subseteq h(\mathfrak{T})$ is a m.c.s. of $\mathfrak{T}$ for each graded maximal ideal $I$ of $\mathfrak{T}$. Suppose that $K$ is a gr-SP-Sub of $\mathfrak{W}$. Since $(K :_{\mathfrak{T}} \mathfrak{W}) \subseteq gr\text{-}Jac(\mathfrak{T}) \subseteq I$, $S_I \cap (K :_{\mathfrak{T}} \mathfrak{W}) = \emptyset$. So by Theorem 2.2, $K$ is a gr-$S_I$-SP-Sub of $\mathfrak{W}$. For the converse, suppose that $K$ is a gr-$S_I = (h(\mathfrak{T}) - I)$-SP-Sub of $\mathfrak{W}$ for each graded maximal ideal $I$ of $\mathfrak{T}$. Let $r_d^n m_j \in K$ where $r_d \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $d, j \in G$, and $n \in \mathbb{N}$. By the assumption, there exits $s_I \in S_I (s_I \notin I)$ with $s_I r_d m_j \in K$. Define the set $L = \{s_I : s_I \notin I$ and $s_I r_d m_j \in K$ for some graded maximal ideal $I$ of $\mathfrak{T}\}$. If $< L > \neq \mathfrak{T}$, then there exists a graded maximal ideal $P$ of $\mathfrak{T}$ containing $L$. So there exits $s_P \in L$ such that $s_P \notin P$. Since $L \subseteq < L > \subseteq P$, then $s_P \in P$ which is a contradiction. Hence $< L > = \mathfrak{T}$ and so there exists $s_{P_1}, s_{P_2}, \cdots, s_{P_n} \in L$ and $x_1, x_2, \cdots, x_n \in \mathfrak{T}$ such that $1 = x_1 s_{P_1} + x_2 s_{P_2} + \cdots + x_n s_{P_n}$. Since $s_{P_i} r_d m_j \in K$ for each $i = 1, 2, \cdots, n$, then $r_d m_j = x_1 s_{P_1} r_d m_j + x_2 s_{P_2} r_d m_j + \cdots + x_n s_{P_n} r_d m_j \in K$. Thus $K$ is a gr-SP-Sub. □

**Definition 2.21.** Let $S \subseteq h(\mathfrak{T})$ be a multiplicatively closed subset of $\mathfrak{T}$. Then $\mathfrak{W}$ is called a graded $S$-reduced module if there exists a fixed $s_t \in S$ and whenever $a_i^n m_j = 0$ for some $a_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $i, j, t \in G$, and $n \in \mathbb{N}$, then $s_t a_i m_j = 0$.

**Theorem 2.22.** *Let $S \subseteq h(\mathfrak{T})$ be a multiplicatively closed subset of $\mathfrak{T}$. Then $\mathfrak{W}$ be a graded $S$-reduced module if and only if the zero gr-Sub is a gr-S-SP-Sub.*

*Proof.* It follows directly from Definition 2.1 and 2.21. □

**Theorem 2.23.** *Let $S \subseteq h(\mathfrak{T})$ be a multiplicatively closed subset of $\mathfrak{T}$ and $K$ be a gr-Sub with $(K :_{\mathfrak{T}} \mathfrak{W}) \cap S = \emptyset$. Then $K$ is a gr-S-SP-Sub if and only if $\mathfrak{W}/K$ is a graded $S$-reduced module.*

*Proof.* It follows directly from Theorem 2.22. □

**Definition 2.24.** A graded $\mathfrak{T}$-module $\mathfrak{W}$ is called graded $S$-torsion-free module if $ann_{\mathfrak{T}}(\mathfrak{W}) \cap S = \emptyset$ and there exists $s_t \in S$ where $t \in G$ such that whenever $r_i m_j = 0$ then $s_t r_i = 0$ or $s_t m_j = 0$ for each $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, and $i, j \in G$.

**Theorem 2.25.** *Let $S \subseteq h(\mathfrak{T})$ be a multiplicatively closed subset of $\mathfrak{T}$. If $\mathfrak{W}$ is a graded $S$-torsion-free module, then $\mathfrak{W}$ is a graded $S$-reduced module*

*Proof.* Suppose that $\mathfrak{W}$ is a graded $S$-torsion-free module. Let $r_i^n m_j = r_i(r_i^{n-1} m_j) = 0$ for some $r_i \in h(\mathfrak{T})$, $m_j \in h(\mathfrak{W})$, $t, i, j \in G$, and $n \in \mathbb{N}$. Since $\mathfrak{W}$ is a graded $S$-torsion-free module, then there exists $s_t \in S$ where $t \in G$ such that $s_t r_i^{n-1} m_j = 0$ or $s_t r_i = 0$. If $s_t r_i = 0$, then $s_t r_i m_j = 0$ which gives that $\mathfrak{W}$ is a graded $S$-reduced module, so assume that $s_t r_i \neq 0$. Now, $r_i(s_t r_i^{n-2} m_j) = s_t r_i^{n-1} m_j = 0$ gives that $s_t^2 r_i^{n-2} m_j = 0$, continue with the same argument until you get $s_t^n m_j = 0$. Since $\mathfrak{W}$ is a graded $S$-torsion-free module then $s_t^{n+1} = 0$ or $s_t m_j = 0$. If $s_t^{n+1} = 0$, then $0 \in S$ which is contradiction. So $s_t m_j = 0$ which gives that $s_t r_i m_j = 0$. Thus $\mathfrak{W}$ is a graded $S$-reduced module. □

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