$\mathcal{I}_g^*$-Continues and $\mathcal{I}_g^*$-irresoluteness

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Abstract

In this paper, $\mathcal{I}_{\mathbf{g}}^*$-closed sets, and $\mathcal{I}_{\mathbf{g}}^*$-open are used to investigate and define a new class of functions is said to be $\mathcal{I}_{\mathbf{g}}^*$-Continues functions, $\mathcal{I}_{\mathbf{g}}^*$-irresolute functions in ideal topological space topological spaces. Morover, I introduce $\mathcal{I}_g^*$-compact spaces and $\mathcal{I}_g^*$-connected spaces, and maximal $\mathcal{I}_g^*$-closed sets. I obtain their characterizations and study their basic properties.

Keywords: Ideal topology; $\mathcal{I}_{\mathbf{g}}^*$-closed sets; $\mathcal{I}_{\mathbf{g}}^*$-irresolute functions; $\mathcal{I}_{\mathbf{g}}^*$-Continues, maximal $\mathcal{I}_g^*$-closed sets; $\mathcal{I}_g^*$-compact spaces

1 Introduction

Ideal topological spaces were studied in terms of ideals during the 1930s. Like most topics in topology, those spaces were analyzed and their Their importance arose through a paper by Vaidyanathaswamy. Decomposable continuity, separation axioms, connectedness, compactness, and resolvability are properties of topology as an algebraic system that Jankovic and Hamlett initiated the generalization of.[9] They used topological ideals to abstract over some of the central concepts in general topology. Many authors have studied these integration topologies via a systematic decomposition or separation of embedded spaces.

The type of generalized closed sets has been researched and is known in the literature of topological spaces. In works,[10,21,24] the concept of generalized closed sets has been utilized in formulating some separation axioms which are considered fundamental in digital topology (the digital line).[12] In the ideal topological space, Dontchev et al.[6] provided several characterizations of extremal disconnectedness in terms of generalized closed sets. This is motivated by the fact that such diagrams can be derived from generalized topological spaces, which is the focus of my research.

A nonempty collection of subsets of $\tilde{W}$ that meets the following criteria is an ideal $\mathcal{I}$ on a nonempty set $\tilde{W}$. According to,[25] if $Q \in \mathcal{I}$ and $B \subset Q$, then $B \in \mathcal{I}$. Additionally, if $Q \in \mathcal{I}$ and $B \in \mathcal{I}$, then $Q \cup B \in \mathcal{I}$. Jankovic and Hamlett,[9] Mukherjee et al.,[17] Arenas et al.,[3] Nasef and Mahmoud,[19] and others further explored applications in various fields.

Given an ideal topological space (W, Γ) with an ideal $\mathcal{I}$ on X, $\wp(\tilde{W})$ is the set of all subsets of X and a set operator $(.)^* : \wp(W) \to \wp(W)$, is said to be a local function[25] of Q with respect to Γ and $\mathcal{I}$, is denoted as follows: for $Q \subseteq X$,

$$Q^*(\mathcal{I}, \Gamma) = \left\{ w \in \tilde{W} \mid U \cap Q \notin \mathcal{I} \text{ for every } U \in \Gamma(w) \right\},$$

where $\Gamma(w) = \{U \in \Gamma \mid x \in U\}$. Furthermore, $Cl^*(Q) = Q \cup Q^*(\mathcal{I}, \Gamma)$ defines a
Kuratowski closure operator for the topology Γ^*, finer than Γ. When there is no chance for confusion, we will

simply write Q^* for $Q^*(\mathcal{I},\Gamma)$. $\tilde{W}^*$ is often a proper subset of $\tilde{W}$. By a space, we always mean a topological space $\left(\tilde{W},\Gamma\right)$ without any separation properties presumed. If $Q \subset \tilde{W}$ and $Cl(Q)$ and $Int(Q)$ are the closure and interior of Q in $\left(\tilde{W},\Gamma\right)$, respectively.

If $Q = Int(Cl^*(Q))$ (resp. $Q = Cl^*(Int(Q))$, then $R_{\mathcal{I}}$-open (resp. $R_{\mathcal{I}}$-closed)[27] is a subset Q of an ideal space $\left(\tilde{W},\Gamma\right)$. Q $\delta\mathcal{I}$-cluster point of Q is defined as a point $\tilde{w} \in \tilde{W}$ if $Int(Cl^*(U)) \cap Qeq\varnothing$ for each open set V containing $\tilde{w}$. $\delta Cl_I(Q)$ represents the $\delta\mathcal{I}$-closure of Q, which is the family of all $\delta\mathcal{I}$-cluster points of Q. $\delta Int_I(Q)$ represents the set $\delta\mathcal{I}$-interior of Q, which is the union of all $R_{\mathcal{I}}$-open sets of $\tilde{W}$ included in Q. If $\delta Cl_I(Q) = Q$,[27] then Q is $\delta\mathcal{I}$-closed.

A set operator $()^{*S} : P(\tilde{W}) \to P(\tilde{W})$ is called a semi local function and Cl^{*S}[1] of Q with respect to Γ and $\mathcal{I}$ are defined as follows:
For $Q \subset \tilde{W}$,

$$Q^{*S}(\mathcal{I},\Gamma) = \left\{\tilde{w} \in \tilde{W} \mid \tilde{U} \cap Q \notin \mathcal{I} \text{ for each semi-open } \tilde{V} \text{ containing } \tilde{w}\right\}$$

and $Cl^{*S} = Q \cup Q^{*S}$.

2 Preliminaries

In their work, Murugalingam[18] presented a new type of sets, known as g^*-closed sets, using a unique method. Then in 1990, S. P. Arya and T. Nour[4] introduced gs-closed sets, which helped in understanding s-normal spaces. Following that, researchers like Dontchev,[6] Gnanambal,[7] and Palaniappan and Rao[22] brought forward gsp-closed sets, gpr-closed sets, and rg-closed sets, respectively. In 1970, Yüksel introduced the concept of g-closed sets,[14] which serves as a broader category that sits between the more familiar closed sets and the class of g-closed sets.

Definition 2.1. Let $(\tilde{X},\Gamma)$ be a topological space and suppose a subset A of X, then A is said to be

1. generalized closed[14] if $Cl(A) \subseteq S$, $A \subseteq S$, and S is open in $(\tilde{X},\Gamma)$. Then the complement is said to be a g-open set of a g-closed set .
2. g semi closed[4] if $sCl(A) \subseteq S$, $A \subseteq S$, and S is open in $(\tilde{X},\Gamma)$.
3. α-generalized closed (briefly αg-closed)[15] if $\alpha Cl(A) \subseteq S$, $A \subseteq S$, and S is α-open in $(\tilde{X},\Gamma)$.
4. sg^*-closed[18] if $Cl(A) \subseteq S$, $A \subseteq S$, and S is semi-open in $(\tilde{X},\Gamma)$.
5. pg^*-closed[18] if $Cl(A) \subseteq S$, $A \subseteq S$, and S is pre-open in $(\tilde{X},\Gamma)$.
6. βg^*-closed[18] if $Cl(A) \subseteq S$, $A \subseteq S$, and S is semi-pre-open in $(\tilde{X},\Gamma)$.

