Coefficient Bounds for Generalized n-Fold Symmetric Neutrosophic Bi-univalent Functions

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Abstract

In this paper, we introduce and investigate new generalized subclasses of neutrosophic n-fold symmetric bi-univalent functions defined in the open unit disk U. These subclasses are characterized via four neutrosophic multi-parameters κ, ρ, γ, and β, which provide a flexible framework to capture the truth, indeterminacy, and falsity components inherent in geometric and analytic behaviors. Within this neutrosophic setting, we derive upper bounds for the initial coefficients $|d_{n+1}|$ and $|d_{2n+1}|$, and establish generalized Fekete–Szegő inequalities for the considered classes. The results obtained extend and unify several existing results in classical and neutrosophic bi-univalent function theory. Examples and corollaries are presented to demonstrate the sharpness and applicability of the results.

Keywords: Neutrosophic analysis; Analytic functions; Univalent functions; Bi-univalent functions; Coefficient bounds; v-fold symmetric functions

1 Introduction

Let $\mathcal{A}$ denote the class of all functions $Y(\tau)$ that are analytic on the open unit disk

$$U = \{\tau : |\tau| < 1\} \tag{1}$$

and have a Maclaurin series expansion of the form:

$$Y(\tau) = \tau + \sum_{k=2}^{\infty} d_k \tau^k. \tag{2}$$

If a function does not take the same value twice, it is called univalent. The subclass of all univalent functions in $\mathcal{A}$ is denoted by S.

Two well-known subclasses of S are the starlike functions of order β, denoted by $S^*(\beta)$, and the convex functions of order β, denoted by $K(\beta)$, which are defined by

$$S^*(\beta) = Y \in S : \Re\left(\frac{\tau Y'(\tau)}{Y(\tau)}\right) > \beta, \quad 0 \leq \beta < 1 \ ,$$

and

$$K(\beta) = Y \in S : \Re\left(\frac{\left(\tau Y^{'}(\tau)\right)^{'}}{Y'(\tau)}\right) > \beta, \quad 0 \leq \beta < 1 \ .$$

Note that

$$Y \in K(\beta) \quad \text{if and only if} \quad \tau Y^{'} \in S^*(\beta).$$

Mocanu[1] further generalized these notions by combining starlike and convex functions, introducing the class of convex functions of order α as follows:

$$M(\alpha) = Y \in S : \Re\left[\alpha\,\frac{\left(\tau Y^{'}(\tau)\right)^{'}}{Y'(\tau)} + (1-\alpha)\frac{\tau Y^{'}(\tau)}{Y(\tau)}\right] > 0, \quad 0 \leq \alpha \leq 1 \ .$$

Recently, motivated by applications in decision theory, information systems, and uncertainty modeling, classical subclasses of analytic and bi-univalent functions have been extended to the neutrosophic environment. Neutrosophic theory, introduced by Smarandache, extends fuzzy and intuitionistic fuzzy concepts. It incorporates three independent components: truth-membership, indeterminacy-membership, and falsity-membership. This framework enables richer modeling of uncertainty and vagueness in real-world problems.

In the context of geometric function theory, this allows us to define neutrosophic analytic functions in the open unit disk, where the analytic behavior of $Y(\tau)$ is studied not only in terms of geometric conditions such as starlikeness and convexity, but also with respect to the neutrosophic parameters that capture uncertainty and partial membership. Thus, classical subclasses like $S^*(\beta)$, $K(\beta)$, and $M(\alpha)$ admit neutrosophic generalizations, which provide more flexibility and applicability in uncertain or imprecise environments.

The present work is devoted to introducing new generalized subclasses of neutrosophic n-fold symmetric bi-univalent functions in U, defined via multi-parameter conditions involving κ, ρ, γ, and β. These parameters are interpreted in a neutrosophic sense, representing varying contributions of truth, indeterminacy, and falsity in shaping the geometric behavior of the considered functions. By deriving coefficient estimates and generalized Fekete–Szegő inequalities, we extend and improve many results from both classical and neutrosophic function theory.

According to the one-quarter theorem of Koebe,[2] every function $Y(z) \in S$ has its inverse function:

$$Y^{-1}(w), \ |w| < r_0(Y), \ \text{ for some } \ r_0(Y) \geq \frac{1}{4},$$

where

$$Y^{-1}(w) = w - d_2 w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2 d_3 + d_4)w^4 + \dots . \tag{3}$$

Let $Y^{-1} : Y(U) \to U$ have an analytic extension to U, and we denote $g = Y^{-1}$ in this extensible case. Throughout this paper, we assume $w \in U$ and $g = Y^{-1}$ given by (3).

A function $Y \in S$ is said to be bi-univalent in U if both Y and Y^{-1} are univalent in U.
Let Σ denote the class of all bi-univalent functions in S. Thus, every function $Y(\tau) \in \Sigma$ is both bi-univalent and analytic in U given by (2). The well-known Koebe function

$$Q(\tau) = \frac{\tau}{(1-\tau)^2}, \quad \tau \in U$$

plays a significant role in the theory of univalent functions due to its extremal properties, such as having the maximum possible growth among all normalized univalent functions in S. However, it is not a bi-univalent function in Σ.

The exploration of bi-univalent functions as a mathematical class originated in the early 1970s, with Lewin[3] introducing the class Σ and establishing the result $|d_2| < 1.51$. within this class. Subsequently, in 1970, Brannan et al.[4] delved into coefficient estimates, specifically $|d_2| < \sqrt{2}$, for a particular class of starlike functions. In 1984, Tan[5] extended these investigations by examining coefficient estimates for functions in the subclass of Σ.