Definition 2.2. A function $g : (\tilde{W},\Gamma) \longrightarrow (K,\sigma)$ is said to be

1. weakly continuous[14] if for all $\tilde{w} \in \tilde{W}$ and for evey open sets S in K containing $g(x)$, there exists an open set S containing $\tilde{w}$ such that $g(S) \subset Cl(S)$.
2. weakly pre-continuous[16] if for all $\tilde{w} \in \tilde{W}$ and for every open sets S in K containing $g(x)$, there exists a pre-open set S containing $\tilde{w}$ such that $g(S) \subset Cl(S)$.
3. weakly semi-continuous[11] if for all $\tilde{w} \in \tilde{W}$ and for all open set S in K containing $g(x)$, there exists an semi-open set S containing $\tilde{w}$ such that $g(S) \subset Cl(S)$.
4. weakly pre-$\mathcal{I}$-continuous[8] if for all $\tilde{w} \in \tilde{W}$ and for all open set S in K containing $g(x)$, there exists a pre-$\mathcal{I}$-open set S containing $\tilde{w}$ such that $g(S) \subset Cl(S)$.

Definition 2.3. 1. $\mathcal{I}_g$-closed[13] if $Q^* \subseteq S$, $Q \subseteq S$, and S is open in $(\tilde{W},\Gamma)$.

2. $sg\mathcal{I}$-closed[13] if $Q^{*S} \subseteq S$, $Q \subseteq S$, and S is semi open in $(\tilde{W}, \Gamma)$.
3. $\mathcal{I}_{s*g}$-closed[13] if $Q^* \subseteq S$, $Q \subseteq S$, and S is semi open in $(\tilde{W}, \Gamma)$.
4. $\mathcal{I} - sg$-closed[21] if $Q^* \subseteq S$, $Q \subseteq S$, and S is semi open in $(\tilde{W}, \Gamma)$.
5. $\mathcal{I} - pg$-closed[21] if $Q^* \subseteq S$, $Q \subseteq S$, and S is pre open in $(\tilde{W}, \Gamma)$.
6. $\mathcal{I} - \alpha g$-closed[21] if $Q^* \subseteq S$, $Q \subseteq S$, and S is α-open in $(\tilde{W}, \Gamma)$.
7. $*g\mathcal{I}$-closed[24] if $Q_* \subseteq S$, $Q \subseteq S$, and S is $\hat{g}$-open in $(\tilde{W}, \Gamma)$.
8. $\mathcal{I}*g$-closed[24] if $Q^* \subseteq S$, $Q \subseteq S$, and S is $\hat{g}$-open in $(\tilde{W}, \Gamma)$.

Definition 2.4. [2] An ideal space $(\tilde{W}, \mathcal{T}, \mathcal{I})$ is said to be

1. *-additive if $\left[\bigcup_{\alpha\in\Omega} F_\alpha\right]^* = \bigcup_{\alpha\in\Omega}(F_\alpha)^*$ for all indexing sets Ω where F_i s are subsets of $\tilde{W}$.
2. $\mathcal{I}^*_{\mathbf{g}}$- finitely additive (resp $\mathcal{I}^*_{\mathbf{g}}$- countable additive, $\mathcal{I}^*_{\mathbf{g}}$- additive) if finite union (resp countable union, arbitrary union) of $\mathcal{I}^*_{\mathbf{g}}$-closed sets is $\mathcal{I}^*_{\mathbf{g}}$-close.
3. *-finitely additive if $\left[\bigcup_{i=1}^{n} F_i\right]^* = \bigcup_{i=1}^{n}(F_i)^*$ for every positive integer n.
4. *-countably additive if $\left[\bigcup_{i=1}^{n} F_i\right]^* = \bigcup_{i=1}^{\infty}(F_i)^*$.

Similarly we define $\mathcal{I}^*_{\mathbf{g}}$- finitely multiplicative (resp$\mathcal{I}^*_{\mathbf{g}}$- multiplicative, $\mathcal{I}^*_{\mathbf{g}}$- countably multiplicative) if finite intersection (resp arbitrary intersection, countable intersection) of $\mathcal{I}^*_{\mathbf{g}}$-closed sets is $\mathcal{I}^*_{\mathbf{g}}$-closed.

$$\begin{tikzcd}
 & \text{closed} \arrow[r] \arrow[dl] & *\text{-closed} \arrow[dl] \\
*g\text{-closed} \arrow[r] \arrow[d] & \mathcal{I}_{*g}\text{-closed} \arrow[r] \arrow[d] & *g\mathcal{I}\text{-closed} \arrow[d] \\
\mathbf{g}\text{-closed} \arrow[r] & \mathcal{I}g\text{-closed} \arrow[r] & g\mathcal{I}\text{-closed} \\
\mathcal{I}_{s*g}\text{-closed} \arrow[r] \arrow[ur] & sg\mathcal{I}\text{-closed} \arrow[ur] &
\end{tikzcd}$$

Diagram 2.2.1 (*Generalized closed sets*)

Definition 2.5. [2] Let $(\tilde{W}, \Gamma)$ be a topological space and suppose a subset R of W, then F is called an $\mathcal{I}^*_{\mathbf{g}}$-closed if $Cl^*(R) \subset S$, $R \subset S$, and S is $\mathbf{g}$-open in $\tilde{W}$.

Definition 2.6. A function $g : (W, \Gamma, \mathcal{I}) \to (\Upsilon, \sigma, \mathcal{J})$ is said to be weakly $\mathcal{I}$-continuous[2] if for for all $w \in W$ and all open set S of Υ with $g(w) \in S$, there is an open set S with w
inS such that $g(R) \subset Cl^*(S)$.

$$\begin{tikzcd}
g^*\text{-closed} \arrow[r] \arrow[dr] & *g\text{-closed} \arrow[r] \arrow[dr] & \mathbf{g}\text{-closed} \arrow[dr] & \\
*\text{-closed} \arrow[r] & \mathcal{I}^*_g\text{-closed} \arrow[r] \arrow[dl] & \mathcal{I}_{*g}\text{-closed} \arrow[r] & \mathcal{I}g\text{-closed} \\
g^{*s}\mathcal{I}\text{-closed} & & &
\end{tikzcd}$$

Diagram 3.1.1 (Relation between generalized closed sets[2])

Theorem 2.7. [2] *If Q and K are $\mathcal{I}_g^*$-closed sets in an ideal space $(W, \Gamma, \mathcal{I})$, then $Q \cup K$ is also an $\mathcal{I}_g^*$-closed set.*

Theorem 2.8. [2] *For any *-finitely additive ideal space $(W, \Gamma, \mathcal{I})$, if V is semi open and Q is $\mathcal{I}_g^*$-open, then $Q \cap V$ is $\mathcal{I}_g^*$-open.*

3 Main Result

In this part, we share some theorems about breaking down continuity using the idea of $\mathcal{I}_g^*$-open sets.

Definition 3.1. A function $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is defined as being $\mathcal{I}_g^*$-continuous if for every $S \in \sigma$, the preimage $g^{-1}(S)$ is $\mathcal{I}_g^*$-open in W. In other words, for every closed set R in K, the preimage $g^{-1}(R)$ is $\mathcal{I}_g^*$-closed in W.

Definition 3.2. A function $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is termed strongly $\mathcal{I}_g^*$-continuous if for any $\mathcal{I}_g^*$-open (or $\mathcal{I}_g^*$-closed) set R in W, the preimage $g^{-1}(R)$ is open (or closed, respectively) in W.

Definition 3.3. A function $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is referred to as weakly $\mathcal{I}_g^*$-continuous if for each point $w \in W$ and for each set $R \in \sigma$ that contains $g(x)$, there exists a $\mathcal{I}_g^*$-open set S in W such that $w \in S$ and $g(S) \subseteq Cl^*(R)$.

Definition 3.4. A function $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is classified as $\mathcal{I}_g^*$-irresolute if for every $\mathcal{I}_g^*$-open (or $\mathcal{I}_g^*$-closed) set R in W, the preimage $g^{-1}(R)$ is $\mathcal{I}_g^*$-open (or $\mathcal{I}_g^*$-closed) in W.

Remark 3.5.

1. Any weakly $\mathcal{I}_g^*$-continuous function remains weakly $\mathcal{I}_g^*$-continuous.
2. Each strongly $\mathcal{I}_g^*$-continuous function is $\mathcal{I}_g^*$-irresolute. A $\mathcal{I}_g^*$-irresolute function is also $\mathcal{I}_g^*$-continuous. See example (3.6).
3. Any continuous function is $\mathcal{I}_g^*$-continuous since all open sets are $\mathcal{I}_g^*$-open. The reverse is not necessarily true, as shown in example (3.6).
4. Each strongly $\mathcal{I}_g^*$-continuous function is continuous, which means it is also $\mathcal{I}_g^*$-continuous. However, the converse does not hold, as shown in example (3.7).
5. Any $\mathcal{I}_g^*$-continuous function is weakly $\mathcal{I}_g^*$-continuous, but the reverse is not true, as illustrated in example (3.7).

Example 3.6. Let $W = \{a, b, c\}$ with a topology $\Gamma = \{\varnothing, W, \{a\}\}$ and an ideal $\mathcal{I} = \{\varnothing, \{a\}\}$, $K = W =, \Gamma = \sigma$. Let $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ be defined by $g(a) = c,\ f(b) = a,\ f(c) = a$. Then g is $\mathcal{I}_g^*$-continuous but not continuous. Since $g^{-1}(\{a\}) = \{b, c\}$ and $A^*(\{b, c\}) = \{b, c\}$.

Example 3.7. Let $W = K$ be indiscrete space and $\mathcal{I} = \{\varnothing, \{w_0\}\} = \mathcal{J}$, where $w_0 \in W$. Let $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ be identity map. K, $\varnothing$, $W - w_0$ are the only $\mathcal{I}_g^*$open sets in K. $g^{-1}(W - w_0) = W - w_0$ is not open in W. Therefore g is not strongly $\mathcal{I}_g^*$-continuous but g is both continuous and $\mathcal{I}_g^*$continuous.

Example 3.8. Let $W = K$ be indiscrete space and $\mathcal{I} = \{\varnothing, \{w_0\}\} = \mathcal{J}$, Let $K = W =, \Gamma = \sigma$. Let $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ be identity function.
Then g is an irresolute function. $A = \{w_0\}$ is $\mathcal{I}_g^*$-closed in K, but $g^{-1}(A) = \{w_0\}$ is not closed in W. Therefore g is not strongly $\mathcal{I}_g^*$-continuous.
Let $w_1 \neq w_0$ be a point of W and $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ be defined by $g(w_0) = w_1$, $g(w_1) = w_0$ and $g(x) = x$ $\forall x \neq w_0, w_1$. Then g is $\mathcal{I}_g^*$-continuous. $A = \{w_0\}$ is $\mathcal{I}_g^*$-closed in K and $g^{-1}(A) = \{w_1\}$ is not $\mathcal{I}_g^*$-closed in W. Therefore g is not $\mathcal{I}_g^*$-irresolute

$$
\begin{tikzcd}
\text{strong } \mathcal{I}_g^*\text{-continuity} \arrow[r] & \text{continuity} \arrow[r] \arrow[d] & \mathcal{I}_g^*\text{-continuity} \arrow[d] \\
 & \text{Weak } \mathcal{I}\text{ -continuity} \arrow[r] & \text{weak } \mathcal{I}_g^*\text{-continuity}
\end{tikzcd}
$$

Diagram 3.1.1 (*$\mathcal{I}_g^*$-continuous function*)

Definition 3.9. An ideal topological space $(W, \Gamma, \mathcal{I})$ is called an $\mathcal{I}_g^*$-multiplicative space if arbitrary intersection of $\mathcal{I}_g^*$-closed sets in $\tilde{W}$ is $\mathcal{I}_g^*$-closed.

Definition 3.10. Let M be a subset of an ideal space $(W, \Gamma, \mathcal{I})$ and $z \in W$ The subset M of W called a $\mathcal{I}_g^*$-open neighbourhood of z if there exists $\mathcal{I}_g^*$-open set R containing z such that $R \subseteq M$.

Theorem 3.11. *Let $(W, \Gamma, \mathcal{I})$ be an ideal space which is $\mathcal{I}_g^*$-multiplicative. Then the following are equivalent.*

1. *g is $\mathcal{I}_g^*$-continuous.*
2. *For each $w \in W$ and each open set R in K with $g(w) \in R$, there exists an $\mathcal{I}_g^*$-open set S containing w such that $g(S) \subset R$.*
3. *For each$w \in W$ and each open set R in K with $g(w) \in R$, $g^{-1}(R)$ is a $\mathcal{I}_g^*$-open neighbourhood of w.*

Proof. Since W is $\mathcal{I}_g^*$-multiplicative, this mean arbitrary union of $\mathcal{I}_g^*$-open sets is $\mathcal{I}_g^*$-open.
$(1) \Rightarrow (2)$ Let g is $\mathcal{I}_g^*$-continuous then for every $R \in K$, $g^{-1}(R)$ is $\mathcal{I}_g^*$-open in W. which mean for every $w \in W$ and R be open in K containing $g(w)$. Then $S = g^{-1}(R)$ is $\mathcal{I}_g^*$-open in W, and then $g(S) \subset R$.

$(2) \Rightarrow (3)$ Let$w \in W$, R open in K containing $g(w)$. By (2), there exists an $\mathcal{I}_g^*$-open set S containing w such that $g(S) \subset R$. Therefore $w \in S \subseteq g^{-1}(R)$ which proves $g^{-1}(R)$ is an $\mathcal{I}_g^*$-open neighbourhood of w.