Currently, Brannan and Taha[6] introduced bi-convex and bi-starlike functions, exploring initial coefficient estimates for these defined classes. Srivastava et al.[7] presented several new subclasses of Σ in 2010, determining the bounds of $|d_2|$ and $|d_3|$ in the subclasses. Frasin and Aouf[8] continued this line of inquiry 2011, examining the bounds for $|d_2|$ and $|d_3|$ for functions belonging to two additional subclasses of Σ. In 2013, Deniz[9] defined four distinct subclasses of bi-univalent functions and investigated coefficient bounds, specifically $|d_2|$ and $|d_3|$, for functions within these subclasses. Further contributions emerged in 2013 when Tang et al.[10] estimated coefficients for new Ma-Minda type bi-univalent functions classes. Frasin[11] extended the exploration in 2014 by investigating two novel subclasses of Σ. For recent developments in the class Σ, the readers are referred to published articles.[12–19]

Let $n \in \mathbb{N} = \{1, 2, 3, ...\}$. If a rotation of the domain D about the origin with an angle $\frac{2\pi}{n}$ maps D to itself, then D is known as an n-fold symmetric domain. An analytic function Y in U is called an n-fold symmetric function if it satisfies:

$$Y\left(e^{i\frac{2\pi}{n}}\tau\right) = e^{i\frac{2\pi}{n}}Y(\tau), \quad \tau \in U.$$

It has been observed that if $Y \in S$, then

$$h(\tau) = \sqrt[n]{Y(\tau^n)} \tag{4}$$

is univalent in the open unit disk U and maps U into an n-fold symmetric domain.

Let S_n represent the class of all n-fold symmetric univalent functions in U.
If $n = 1$, then $S_1 = S$. Every function $Y \in S_n$ ([21,22]) has a Maclaurin series expansion of the form:

$$Y(\tau) = \tau + \sum_{k=1}^{\infty} d_{nk+1}\tau^{nk+1}. \tag{5}$$

Srivastava et al.[24] explored a natural extension of S_n and introduced the class of n-fold symmetric bi-univalent functions in U. The Maclaurin series expansion for each $g = Y^{-1}$ is obtained by them as follows:

$$\begin{aligned} g(w) &= w - d_{n+1}w^{n+1} + ((n+1)d_{n+1}^2 - d_{2n+1})w^{2n+1} \\ &\quad -\{\frac{1}{2}(n+1)(3n+2)d_{n+1}^3 - (3n+2)d_{n+1}d_{2n+1} + d_{3n+1})\}w^{3n+1} + \dots . \end{aligned} \tag{6}$$

Let Σ_n denote the subclass of all n-fold symmetric bi-univalent functions in S_n. If $n = 1$, then equations (3) and (6) are equivalent. The following are three examples of n-fold symmetric bi-univalent functions in Σ_n:

$$h_1(\tau) = \left(\frac{\tau^n}{1-\tau^n}\right)^{\frac{1}{n}}, \quad h_2(\tau) = [-\log(1-\tau^n)]^{\frac{1}{n}}, \quad \text{and } h_3(\tau) = \left[\frac{1}{2}\log\left(\frac{1+\tau^n}{1-\tau^n}\right)\right]^{\frac{1}{n}}.$$

In recent years, Srivastava et al.[25,26] led to a demand for research on subclasses of S_n, which consequently developed several articles on S_n. In 2018, Srivastava et al.[27] tackled a new subclass of S_n in estimating the initial coefficients of the Maclaurin series of the functions. In,[28] Sakar and Tasar introduced new subclasses of S_n and obtained the initial coefficient bounds for these subclasses in.[29] In addition,[30] examined a comprehensive subclass of S_n by applying the subordination principle (see[20] for more details about subordination). Most recently, Swamy et al.[31] looked at a particular subclass of S_n.

The study of univalent and bi-univalent functions has traditionally been confined to deterministic analytic frameworks. However, many real-world problems in applied sciences, engineering, and decision theory are inherently uncertain and imprecise. Classical approaches cannot fully capture such uncertainty, which motivates the integration of neutrosophic analysis into geometric function theory. Neutrosophy, pioneered by Smarandache, extends fuzzy and intuitionistic fuzzy models by considering three independent components—truth-membership, indeterminacy-membership, and falsity-membership. Embedding these components into the structure of analytic and bi-univalent functions enables a richer modeling of functions under conditions of vagueness or incomplete information.

In particular, the four parameters κ, ρ, γ, and β introduced in this paper provide a neutrosophic multi-parameter framework for controlling analytic and geometric behaviors of n-fold symmetric bi-univalent functions. These parameters can be interpreted as flexible weights that regulate the influence of truth, indeterminacy, and falsity on the growth, distortion, and coefficient properties of the considered classes. This not only extends several well-known subclasses in the literature but also offers a natural platform for exploring new inequalities and coefficient bounds under neutrosophic conditions. Therefore, our work contributes both to the development of classical geometric function theory and to its extension within the neutrosophic environment.

This paper seeks to broaden the range of existing classes in the literature. We introduce four parameters: $0<\kappa\leq 1$, $0\leq\beta<1$, $0<\rho\leq 1$, and $0\leq\gamma\leq 1$ to construct the general subclasses of n-fold symmetric bi-univalent functions in Σ_n. First, we include some subclasses of bi-univalent functions in Σ, which are important and helpful in our present investigations.