$(3) \Rightarrow (1)$ Let R be open in K and $w \in g^{-1}(R)$. Then $g^{-1}(R)$ is a $\mathcal{I}_g^*$-open neighbourhood of w. Thus for each $w \in g^{-1}(R)$, there exists an $\mathcal{I}_g^*$-open set S_w containing w such that $w \in S_w \subseteq g^{-1}(R)$. Therefore $g^{-1}(R) = \bigcup_{w \in g^{-1}(R)} \bigcup_w$ is a $\mathcal{I}_g^*$-open this implies that $\mathcal{I}_g^*$ is continuous. □

Theorem 3.12. *Let $(W, \Gamma, \mathcal{I})$ be a $\mathcal{I}_g^*$-multiplicative ideal space in which every open set is *-closed. Then a function $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is weakly $\mathcal{I}_g^*$-continuous if and only if it is $\mathcal{I}_g^*$-continuous.*

Proof. Obviously $\mathcal{I}_g^*$-continuity $\Longrightarrow$ weakly $\mathcal{I}_g^*$-continuity. Conversely, let g be weakly $\mathcal{I}_g^*$-continuous. Then for each $w \in W$ and open set R containing $g(x)$, there exists $\mathcal{I}_g^*$-open set S such that $x \in S$ and $g(S) \subseteq Cl^*(R) = R$. Therefore by theorem 3.11, g is continuous □

Theorem 3.13. *Let $g : (W, \Gamma, \mathcal{I}_1) \longrightarrow (K, \sigma, \mathcal{I}_2)$ and $f : (K, \sigma, \mathcal{I}_2) \longrightarrow (Z, \eta, \mathcal{I}_3)$ be any two functions.Then the following satisfied.*

1. *$g \circ f$ is strongly $\mathcal{I}_g^*$-continuous if g is $\mathcal{I}_g^*$-continuous and g is continuous.*
2. *If g is $\mathcal{I}_g^*$-continuous and g is continuous this implies that $f \circ g$ is $\mathcal{I}_g^*$-continuous.*
3. *If g is strongly $\mathcal{I}_g^*$-continuous and g is $\mathcal{I}_g^*$-irresolute this implies that $f \circ g$ is strongly $\mathcal{I}_g^*$-continuous and irresolute.*
4. *If both g and g are $\mathcal{I}_g^*$-irresolute then $f \circ g$ is $\mathcal{I}_g^*$-irresolute.*

Proof. (2) and (4) are Obvious from definition.
(1) Let R be closed in Z. Because g is continuous, $g^{-1}(R)$ is closed in K. $\mathcal{I}_g^*$-continuity of g so $g^{-1}(g^{-1}(R))$ is $\mathcal{I}_g^*$-closed in W and thus $g \circ f$ is $\mathcal{I}_g^*$-continuous.
(3) Let R be $\mathcal{I}_g^*$-closed in Z. Hence g is $\mathcal{I}_g^*$-irresolute, $g^-1(R)$ is $\mathcal{I}_g^*$-closed in K. Because g is $\mathcal{I}_g^*$-irresolute, $g^{-1}(g^{-1}(R))$ is $\mathcal{I}_g^*$-closed in W and thus $f \circ g$ is $\mathcal{I}_g^*$-irresolute. □

Definition 3.14. A function $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is said to be strongly $\mathcal{I}_g^*$- continuous if for every $\mathcal{I}_g^*$ – open set R in $(K, \sigma, \mathcal{J})$, $g^{-1}(R)$ is open (W, Γ).

Theorem 3.15. *If $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ strongly $\mathcal{I}_g^*$- irresolute, then it is continuous.*

Theorem 3.16. *Let $(W, \Gamma, \mathcal{I})$ be an ideal topological space. $(K, \sigma, \mathcal{J})$ be a $G{*_I}$- space and let $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ be any function. Then, the following are comparable:*
(1) The function g is strongly $\mathcal{I}_g^*$- irresolute .
(2) The function g is continuous.

Proof. $(1) \Longrightarrow (2)$ Follows from the Theorem 3.15.
$(2) \Longrightarrow (1)$ Let S be any $\mathcal{I}_g^*$–open set in $(K, \sigma, \mathcal{J})$. Since $(K, \sigma, \mathcal{J})$ is a $G{*_I}$- space, S is open set in $(K, \sigma, \mathcal{J})$ and by hypothesis, $g^{-1}(S)$ is open set in $(W, \Gamma, \mathcal{I})$. Hence the function g is strongly $\mathcal{I}_g^*$- irresolute. □

Theorem 3.17. *A function is $g : (W, \Gamma) \to (K, \sigma, \mathcal{J})$ strongly $\mathcal{I}_g^*$- irresolute if and only if the inverse image of every $\mathcal{I}_g^*$– closed set in $(K, \sigma, \mathcal{J})$ is closed in (W, Γ).*

Theorem 3.18. *If $g : (W, \Gamma) \to (K, \sigma, \mathcal{J})$ is strongly continuous, then it is strongly $\mathcal{I}_g^*$- irresolute.*

Definition 3.19. An ideal topological space $(W, \Gamma, \mathcal{I})$ is called an $\mathcal{I}_g^*$- space if every subset in it is $\mathcal{I}_g^*$– closed .That is, $\Gamma^{\mathcal{I}_g^*} = P(W)$.

Theorem 3.20. *Suppose (W, Γ) be a discrete topological space, $(K, \sigma, \mathcal{J})$ be an $\mathcal{I}_g^*$- space and $g : (W, \Gamma) \to (K, \sigma, \mathcal{J})$ be any function. Therefore the following statements are comparable:*
(1) The function g is strongly continuous.
(2) The function g is strongly $\mathcal{I}_g^*$- irresolute.

Proof. $(1) \implies (2)$ Follows from Theorem 3.18.
$(2) \implies (1)$ Let S be any $\mathcal{I}_g^*$- open set in $(K, \sigma, \mathcal{J})$. Since $(K, \sigma, \mathcal{J})$ is an $\mathcal{I}_g^*$- space, S is an $\mathcal{I}_g^*$- open subset of $(K, \sigma, \mathcal{J})$ and by hypothesis, $g^{-1}(S)$ is open in (W, Γ). But (W, Γ) is a discrete topological space and then $g^{-1}(S)$ is also closed in (W, Γ). That is, $g^{-1}(S)$ is both closed and open in (W, Γ) and hence g is strongly continuous. ☐

Definition 3.21. A function $g : (W, \Gamma,$
$I) \to (K, \sigma,$
$J)$ is said to be perfectly $\mathcal{I}_g^*$-continuous if the preimage of each $\mathcal{I}_g^*$ – open set in $(K, \sigma, \mathcal{J})$ is both $\mathcal{I}_g^*$ open and $\mathcal{I}_g^*$ closed in $(W, \Gamma$.

Theorem 3.22. *If $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is perfectly $\mathcal{I}_g^*$- continuous, then it is $\mathcal{I}_g^*$-irresolute.*

Proof. As $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is ideally $\mathcal{I}_g^*$-continuous, $g^{-1}(S)$ is $\mathcal{I}_g^*$-open and $\mathcal{I}_g^*$-closed in (W, Γ) for each $\mathcal{I}_g^*$– open set S in $(K, \sigma, \mathcal{J})$. Thus, g is $\mathcal{I}_g^*$-irresolute. ☐

The following example demonstrates that the above theorem's converse is untrue.