A function $Y\in\mathcal{A}$ is said to be strongly starlike of order β if

$$\left|\arg\left(\frac{\tau Y^{'}(\tau)}{Y(\tau)}\right)\right|<\frac{\beta\pi}{2}.$$

Brannan and Taha[32] (also see[33]) investigated the subclass of the strongly bi-starlike functions of order β, defined by

$$S^{*}_{\Sigma}(\beta)=\left\{Y\in\Sigma:\ \left|\arg\left(\frac{\tau Y^{'}(\tau)}{Y(\tau)}\right)\right|<\frac{\beta\pi}{2}\ \text{ and }\ \left|\arg\left(\frac{wg^{'}(w)}{g(w)}\right)\right|<\frac{\beta\pi}{2},\quad 0<\beta\leq 1\right\}.$$

Similarly, the subclass of the strongly bi-convex function of order β can be defined as follows,

$$K_{\Sigma}(\beta)=\left\{Y\in\Sigma:\ \left|\arg\frac{\left(\tau Y^{'}(\tau)\right)^{'}}{Y^{'}(\tau)}\right|<\frac{\beta\pi}{2},\ \text{and}\ \left|\arg\frac{\left(wg^{'}(\tau)\right)^{'}}{g^{'}(\tau)}\right|<\frac{\beta\pi}{2},\quad 0<\beta\leq 1\right\}.$$

Let $Y,g=Y^{-1}\in\Sigma$ and $\tau,w\in U$; we denote

$$Q_1(Y,\tau)=\frac{\tau Y^{'}(\tau)}{Y(\tau)}\quad\text{and}\quad Q_2(g,\tau)=\frac{wg^{'}(w)}{g(w)}.$$

Then $Y\in S^{*}_{\Sigma}(\beta)$ if and only if

$$|\arg Q_1(Y,\tau)|<\frac{\beta\pi}{2}\quad\text{and}\quad|\arg Q_2(g,w)|<\frac{\beta\pi}{2},$$

and $Y\in K_{\Sigma}(\beta)$ if and only if

$$\tau Y^{'}(\tau)\in S^{*}_{\Sigma}(\beta).$$

Furthermore, we denote

$$Q_1^{\rho}(Y,\tau)=\frac{1}{2}\left(\frac{\tau Y^{'}(\tau)}{Y(\tau)}+\left(\frac{\tau Y^{'}(\tau)}{Y(\tau)}\right)^{\frac{1}{\rho}}\right),$$

$$Q_2^{\rho}(Y,\tau)=\frac{1}{2}\left(\frac{\left(\tau Y^{'}(\tau)\right)^{'}}{Y^{'}(\tau)}+\left(\frac{\left(\tau Y^{'}(\tau)\right)^{'}}{Y^{'}(\tau)}\right)^{\frac{1}{\rho}}\right),$$

$$Q_3^\rho(g,w)=\frac{1}{2}\left(\frac{wg^{'}(w)}{g(w)}+\left(\frac{wY^{'}(w)}{g(w)}\right)^{\frac{1}{\rho}}\right),$$

and

$$Q_4^\rho(g,w)=\frac{1}{2}\left(\frac{\left(wg^{'}(w)\right)^{'}}{g^{'}(w)}+\left(\frac{\left(wg^{'}(w)\right)^{'}}{g^{'}(w)}\right)^{\frac{1}{\rho}}\right).$$

Notice that $\frac{1}{\rho}\geq 1$. If $\frac{1}{\rho}\notin\mathbb{N}$, then it is necessary to assume that

$$\frac{\tau Y^{'}(\tau)}{Y(\tau)}\neq 0,\ \frac{\left(\tau Y^{'}(\tau)\right)^{'}}{Y^{'}(\tau)}\neq 0,$$

$$\frac{wg^{'}(w)}{g(w)}\neq 0,\ \frac{\left(wg^{'}(w)\right)^{'}}{g^{'}(w)}\neq 0.$$

2 Definitions and Preliminaries

Let $\mathbb{U}=\{\tau\in\mathbb{C}:|\tau|<1\}$ denote the open unit disk in the complex plane. Consider an analytic function of the form

$$Y(\tau)=\tau+a_2\tau^2+a_3\tau^3+\cdots,\quad \tau\in\mathbb{U}.$$

The class Σ of bi-univalent functions consists of functions Y for which both Y and its inverse Y^{-1} are univalent in $\mathbb{U}$.

In this work, we focus on n-fold symmetric bi-univalent functions in the neutrosophic setting. To prepare for our main results, we recall the following definitions and lemmas.

Let Y be an analytic n-fold symmetric function in $\mathbb{U}$ of the form

$$Y(\tau)=\tau+\sum_{k=1}^{\infty}d_{nk+1}\tau^{nk+1}.$$

Its inverse $g=Y^{-1}$ has an expansion of the form

$$g(w)=w-d_{n+1}w^{n+1}+\big((n+1)d_{2n+1}-d_{n+1}^2\big)w^{2n+1}+\cdots.$$

For analytic functions Y and $g=Y^{-1}$, we define the following operators:

$$Q_1^\rho(Y,\tau)=\frac{1}{2}\left(\frac{\tau Y'(\tau)}{Y(\tau)}+\left(\frac{\tau Y'(\tau)}{Y(\tau)}\right)^{\frac{1}{\rho}}\right),$$

$$Q_2^\rho(Y,\tau)=\frac{1}{2}\left(\frac{(\tau Y'(\tau))'}{Y'(\tau)}+\left(\frac{(\tau Y'(\tau))'}{Y'(\tau)}\right)^{\frac{1}{\rho}}\right),$$

$$Q_3^\rho(g,w)=\frac{1}{2}\left(\frac{wg'(w)}{g(w)}+\left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\rho}}\right),$$

$$Q_4^{\rho}(g,w)=\frac{1}{2}\left(\frac{(wg'(w))'}{g'(w)}+\left(\frac{(wg'(w))'}{g'(w)}\right)^{\frac{1}{\rho}}\right).$$

Note that $\frac{1}{\rho}\geq 1$. If $\frac{1}{\rho}\notin\mathbb{N}$, then it is necessary to assume that

$$\frac{\tau Y'(\tau)}{Y(\tau)}\neq 0,\quad \frac{(\tau Y'(\tau))'}{Y'(\tau)}\neq 0,\quad \frac{wg'(w)}{g(w)}\neq 0,\quad \frac{(wg'(w))'}{g'(w)}\neq 0.$$

Definition 2.1. A function Y is said to be in the subclass of Σ^n, denoted by $S\Sigma^n(\kappa,\rho,\gamma)$, if

$$\left|\arg\left[(1-\gamma)Q_1^{\rho}(Y,\tau)+\gamma Q_2^{\rho}(Y,\tau)\right]\right|<\frac{\kappa\pi}{2},$$

and

$$\left|\arg\left[(1-\gamma)Q_3^{\rho}(g,w)+\gamma Q_4^{\rho}(g,w)\right]\right|<\frac{\kappa\pi}{2},$$

where $0<\kappa\leq 1$, $0<\rho\leq 1$, and $0\leq\gamma\leq 1$. From a neutrosophic viewpoint, κ controls truth-membership, ρ balances indeterminacy, and γ measures falsity-membership.

Definition 2.2. A function Y is said to be in the subclass of Σ^n, denoted by $T\Sigma^n(\beta,\rho,\gamma)$, if

$$\Re\left[(1-\gamma)Q_1^{\rho}(Y,\tau)+\gamma Q_2^{\rho}(Y,\tau)\right]>\beta,$$

and

$$\Re\left[(1-\gamma)Q_3^{\rho}(g,w)+\gamma Q_4^{\rho}(g,w)\right]>\beta,$$

where $0\leq\beta<1$, $0<\rho\leq 1$, $0\leq\gamma\leq 1$. In the neutrosophic setting, β is interpreted as a flexible order-threshold that interacts with truth, indeterminacy, and falsity components.

Remark 2.3. Definitions 2 and 3 extend many existing classes. Examples:

1. For $\gamma=0$: subclasses reduce to cases with only truth and indeterminacy components.
2. For $\gamma=0$ and $\rho=1$: subclasses reduce to deterministic truth-based models.
3. For $\gamma=0$, $n=1$, and $\rho=1$: subclasses reduce to the classical strongly bi-starlike classes of order κ and β.

Remark 2.4. The class $S\Sigma^n(\kappa,\rho,\gamma)$ is not empty. For example, let

$$p^*(\tau)=1+0.35\tau+0.1\tau^2,$$

which satisfies $\Re(p^*(\tau))>0$ for $\tau\in\mathbb{U}$. Define

$$Y^*(\tau)=\frac{\tau}{1+0.2\tau}\in S_n,\qquad g^*(w)=\frac{w}{1-0.2w}\in S_n.$$

Then $Y^*\in\Sigma^n$, and for $\alpha=0.5$, $\kappa=1$, $\rho=1$, it follows that $Y^*\in S\Sigma^n(1,1,0.5)$, showing non-emptiness of the class.

Let $\mathcal{P}$ denote the class of analytic functions p in $\mathbb{U}$ such that $p(0)=1$ and $\Re(p(z))>0$, with series expansion

$$p(\tau)=1+\sum_{n=1}^{\infty}p_n\tau^n.$$

For n-fold symmetric functions in $\mathcal{P}$, we have

$$p(\tau)=1+p_{n+1}\tau^{n+1}+p_{2n+1}\tau^{2n+1}+\cdots.$$

Lemma 2.5 ([22]). *If $p(\tau)\in\mathcal{P}$, then*

$$|p_n|\leq 2,\quad n\in\mathbb{N},\qquad \textit{and}\qquad \left|\frac{p_2-p_1^2}{2}\right|\leq\frac{2-|p_1|^2}{2}.$$

Lemma 2.6 ([23]). *Let $l_1,l_2\in\mathbb{R}$ and $\varepsilon_1,\varepsilon_2\in\mathbb{C}$. If $|\varepsilon_1|<R$ and $|\varepsilon_2|<R$, then*

$$|(l_1+l_2)\varepsilon_1+(l_1-l_2)\varepsilon_2|\leq\begin{cases}2|l_1|R, & |l_1|\geq|l_2|,\\ 2|l_2|R, & |l_1|\leq|l_2|.\end{cases}$$

3 Main Results

This section addresses several well-known coefficient problems for n-fold symmetric bi-univalent functions in Σ_n, but now formulated within a *neutrosophic analytic framework*. In particular, we focus on the determination of sharp upper bounds for the initial Maclaurin coefficients and the establishment of generalized *neutrosophic Fekete–Szegő inequalities*. These problems are studied under the influence of the neutrosophic multi-parameters κ, ρ, γ, and β, which provide a flexible mechanism to regulate the contributions of truth-membership, indeterminacy-membership, and falsity-membership to the analytic and geometric behavior of the considered subclasses. In this way, the classical coefficient inequalities are extended, enriched, and unified through neutrosophic conditions, offering new insights into the structural properties of n-fold symmetric bi-univalent functions.

Fekete and Szegő were the first to present the Fekete–Szegő inequality[42] as follows.

$$\left|d_3 - \alpha d_2^2\right| \leq \begin{cases} 3-4\alpha, & \text{if } \alpha \leq 0, \\ 1+2e^{\frac{-2\alpha}{1-\alpha}}, & \text{if } 0 \leq \alpha \leq 1, \\ 4\alpha - 3, & \text{if } \alpha \geq 1. \end{cases}$$

The Fekete–Szegő inequality is applicable for $\left|a_{2n+1} - \alpha a_{n+1}^2\right|$ in the newly defined classes. Moreover, the theorems establish generalized Fekete–Szegő inequalities.

Theorem 3.1. *Let Y be given by (5) and $Y \in S_{\Sigma_n}(\kappa, \rho, \gamma)$, then*

$$|d_{n+1}| \leq \frac{4\kappa\rho}{n\left\{\sqrt{\Psi_1(\gamma, n, \rho) + 2\kappa(1-\rho)\left\{1+\gamma n(n+2)\right\}}\right\}},$$

$$|d_{2n+1}| \leq \frac{2\kappa\rho}{n(1+\rho)(1+\gamma n)} + \frac{8\kappa^2\rho^2(n+1)(1+2\gamma n)}{n^2(1+\rho)^2(1+\gamma n)^3},$$

and

$$\left|d_{2n+1} - \frac{\left\{1+\gamma n(n+2)\right\}(n-2\rho)}{4\rho(1+\gamma n)} d_{n+1}^2\right| \leq \frac{\rho\left(2\kappa + 2\kappa(\kappa-1)\right)}{n(1+\rho)(1+\gamma n)},$$

where $\Psi_1(\gamma, n, \rho)$ is given by (21).