Example 3.23. Let $W = K = \{a, b, c\}$ with a topology $\Gamma = \{\varnothing, W, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{\varnothing, \{a, b\}, K\}$ and an ideal $\mathcal{J} = \{\varnothing, \{a\}\}$.Defined $g : (W, \Gamma) \to (K, \sigma, \mathcal{J})$ be an identity function. This implies g is $\mathcal{I}_g^*$- irresolute. But the map g is not perfectly $\mathcal{I}_g^*$- continuous

Theorem 3.24. *Suppose $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ be any function and both $(W, \Gamma, \mathcal{I})$ and $(K, \sigma, \mathcal{J})$ be $G*_I$- spaces. Therefore the following statements are comparable:*
(1) g is strongly $\mathcal{I}_g^*$- irresolute .
(2) g is continuous.
(3) g is $\mathcal{I}_g^*$- irresolute .
(4) g is $\mathcal{I}_g^*$- continuous.

Theorem 3.25. *If $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is strongly continuous map, therefore it is perfectly $\mathcal{I}_g^*$- continuous.*

Proof. Since $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is strongly continuous, $g^{-1}(S)$ is both closed and open in (W, Γ) , for each $\mathcal{I}_g^*$– open set S in $(K, \sigma, \mathcal{J})$. Therefore, g is perfectly $\mathcal{I}_g^*$- continuous. ☐

Theorem 3.26. *Let (W, Γ) be a discrete topological space, $(K, \sigma, \mathcal{J})$ be any ideal topological space and $g : (W, \Gamma) \to (K, \sigma, \mathcal{J})$ be any function. Then the following statements are comparable:*
(1) g is strongly $\mathcal{I}_g^*$- irresolute . (2) g is perfectly $\mathcal{I}_g^*$- continuous.

Proof. $(1) \implies (2)$ Let S be any $\mathcal{I}_g^*$– open set in $(K, \sigma, \mathcal{J})$. By theory $g^{-1}(S)$ is open in (W, Γ). Hence (W, Γ) is a discrete space, $g^{-1}(S)$ is also closed in (W, Γ). This implies , $g^{-1}(S)$ is both closed and open in (W, Γ) and hence g is perfectly $\mathcal{I}_g^*$- continuous.
$(2) \implies (1)$ Its follows from Theorem 3.22. ☐

Theorem 3.27. *Let (W, Γ) be a discrete topological space and $(K, \sigma, \mathcal{J})$ be an $\mathcal{I}_g^*$- space and suppose the function $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$. Then the following statements are comparable:*
(1) g is strongly continuous.
(2) g is strongly $\mathcal{I}_g^*$- irresolute.
(3) g is perfectly $\mathcal{I}_g^*$- continuous.

Theorem 3.28. *Let $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ and $f : (K, \sigma, \mathcal{J}) \to (Z, \eta, K)$ are perfectly $\mathcal{I}_g^*$-continuous, where $\mathcal{I}$, $\mathcal{J}$ and K are three ideals on W, K and Z respectively. Then their composition $f \circ g : (W, \Gamma, \mathcal{I}) \to (Z, \eta, K)$ is also perfectly $\mathcal{I}_g^*$- continuous.*

Theorem 3.29. *Let $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ and $f : (K, \sigma, \mathcal{J}) \to (Z, \eta, K)$ be two functions, where $\mathcal{I}$, $\mathcal{J}$ and K are ideals on W, K and Z correspondingly. Then their composition $g \circ f : (W, \Gamma, \mathcal{I}) \to (Z, \eta, K)$ is*
(i) $\mathcal{I}_g^*$- continuous if g is$\mathcal{I}_g^*$- continuous and f is strongly continuous.
(ii) $\mathcal{I}_g^*$- irresolute if g is $\mathcal{I}_g^*$- continuous (or g is $\mathcal{I}_g^*$- irresolute) and f is perfectly $\mathcal{I}_g^*$- continuous.
(iii) strongly $\mathcal{I}_g^*$- irresolute if g is continuous (or g is strongly $\mathcal{I}_g^*$- continuous) and f is perfectly $\mathcal{I}_g^*$- continuous.
(iv) perfectly $\mathcal{I}_g^*$- continuous if g is perfectly $\mathcal{I}_g^*$- continuous and f is strongly continuous.

Theorem 3.30. *$g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is bijective, $*$- closed and $\mathcal{I}$ -irresolute, then the inverse function $g^{-1} : (K, \sigma, \mathcal{J}) \to (W, \Gamma, \mathcal{I})$ is $\mathcal{I}_g^*$- irresolute.*

Proof. Let A be $\mathcal{I}_g^*$- closed in $(W, \Gamma, \mathcal{I})$. Let $(g^{-1})^{-1}(A) = f(A) \subset S$ where S is semi- $\mathcal{I}$- open in $(K, \sigma, \mathcal{J})$. Then, $A \subseteq g^{-1}(S)$ holds. Since $g^{-1}(S)$ is semi- $\mathcal{I}$- open in $(W, \Gamma, \mathcal{I})$ and A is $\mathcal{I}_g^*$- closed in $(W, \Gamma, \mathcal{I})$, $cl^*(A) \subseteq g^{-1}(S)$ and hence $g(cl^*(A)) \subseteq S$. Since g is $*$-closed and $Cl^*(A)$ is $*$- closed in $(W, \Gamma, \mathcal{I})$, $g(cl^*(A))$ is $*$- closed in $(K, \sigma, \mathcal{J})$ and so $g(cl^*(A))$ is $\mathcal{I}_g^*$- closed in $(K, \sigma, \mathcal{J})$. Therefore $cl^*(g(cl^*(A))) \subseteq S$ and hence $cl^*(g(A)) \subseteq S$. Thus $g(A)$ is $\mathcal{I}_g^*$-closed in the space $(K, \sigma, \mathcal{J})$ and so g^{-1} is $\mathcal{I}_g^*$-irresolute □

Theorem 3.31. *Let $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{I})$ and $f : K, \sigma, \mathcal{I}) \implies (Z, \eta, K)$ are $\mathcal{I}_g^*$- irresolute, where $\mathcal{I}$, $\mathcal{J}$ and K are three ideals on W, K and Z correspondingly. Then their composition $f \circ g : (W, \Gamma, \mathcal{I}) \implies (Z, \eta, K)$ is also $\mathcal{I}_g^*$ - irresolute .*

Proof. Let S be an $\mathcal{I}_g^*$-open set in the space (Z, η, K). Since f is $\mathcal{I}_g^*$ - irresolute, $f^{-1}(S)$ is open in $(K, \sigma, \mathcal{I})$. Since $f^{-1}(S)$ is open, it is $\mathcal{I}_g^*$– open in $(K, \sigma, \mathcal{I})$. As g is also $\mathcal{I}_g^*$- irresolute $g^{-1}f^{-1}(S) = (f \circ g)^{-1}(S)$ is open in $(W, \Gamma, \mathcal{I})$ and so $(f \circ g)$ is $\mathcal{I}_g^*$- irresolute □

Theorem 3.32. *Let $(W, \Gamma, \mathcal{I})$ be *-finitely additive. Let $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma)$ be $\mathcal{I}_g^*$-continuous and S be $\mathcal{I}_g^*$-open in W. Then $g/S : (W, \Gamma_S, \mathcal{I}_S) \to (K, \sigma)$ is $\mathcal{I}_g^*$-continuous.*