Proof. Let $Y \in S_{\Sigma_n}(\kappa, \rho, \gamma)$, then

$$(1-\gamma)Q_1^\rho(Y,\tau) + \gamma Q_2^\rho(Y,\tau) = [p(\tau)]^\kappa \tag{7}$$

and for its inverse map $g = Y^{-1}$, we have

$$(1-\gamma)Q_3^\rho(g,w) + \gamma Q_4^\rho(g,w) = [q(w)]^\kappa, \tag{8}$$

where $p(\tau)$ and $q(w)$ have the following forms:

$$p(\tau) = 1 + p_n\tau^n + p_{2n}\tau^{2n} + ... \tag{9}$$

and

$$q(w) = 1 + q_n w^n + q_{2n} w^{2n} + \dots . \tag{10}$$

Now equating the coefficients in (7) and (8) we obtain

$$\frac{n(1+\rho)(1+\gamma n)}{2\rho} d_{n+1} = \kappa p_n, \tag{11}$$

$$\begin{aligned} &\frac{n(1+\rho)}{2\rho}\left\{2(1+\gamma n)d_{2n+1} - \mathcal{J}_3(n,\gamma)\, d_{n+1}^2\right\} + \mathcal{J}_2(n,\gamma)\, d_{n+1}^2 + ... \\ = \quad & \kappa p_{2n} + \frac{\kappa(\kappa-1)}{2} p_n^2 + ..., \end{aligned} \tag{12}$$

$$-\frac{n(1+\rho)(1+\gamma n)}{2\rho}d_{n+1}=\kappa q_n, \tag{13}$$

and

$$\begin{aligned}&\frac{n(1+\rho)}{2\rho}\left\{\mathcal{J}_1\left(n,\gamma\right)d_{n+1}^2-2\left(1+\gamma n\right)\right)d_{2n+1}\right\}+\mathcal{J}_2\left(n,\gamma\right)d_{n+1}^2\\ =\ &\kappa q_{2n}+\frac{\kappa(\kappa-1)}{2}q_n^2, \end{aligned}\tag{14}$$

where

$$\mathcal{J}_1\left(n,\gamma\right) = (2n+1)+\gamma n(3n+2), \tag{15}$$

$$\mathcal{J}_2\left(n,\gamma\right) = \frac{n^2(1-\rho)}{4\rho^2}\left\{1+\gamma n(n+2)\right\}, \tag{16}$$

$$\mathcal{J}_3\left(n,\gamma\right) = 1+\gamma n(n+2). \tag{17}$$

From (11) and (13) we obtain

$$p_n=-q_n, \tag{18}$$

and

$$\frac{n^2(1+\rho)^2(1+\gamma n)^2}{2\rho^2}d_{n+1}^2=\kappa^2(p_n^2+q_n^2). \tag{19}$$

Also form (12), (14), and (19) we have

$$\begin{aligned}&\left\{\frac{n^2(1+\rho)(1+\gamma n)}{\rho}+\frac{n^2(1-\rho)}{2\rho^2}\left\{1+\gamma n(n+2)\right\}\right\}d_{n+1}^2\\ =\ &\kappa(p_{2n}+q_{2n})+\frac{\kappa(\kappa-1)}{2}(p_n^2+q_n^2)\\ =\ &\kappa(p_{2n}+q_{2n})+\frac{\kappa(\kappa-1)}{2}\times\\ &\left\{\frac{n^2(1+\rho)^2(1+\gamma n)^2}{2\rho^2\kappa^2}d_{n+1}^2\right\}.\end{aligned}$$

Therefore we have

$$d_{n+1}^2=\frac{4\kappa^2\rho^2(p_{2n}+q_{2n})}{n^2\left[\Psi_1\left(\gamma,n,\rho\right)+2\kappa(1-\rho)\left\{1+\gamma n(n+2)\right\}\right]}, \tag{20}$$

where

$$\Psi_1\left(\gamma,n,\rho\right)=(1+\rho)(1+\gamma n)\left\{4\kappa\rho+(1+\rho)(1-\kappa)(1+\gamma n)\right\}. \tag{21}$$

Applying Lemma 1 on equation (20) to the coefficients p_{2n} and q_{2n}, we obtain

$$|d_{n+1}|\leq\frac{4\kappa\rho}{n\left\{\sqrt{\Psi_1\left(\gamma,n,\rho\right)+2\kappa(1-\rho)\left\{1+\gamma n(n+2)\right\}}\right\}}.$$

Next, in order to find the bound on $|d_{2n+1}|$, by subtracting (14) from (12), we obtain

$$\begin{aligned}&\frac{2n(1+\rho)(1+\gamma n)}{\rho}d_{2n+1}-\frac{n(1+\rho)(n+1)(1+2\gamma n)}{\rho}d_{n+1}^2\\ =\ &\kappa(p_{2n}-q_{2n})+\frac{\kappa(\kappa-1)}{2}(p_n^2-q_n^2).\end{aligned}$$

Then, in view of (18)

$$\frac{2n(1+\rho)(1+\gamma n)}{\rho}d_{2n+1}=\frac{n(1+\rho)(n+1)(1+2\gamma n)}{\rho}d_{n+1}^2+\kappa(p_{2n}-q_{2n}). \tag{22}$$

Using (19), and applying Lemma 2 to the coefficients p_{2n}, q_{2n}, p_n and q_n we have

$$|d_{2n+1}|\leq\frac{2\kappa\rho}{n(1+\rho)(1+\gamma n)}+\frac{8\kappa^2\rho^2(n+1)(1+2\gamma n)}{n^2(1+\rho)^2(1+\gamma n)^3}.$$

From (12), we have

$$\begin{aligned}&\frac{n(1+\rho)(1+\gamma n)}{\rho}\left[d_{2n+1}-\frac{\{1+\gamma n(n+2)\}}{2(1+\gamma n)}\left(\frac{n-2\rho}{2\rho}\right)d_{n+1}^2\right]\\=\;&\kappa p_{2n}+\frac{\kappa(\kappa-1)}{2}p_n^2+\ldots.\end{aligned}$$

Applying Lemma 1 to the coefficients p_{2n}, p_n and q_n we have

$$\begin{aligned}&\left|d_{2n+1}-\frac{\{1+\gamma n(n+2)\}}{2(1+\gamma n)}\left(\frac{n-2\rho}{2\rho}\right)d_{n+1}^2\right|\\\leq\;&\frac{\rho}{n(1+\rho)(1+\gamma n)}\left(2\kappa+2\kappa(\kappa-1)\right),\end{aligned}$$

which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $Y\in S_{\Sigma_n}(\kappa,\rho,\gamma)$, where Y is of the form (5). Then*

$$\begin{aligned}&\left|d_{2n+1}-\varphi(\kappa,\rho,\gamma,n)\,d_{n+1}^2\right|\\\leq\;&\begin{cases}2\varphi(\kappa,\rho,\gamma,n) & \text{for } |\mathcal{Q}(n,\gamma)-1|\geq\frac{n[\Psi_1(\gamma,n,\rho)+2\kappa(1-\rho)\{1+\gamma n(n+2)\}]}{8\kappa\rho(1+\rho)(1+\gamma\rho)},\\\frac{\kappa\rho}{n(1+\rho)(1+\gamma\rho)} & \text{for } |\mathcal{Q}(n,\gamma)-1|\leq\frac{n[\Psi_1(\gamma,n,\rho)+2\kappa(1-\rho)\{1+\gamma n(n+2)\}]}{8\kappa\rho(1+\rho)(1+\gamma\rho)},\end{cases}\end{aligned}$$

where $\mathcal{Q}(n,\gamma)$ and $\varphi(\kappa,\rho,\gamma,n)$ is given by (23) and (24).

Proof. From (20) and(22) it follows that

$$\begin{aligned}&d_{2n+1}-\varphi(\kappa,\rho,\gamma,n)\,d_{n+1}^2\\=\;&\left(\varphi(\kappa,\rho,\gamma,n)+\frac{\kappa\rho}{2n(1+\rho)(1+\gamma\rho)}\right)p_{2n}\\&+\left(\varphi(\kappa,\rho,\gamma,n)-\frac{\kappa\rho}{2n(1+\rho)(1+\gamma\rho)}\right)q_{2n},\end{aligned}$$

where

$$\mathcal{Q}(n,\gamma)=\frac{(n+1)(1+2\gamma n)}{2(1+\gamma n)} \tag{23}$$

and

$$\varphi(\kappa,\rho,\gamma,n)=\frac{4\kappa^2\rho^2(1-\mathcal{Q}(n,\gamma))}{n^2[\Psi_1(\gamma,n,\rho)+2\kappa(1-\rho)\{1+\gamma n(n+2)\}]}. \tag{24}$$

Then, applying for Lemma 1 and Lemma 2, we get

$$\begin{aligned}&\left|d_{2n+1}-\varphi(\kappa,\rho,\gamma,n)\,d_{n+1}^2\right|\\\leq\;&\begin{cases}2\varphi(\kappa,\rho,\gamma,n) & \text{if } |\varphi(\kappa,\rho,\gamma,n)|\geq\frac{\kappa\rho}{2n(1+\rho)(1+\gamma\rho)},\\\frac{\kappa\rho}{n(1+\rho)(1+\gamma\rho)} & \text{if } |\varphi(\kappa,\rho,\gamma,v)|\leq\frac{\kappa\rho}{2n(1+\rho)(1+\gamma\rho)},\end{cases}\end{aligned}$$

which yields the desired inequality. □

Theorem 3.3. *Let Y be given by (5) and Y is in the class $T_{\Sigma_n}(\beta,\rho,\gamma)$, then*

$$|d_{n+1}|\leq\frac{2\rho\sqrt{2(1-\beta)}}{n\left\{\sqrt{(2\rho^2+\rho+1)+\gamma n\{(1-\rho)n+2\rho^2\}}\right\}}, \tag{25}$$

and

$$|d_{2n+1}|\leq\frac{2\rho(1-\beta)}{n(1+\rho)(1+\gamma n)}+\frac{8\rho^2(n+1)(1+2\gamma n)(1-\beta)^2}{n^2(1+\rho)^2(1+\gamma n)^3}. \tag{26}$$

Proof. Let $Y \in T_{\Sigma_n}(\beta, \rho, \gamma)$, then

$$(1-\gamma)Q_1^{\rho}(Y,\tau) + \gamma Q_2^{\rho}(Y,\tau) = \beta + (1-\beta)p(\tau), \tag{27}$$

and for its inverse map $g = Y^{-1}$, we have

$$(1-\gamma)Q_3^{\rho}(g,w) + \gamma Q_4^{\rho}(g,w) = \beta + (1-\beta)q(w). \tag{28}$$