Proof. Since $(W, \Gamma, \mathcal{I})$ is *-finitely additive, finite intersection $\mathcal{I}_g^*$-open sets is $\mathcal{I}_g^*$-open. Let R be open in (K, σ). Then $g^{-1}(R)$ is $\mathcal{I}_g^*$-open in W. Therefore $(g/S)^{-1}(R) = g^{-1}(R) \cap S$ is $\mathcal{I}_g^*$-open.Therefore $g(S)$ is $\mathcal{I}_g^*$-continuous.
Note: The result is true if $\mathcal{I}_g^*$-continuous is replaced by $\mathcal{I}_g^*$-irresolute. □

Theorem 3.33. *Let $(W, \Gamma, \mathcal{I})$ be an ideal space which is *-multiplicative finitely additive and $\mathcal{I}_g^*$-multiplicative. Then $f : (W, \Gamma, \mathcal{I}) \to (K, \sigma)$ is $\mathcal{I}_g^*$-continuous if and only if the graph of the mapping $f : W \to W \times K$ denoted as $f(x) = (w, g(w))$ for every $w \in W$ is $\mathcal{I}_g^*$-continuous*

Proof. Necessity: Let $w \in W$ and A be any open set in $W \times K$ containing $f(x) = (w, g(w))$. Then there exists basic open set $S \times R$ such that $f(w) \in S \times R \subseteq A$. Hence $f(w) \in R$. Since f is $\mathcal{I}_g^*$-continuous, there exists $\mathcal{I}_g^*$-open set S_1 containing W such that $x \in S_1$ and $g(S_1) \subset R$ (by theorem 3.11) and By theorem 2.8 $S_1 \cap R$ is $\mathcal{I}_g^*$-open in W. Then $w \in S_1 \cap R$ and $f(S_1 \cap R) \subseteq S \cap R \subset A$. Therefore f is $\mathcal{I}_g^*$-continuous.

Sufficiency: Suppose that there exists $g : W \to W \times K$ which is $\mathcal{I}_g^*$-continuous. Suppose that $w \in W$ and R is an open set in K such that $g(w) \in R$. Then $W \times R$ is an open set in $W \times K$. Since g is $\mathcal{I}_g^*$-continuous, there exists $\mathcal{I}_g^*$-open set S in W such that $w \in S$ and $g(S) \subseteq W \times R$. Thus $w \in S$ and $g(S) \subseteq R$ which proves that g is $\mathcal{I}_g^*$-continuous. □

Theorem 3.34. *Let $\{W_\gamma / \gamma \in \nabla\}$ be family of any topological spaces. If $g : (W, \Gamma, \mathcal{I}) \to \prod_{\gamma \in \nabla} W_\gamma$ is a $\mathcal{I}_g^*$-continuous, function, this implies $P_\gamma \circ g : W \to W_\gamma$ is $\mathcal{I}_g^*$-continuous for every $\gamma \in \nabla$ where P_γ is projection of $\prod W_\gamma$ onto W_γ.*

Proof. Consider a fixed $\gamma_0 \in \nabla$. Let G_{γ_0} be an open set in W_{γ_o}.Therefore $P_{\gamma_0}^{-1} G_{\gamma_0}$ is open in W_{γ_o}. (P_{γ_o} is continuous). Then $g^{-1}\left[P_{\gamma_0}^{-1} G_{\gamma_0}\right]$ is $\mathcal{I}_g^*$-open in W. This implies $P_{\gamma_o} \circ F$ is continuous. □

Theorem 3.35. *For any bijection $g^{-1} : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ the following statements are comparable.*

1. *$g^{-1} : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ is $\mathcal{I}_g^*$-continuous.*
2. *is $\mathcal{I}_g^*$-open in K for each open set S in W.*
3. *$g(S)$ is $\mathcal{I}_g^*$-closed in K for each closed set S in W.*

Proof. Obvious. □

4 $\mathcal{I}_g^*$-compact spaces and $\mathcal{I}_g^*$-connected spaces

In this chapter, we investigate ab=nd defined new types of functions called $\mathcal{I}_g^*$-compact and $\mathcal{I}_g^*$-connected. We consider and examine some characterizations and fundamental properties of these function classes. We also explore the relationships between these classes and the classes of $\mathcal{I}_g^*$-continuous.

Definition 4.1. The collection set $\left\{C_\gamma/\gamma \in \Omega\right\}$ of $\mathcal{I}_g^*$-open set in an ideal topological space W is called $\mathcal{I}_g^*$-open cover of a subset B of W if $B \subseteq \bigcup \left\{C_\gamma/\gamma \in \Omega\right\}$

Definition 4.2. An ideal topological space $(W, \Gamma, \mathcal{I})$ is called $\mathcal{I}_g^*$- compact modules $\mathcal{I}$, if for every $\mathcal{I}_g^*$-open cover $\left\{C_\gamma/\gamma \in \Omega\right\}$ of $(W, \Gamma, \mathcal{I})$, there exists a finite subset ω_0 and ω such that $W - \bigcup \left\{C_\gamma/\gamma \in \triangle_0\right\} \in \mathcal{I}$.

Theorem 4.3. *The image of $\mathcal{I}_g^*$-compact modulo $\mathcal{I}$ space $(W, \Gamma, \mathcal{I})$ under a $\mathcal{I}_g^*$-continuous subjective function g is $g(\mathcal{I})$-compact*

Proof. Let $(W, \Gamma, \mathcal{I})$ be a $\mathcal{I}_g^*$-compact modulo $\mathcal{I}$ space and $g : (W, \Gamma, \mathcal{I}) \longrightarrow (K, \sigma)$ be a subjective $\mathcal{I}_g^*$-continuous function. Then $g(\mathcal{I})$ is an ideal in (K, σ).
Let $\left\{C_\gamma/\gamma \in \Omega\right\}$ be an open cover for $(K, \sigma, f(\mathcal{I}))$
Then $g^{-1}(C_\gamma)$ is $\mathcal{I}_g^*$-open in W for every $\omega \in \gamma$ so $\left\{g^{-1}(C_\gamma)/\gamma \in \Omega\right\}$ is a $\mathcal{I}_g^*$-open cover for W. Since $(W, \Gamma, \mathcal{I})$ is $\mathcal{I}_g^*$-compact modulo $\mathcal{I}$, there exists a finite subset ω_0 of ω such that $W - \bigcup \left\{g^{-1}(C_\gamma)/\gamma \in \Omega_0\right\} \in \mathcal{I}$. Therefore $K - \bigcup \left\{g^{-1}(C_\gamma)/\gamma \in \Omega_0\right\} \in \mathcal{I}$, this proves that $(K, \sigma, f(\mathcal{I}))$ is compact modulo $g(\mathcal{I})$.
The following examples prove that there exist spaces which are $\mathcal{I}_g^*$-compact □

Example 4.4. Consider the space in example 3.6. To find $\mathcal{I}_g^*$-open covers we have to find $\left\{C_\gamma/\gamma \in \Omega\right\}$ then the only cover are $\{W\}$ and $\{W,\ W - \{x_0\}\}$. Hence the space is $\mathcal{I}_g^*$-compact modulo $\mathcal{I}$.