Where $p, q \in \mathcal{P}$ and $g = Y^{-1}$. Now, equating the coefficients in (27) and (28), we obtain

$$\frac{n(1+\rho)(1+\gamma n)}{2\rho} d_{n+1} = (1-\beta)p_n, \tag{29}$$

$$\begin{aligned} & \frac{n(1+\rho)}{2\rho}\left\{2(1+\gamma n)d_{2n+1} - \mathcal{J}_3(n,\gamma)\, d_{n+1}^2\right\} + \mathcal{J}_2(n,\gamma)\, d_{n+1}^2 \\ = \;& (1-\beta)p_{2n}, \end{aligned} \tag{30}$$

$$-\frac{n(1+\rho)(1+\gamma n)}{2\rho} d_{n+1} = (1-\beta)q_n, \tag{31}$$

$$\begin{aligned} & \frac{n(1+\rho)}{2\rho}\left\{\mathcal{J}_1(n,\gamma)\, d_{n+1}^2 - 2\left(1+\gamma n\right)\right) d_{2n+1}\right\} + \mathcal{J}_2(n,\gamma)\, d_{n+1}^2 \\ = \;& (1-\beta)q_{2n}, \end{aligned} \tag{32}$$

where$\mathcal{J}_1(n,\gamma)$, $\mathcal{J}_2(n,\gamma)$,and $\mathcal{J}_3(n,\gamma)$ is given by (15), (16), (17). From (29) and (31) we obtain

$$p_n = -q_n \tag{33}$$

and

$$\frac{n^2(1+\rho)^2(1+\gamma n)^2}{2\rho^2} d_{v+1}^2 = (1-\beta)^2(p_v^2 + q_v^2). \tag{34}$$

Adding (30) and (32), we have

$$\left\{\frac{n^2(1+\rho)(1+\gamma n)}{\rho} + \frac{n^2(1-\rho)}{2\rho^2}\left\{1+\gamma n(n+2)\right\}\right\} d_{n+1}^2 = (1-\beta)(p_{2n} + q_{2n}),$$

therefore we have

$$d_{v+1}^2 = \frac{2\rho^2(1-\beta)(p_{2n}+q_{2n})}{n^2\left\{(2\rho^2+\rho+1) + \gamma n\left\{(1-\rho)n + 2\rho^2\right\}\right\}}. \tag{35}$$

Applying Lemma 1 on equation (35) to the coefficients p_{2n} and q_{2n}, we obtain

$$|d_{n+1}| \le \frac{2\rho\sqrt{2(1-\beta)}}{n\left\{\sqrt{(2\rho^2+\rho+1) + \gamma n\left\{(1-\rho)n + 2\rho^2\right\}}\right\}}.$$

Next, in order to find the bound on $|d_{2n+1}|$, by subtracting (32) from (30), we obtain

$$\begin{aligned} & \frac{2n(1+\rho)(1+\gamma n)}{\rho} d_{2n+1} - \frac{n(1+\rho)(n+1)(1+2\gamma n)}{\rho} d_{n+1}^2 \\ = \;& (1-\beta)(p_{2n} - q_{2n}), \end{aligned} \tag{36}$$

$$d_{2n+1} = \frac{(n+1)(1+2\gamma n)}{2(1+\gamma n)} d_{n+1}^2 + \frac{\rho(1-\beta)(p_{2n}-q_{2n})}{2n(1+\rho)(1+\gamma n)}. \tag{37}$$

Then, in view of (33) and (34) , and applying Lemma 1 on equation (36) to the coefficients p_{2n}, q_{2n}, p_n and q_n we have

$$|d_{2n+1}| \le \frac{2\rho(1-\beta)}{n(1+\rho)(1+\gamma n)} + \frac{8\rho^2(1-\beta)^2(n+1)(1+2\gamma n)}{n^2(1+\rho)^2(1+\gamma n)^3}.$$

which completes the proof of Theorem 3.3. □

Theorem 3.4. *Let Y be given by (5) and Y is in the class $T_{\Sigma_n}(\beta,\rho,\gamma)$, then*

$$\begin{aligned}&\left|d_{2n+1}-\Upsilon(\kappa,\rho,\gamma,n)\,d_{n+1}^2\right|\\ \le&\begin{cases}2\Upsilon(\kappa,\rho,\gamma,n), & \text{for } |\mathcal{L}(n,\gamma)-1|\ge\frac{n\left\{(2\rho^2+\rho+1)+\gamma n\left\{(1-\rho)n+2\rho^2\right\}\right\}}{4(1+\rho)(1+\gamma n)\rho},\\ \frac{\rho(1-\beta)}{n(1+\rho)(1+\gamma n)}, & \text{for } |\mathcal{L}(n,\gamma)-1|\le\frac{n\left\{(2\rho^2+\rho+1)+\gamma n\left\{(1-\rho)n+2\rho^2\right\}\right\}}{4(1+\rho)(1+\gamma n)\rho}.\end{cases}\end{aligned}$$

Proof. From (35) and (37), we have

$$\begin{aligned}&d_{2n+1}-\Upsilon(\kappa,\rho,\gamma,n)\,d_{n+1}^2\\ =&\left(\Upsilon(\kappa,\rho,\gamma,n)+\frac{\rho(1-\beta)}{2n(1+\rho)(1+\gamma n)}\right)p_{2n}\\ &+\left(\Upsilon(\kappa,\rho,\gamma,n)-\frac{\rho(1-\beta)}{2n(1+\rho)(1+\gamma n)}\right)q_{2n},\end{aligned}$$

where

$$\mathcal{L}(n,\gamma)=\frac{(n+1)(1+2\gamma n)}{2(1+\gamma n)},$$

and

$$\Upsilon(\kappa,\rho,\gamma,n)=\frac{2\rho^2(1-\beta)\left(1-\mathcal{L}(n,\gamma)\right)}{n^2\left\{(2\rho^2+\rho+1)+\gamma n\left\{(1-\rho)n+2\rho^2\right\}\right\}}.$$

Using Lemma 1 and Lemma 2, we get

$$\begin{aligned}&\left|d_{2n+1}-\Upsilon(\kappa,\rho,\gamma,n)\,d_{n+1}^2\right|\\ \le&\begin{cases}2\varphi(\kappa,\rho,\gamma,n) & \text{if } |\varphi(\kappa,\rho,\gamma,n)|\ge\frac{\rho(1-\beta)}{2n(1+\rho)(1+\gamma n)},\\ \frac{\kappa\rho}{n(1+\rho)(1+\gamma\rho)} & \text{if } |\varphi(\kappa,\rho,\gamma,n)|\le\frac{\rho(1-\beta)}{2n(1+\rho)(1+\gamma n)},\end{cases}\end{aligned}$$

which yields the desired inequality. □

Remark 3.5. If $\gamma=0$, then Theorem 3.1 and Theorem 3.3 give the estimates of $|d_{n+1}|$ and $|d_{2n+1}|$ proved in.[43]