Example 4.5. Let $W = Z$, Γ-cofinite topology and $\mathcal{I} = \{\varnothing\}$.
Let for each positive integer n, $A_n = \{-n, -n+1, ...0, 1...., ..., n-1, n\}$. Then $\left\{A_n^c\right\}_{n=1}^{\infty}$ is a $\mathcal{I}_g^*$-open cover for Z. Suppose there exists a finite subset $\{\gamma_1, ...\gamma_k\}$ of positive integers such that $W - \bigcup_{i=1}^{k} A_{\gamma_i}^c = \varnothing$ then $W = \bigcup_{i=1}^{k} A_{\gamma_i}^c$ and hence $\bigcap_{i=1}^{n} A_{\gamma_i} = A_\gamma$ where $\gamma = \{\gamma_1, ...\gamma_k\}$ which is not true, since $C_\gamma = \{-\gamma, -\gamma+1, ...0, ..., ..., \gamma\}$. Therefore this space is not $\mathcal{I}_g^*$-compact modulo $\mathcal{I}$.

Definition 4.6. An ideal topological space $(W, \Gamma, \mathcal{I})$ is called $\mathcal{I}_g^*$-connected if W cannot be expressed as the disjoint union of two non-empty $\mathcal{I}_g^*$-open sets. A subset of W is considered $\mathcal{I}_g^*$-connected if it is $\mathcal{I}_g^*$-connected when viewed as a subspace. A space that is not $\mathcal{I}_g^*$-connected is referred to as $\mathcal{I}_g^*$-disconnected.

Remark 4.7. An ideal space $(W, \Gamma, \mathcal{I})$ is $\mathcal{I}_g^*$-disconnect if and only if there exists a proper subset which is both $\mathcal{I}_g^*$-open and $\mathcal{I}_g^*$-closed.

Theorem 4.8. $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ *be an onto function.*

1. *If g is continuous and W is $\mathcal{I}_g^*$-connected, this implies K is connected.*
2. *If g is irresolute and W is connected, therefore K is also $\mathcal{I}_g^*$-connected.*
3. *If g is strongly $\mathcal{I}_g^*$-continuous and W is connected, this implies K is $\mathcal{I}_g^*$-connected.*

Proof. (1) Suppose K is disconnected the K can be written as disjoint union of open sets A and B. Then $W = g^{-1}(K) = g^{-1}(A) \cup g^{-1}(B)$ which is a disjoint union of $\mathcal{I}_g^*$-open sets. This is a contradiction to the fact that W is $\mathcal{I}_g^*$-connected. Therefore K is connected.
(2) Similar to the proof of 1.
(3) Similar to the proof of 1 and 2. □

Definition 4.9. An ideal topological space $(W, \Gamma, \mathcal{I})$ is called $\mathcal{I}_g^*$-normal if for every pair of disjoint closed sets A and B of subset of $(W, \Gamma, \mathcal{I})$ there exists disjoint $\mathcal{I}_g^*$-open sets $S, R \subseteq W$ such that $A \subseteq S$ and $B \subseteq R$.

We now give examples of spaces which are $\mathcal{I}_g^*$-normal and not $\mathcal{I}_g^*$-normal.

Example 4.10. Let W be an finite set Γ-cofinite topology and $\mathcal{I} = \{\varnothing\}$. Here $\mathcal{I}_g^*$-open=$\{\varnothing, W, A/A^c \, is finite\}$. Suppose S and R are two disjoint $\mathcal{I}_g^*$-open sets then $S \cap R = \varnothing$. Therefore $S^c \cup S^c = W$ which is a contradiction since S^c and R^c are finite. Hence $(W, \Gamma, \mathcal{I})$ is not $\mathcal{I}_g^*$-normal.

In the above example, if $\mathcal{I} = P(W)$ then $(W, \Gamma, \mathcal{I})$ is $\mathcal{I}_g^*$-normal.

Theorem 4.11. *$g : (W, \Gamma, \mathcal{I}) \to (K, \sigma, \mathcal{J})$ be a injective and closed function.*

1. *If g is $\mathcal{I}_g^*$-continuous then K is normal $\Longrightarrow W$ is $\mathcal{I}_g^*$-normal.*
2. *If g is $\mathcal{I}_g^*$-irresolute then K is $\mathcal{I}_g^*$-normal $\Longrightarrow W$ is $\mathcal{I}_g^*$-normal.*
3. *If g is strongly $\mathcal{I}_g^*$-continuous then K is $\mathcal{I}_g^*$ $\Longrightarrow W$ is normal.*

Proof. (1) Let F_1 and F_2 be two disjoint closed sets in W. Since g is closed and injective, then $g(F_1)$ and $g(F_2)$ are disjoint closed sets in K. Then $\exists$ an open sets R_1 and R_2 in K such that $g(F_1) \subseteq R_1$ and $g(F_2) \subseteq R_2$. Then $F_1 \subseteq g^{-1}(R_1)$, $F_2 \subseteq g^{-1}(R_2)$ where $g^{-1}(R_1)$, $g^{-1}(R_2)$ are two disjoint $\mathcal{I}_g^*$-open sets in W. Hence W is $\mathcal{I}_g^*$-normal.
(2) The proof is similar of 1.
(3) The proof is similar of 1 and 2. □

$$\begin{tikzcd}
\mathcal{I}_g^*\text{-continuous} \arrow[r] & \text{continuity} \arrow[r] & \mathcal{I}_g^*\text{-normal} \arrow[r] & \text{normal} \\
 & \mathcal{I}_g^*\text{-irresolute} \arrow[u] \arrow[dr] & \text{strongly } \mathcal{I}_g^*\text{-continuous} \arrow[ur] \arrow[d] & \\
 & & \mathcal{I}_g^*\text{-connected} &
\end{tikzcd}$$

Diagram 4.1.1 ($\mathcal{I}_g^*$*-irresolute, connected, normal function*)

5 Maximal $\mathcal{I}_g^*$-closed sets

Definition 5.1. Let $\mathcal{I}_g^*$-closed S be a proper nonempty subset S of an ideal topological space $(W, \Gamma, \mathcal{I})$ is called maximal $\mathcal{I}_g^*$-closed if for each $\mathcal{I}_g^*$-closed set which is $\supseteq S$ will be W or S.

Example 5.2. Let $W = \{a, b, c\}$ with a topology $\Gamma = \{\varnothing, W, \{c\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\varnothing, \{a\}\}$. Then $\mathcal{I}_g^*$-closed sets are $\varnothing, W, \{a\}, \{a, b\}, \{a, c\}$. Here $\{a, b\}, \{a, c\}$ both of then are maximal $\mathcal{I}_g^*$-closed.

Remark 5.3. In fact, a maximal $\mathcal{I}_g^*$-closed set must certainly be a $\mathcal{I}_g^*$-closed set; however, conversely, a $\mathcal{I}_g^*$-closed set need not be a maximal $\mathcal{I}_g^*$-closed set.

Example 5.4. example5.2 R is a $\mathcal{I}_g^*$-closed set but not a maximal $\mathcal{I}_g^*$-closed set.