Remark 3.6. If $\gamma=0$ and $\rho=1$, then Theorem 3.1 and Theorem 3.3 give the estimates of $|d_{n+1}|$ and $|d_{2n+1}|$ proved in.[39]

Remark 3.7. If $\gamma=0$, $n=1$, and $\rho=1$, then Theorem 3.1 and Theorem 3.3 give the Corollary 10 and Corollary 11 proved in.[43]

Furthermore, we can obtain two new results in Corollary 1 and Corollary 2 by taking $\gamma=1$ in Theorem 3.1 and Theorem 3.3.

Corollary 3.8. *Let Y given by (5) be in the class $S_{\Sigma_n}(\kappa,\rho,1)$, then*

$$|d_{n+1}|\le\frac{4\kappa\rho}{n\left\{\sqrt{\Psi_2(\gamma,n,\rho)+2\kappa(1-\rho)\left\{1+n(n+2)\right\}}\right\}}$$

and

$$|d_{2n+1}|\le\frac{2\kappa\rho}{n(1+\rho)(1+n)}+\frac{8\kappa^2\rho^2(1+2n)}{n^2(1+\rho)^2(1+n)^2},$$

where

$$\Psi_2(\gamma,n,\rho)=(1+\rho)(1+n)\left\{4\kappa\rho+(1+\rho)(1-\kappa)(1+n)\right\}.$$

Corollary 3.9. *Let Y given by (5) be in the class $S_{\Sigma_n}(\beta,\rho,1)$, then*

$$|d_{n+1}|\le\frac{2\rho\sqrt{2(1-\beta)}}{n\left\{\sqrt{(2\rho^2+\rho+1)+n\left\{(1-\rho)n+2\rho^2\right\}}\right\}},$$

and

$$|d_{2n+1}|\le\frac{2\rho(1-\beta)}{n(1+\rho)(1+n)}+\frac{8\rho^2(1+2n)(1-\beta)^2}{n^2(1+\rho)^2(1+n)^2}.$$

Next, two corollaries below are obtained by taking $\gamma = 1$ and $\rho = 1$ in Theorem 3.1 and Theorem 3.3.

Corollary 3.10. *Let Y be given by (5) and $Y \in S_{\Sigma_n}(\kappa, 1, 1)$, then*

$$|d_{n+1}| \leq \frac{4\kappa}{n\left\{\sqrt{4(1+n)\left\{2\kappa + (1-\kappa)(1+n)\right\}}\right\}}$$

and

$$|d_{2n+1}| \leq \frac{\kappa}{n(1+n)} + \frac{2\kappa^2(1+2n)}{n^2(1+n)^2}.$$

Corollary 3.11. *Let Y given by (5) be in $S_{\Sigma_n}(\beta, 1, 1)$, then*

$$|d_{n+1}| \leq \frac{2}{n}\sqrt{\frac{(1-\beta)}{2+n}}$$

and

$$|d_{2n+1}| \leq \frac{(1-\beta)}{n(1+n)} + \frac{2(1+2n)(1-\beta)^2}{n^2(1+n)^2}.$$

In addition, if we take $n = 1$, $\rho = 1$, and $\gamma = 1$ in the previous Corollary 3 and Corollary 4, we can obtain the new results in the following two corollaries.

Corollary 3.12. *Let Y given by (2) be in $S_{\Sigma_1}(\kappa, 1, 1)$, then*

$$|d_2| \leq \kappa$$

and

$$|d_3| \leq \frac{\kappa}{2}(1+3\kappa).$$

Corollary 3.13. *Let Y given by (2) be in $S_{\Sigma_1}(\beta, 1, 1)$, then*

$$|d_2| \leq 2\sqrt{\frac{(1-\beta)}{3}},$$

$$|d_3| \leq \frac{(1-\beta)}{2}\left\{1 + 3(1-\beta)\right\}.$$

4 Conclusion

This study advances recent developments in Geometric Function Theory, particularly in the area of n-fold symmetric functions defined in the open unit disk, by introducing and exploring new classes.

The study was organized into three main parts.
The first part reviewed essential concepts in analytic functions, bi-univalent functions, and n-fold symmetric functions.
The second part constructed two subclasses of symmetric bi-univalent functions, laying the groundwork for subsequent analyses. Two established lemmas were also presented as analytical tools to examine our primary findings.
The core study in the third part addressed the coefficient-related problems for the newly defined subclasses of n-fold symmetric bi-univalent functions in the open unit disk. In particular, it revealed the maximum values of $|d_{n+1}|$ and $|d_{2n+1}|$, as well as the generalized Fekete–Szegő inequalities for these classes.

In addition to the classical framework, this work integrated a *neutrosophic perspective*. The four parameters κ, ρ, γ, and β were interpreted as neutrosophic weights capturing truth-membership, indeterminacy-membership, and falsity-membership. This interpretation provides a richer analytic environment for n-fold symmetric bi-univalent functions under uncertainty and incomplete information, thereby extending traditional geometric function theory into a neutrosophic setting. Such an approach not only generalizes existing subclasses but also establishes a flexible platform for exploring inequalities and coefficient bounds under vagueness.

Several promising avenues for future exploration emerge from this study. Potential directions include the application of neutrosophic analysis to specific classes of bi-univalent functions involving the Hohlov operator associated with the Legendre polynomial, integrodifferential operators, and the q-derivative operator. These extensions are expected to further enrich the understanding of symmetric bi-univalent functions, especially when examined within neutrosophic environments that model uncertainty and indeterminacy in complex systems.

Author contributions

All authors contribute equally to this study. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interest.

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