Theorem 5.5. *The following statements are true for arbitrary ideal space $(W, \Gamma, \mathcal{I})$.*

1. *Let S represent a maximal $\mathcal{I}_g^*$-closed set and R denote a $\mathcal{I}_g^*$-closed set. Then it follows that $S \cup R = W$ or $R \subseteq S$.*
2. *Let both S and R be maximal $\mathcal{I}_g^*$-closed sets. In this case, $S \cup R = W$ or $S = R$.*

Proof. (1) Assume S is a maximal $\mathcal{I}_g^*$-closed set and R is a $\mathcal{I}_g^*$-closed set. If $S \cup R = W$, there is nothing further to demonstrate. Suppose instead that $S \cup R \neq W$. It follows that $S \subset S \cup R$. According to theorem 2.7, $S \cup R$ is a $\mathcal{I}_g^*$-closed set. Riven that S is a maximal $\mathcal{I}_g^*$-closed set, we conclude that $S \cup R = W$ or $S \cup R = S$. Therefore, we have $S \cup R = S$, which implies that $R \subset S$.

(2) Now, let S and R be maximal $\mathcal{I}_g^*$-closed sets. If $S \cup R = W$, there is nothing further to prove. Assuming $S \cup R \neq W$, it follows from (1) that $S \subseteq R$ and $R \subseteq S$, leading to the conclusion that $S = R$. □

Definition 5.6. A space $(W, \Gamma, \mathcal{I})$ is classified as a gT^*-space given that each nonempty proper $\mathcal{I}_g^*$-closed subset of W qualifies as a maximal $\mathcal{I}_g^*$-closed set.

Example 5.7. Let $W = \{a, b, c, d\}, \Gamma = \{\varnothing, W, \{a\}, \{b, c, d\}\}, \mathcal{I} = \{\varnothing\}$. Then $\mathcal{I}_g^*$-closed sets are $\{\varnothing, W, \{a\}, \{b, c, d\}\}$. Here $\{a\}$ and $\{b, c, d\}$ are maximal $\mathcal{I}_g^*$-closed sets. Hence $(W, \Gamma, \mathcal{I})$ is a gT^*-space.

Definition 5.8. Suppose $g : (W, \Gamma) \to (K, \sigma, \mathcal{I})$ be any function then g is said to be maximal $\mathcal{I}_g^*$- continuous if $g^{-1}(R)$ is a maximal $\mathcal{I}_g^*$-closed set in W for every nonempty proper closed set R of K.

Theorem 5.9. *Every surjective maximal $\mathcal{I}_g^*$-continuous function is $\mathcal{I}_g^*$-continuous.*

Proof. Let $g : (W, \Gamma) \to (K, \sigma, \mathcal{I})$ be a surjective maximal $\mathcal{I}_g^*$-continuous function. The inverse image of $\varnothing$ and K are always $\mathcal{I}_g^*$-closed sets in W. Let R be a proper closed set in K. Now g is a maximal $\mathcal{I}_g^*$-continuous function implies $g^{-1}(R)$ is a maximal $\mathcal{I}_g^*$-closed set in W. Since every maximal $\mathcal{I}_g^*$-closed set is a $\mathcal{I}_g^*$-closed set, the proof follows. □

Remark 5.10. The converse of the aforementioned theorem is not necessarily true, as demonstrated by the following example.

Example 5.11. Let $W = K = \{a, b, c\}, \Gamma = \{\varnothing, W, \{a, b\}\}, \mathcal{I} = \{\varnothing, \{a\}\}$ and $\sigma = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, K\}$. Then the function $g : (W, \Gamma) \to (K, \sigma)$ defined by $g(a) = c$, $g(b) = a$ and $g(c) = b$ is $\mathcal{I}_g^*$-continuous, but not maximal $\mathcal{I}_g^*$-continuous.

Theorem 5.12. *If we have a function $g : (W, \Gamma) \to (K, \sigma)$ that is surjective and continuous in the sense of $\mathcal{I}_g^*$, with W being a gT^*-space, then we can say that g is a maximal continuous function in the same sense.*

Proof. Take a closed subset R of K that is both proper and not empty. Since g covers K entirely and maintains continuity under $\mathcal{I}_g^*$, the inverse image $g^{-1}(R)$ forms a proper closed subset of W that is also not empty in terms of $\mathcal{I}_g^*$. Given that W is a gT^*-space, it follows that $g^{-1}(R)$ is the largest closed subset in W relating to $\mathcal{I}_g^*$. Therefore, we conclude that g is a maximal continuous function in the sense of $\mathcal{I}_g^*$. □

Theorem 5.13. *If we have a $g : (W, \Gamma) \to (K, \sigma)$ that is maximal $\mathcal{I}_g^*$-continuous function and $f : (K, \sigma, \mathcal{J}) \longrightarrow (Z, \eta)$ be a continuous, surjective function. Then $g \circ f : (W, \Gamma, \mathcal{I}) \longrightarrow (Z, \eta)$ is a maximal $\mathcal{I}_g^*$-continuous function.*

Proof. Assume that R is a proper closed set in Z that is not empty. Given that f is continuous, $f^{-1}(R)$ is a nonempty proper closed set in K. It is implied that $g^{-1}(f^{-1}(R)) = (f \circ g)^{-1}(R)$ is a maximal $\mathcal{I}_g^*$-closed set in W since g is now maximal $\mathcal{I}_g^*$-continuous. A maximum $\mathcal{I}_g^*$-continuous function is thus $f \circ g$. □

Theorem 5.14. *If we have a $g : (W, \Gamma, \mathcal{I}) \to (K, \sigma)$ that is surjective, and $\mathcal{I}_g^*$-continuous function, where $f : (K, \sigma, \mathcal{J}) \longrightarrow (Z, \eta)$ is a surjective,continuous function and W is a gT^*-space. This implies $g \circ f : (W, \Gamma, \mathcal{I}) \longrightarrow (Z, \eta)$ is a maximal $\mathcal{I}_g^*$-continuous function.*

Proof. Assume that R is a proper closed set in Z that is not empty. $f^{-1}(R)$ is proper closed set in K that is not empty since f is continuous. g is now $\mathcal{I}_g*$-continuous denotes $g^{-1}(f^{-1}(R)) = (f \circ g)^{-1}(R)$ is a proper $\mathcal{I}_g^*$-closed set in W. $(f \circ g)^{-1}(R)$ is a maximal $\mathcal{I}_g^*$-closed set in W hence W is a gT^*-space. Therefore, $f \circ g$ is a maximal $\mathcal{I}_g^*$-continuous function. □

6 Conclusion

In this thesis, we developed the theory of ideal topology by constructing some new classes of operator and sets in order to study some characterizations and basic properties of these classes of operator and sets. The main achievements can be summarized as follows:

(a) We extended the class of g^*-closed sets to the class of $\mathcal{I}_g^*$-closed sets by using the notion of g-closed sets in topological space.

(b) A definitions of $\mathcal{I}_g^*$-closed sets has been obtained using the concept of g^*-closed sets and g-closed to obtain a decomposition of continuity via idealization, and investigate properties of $\mathcal{I}_g^*$-closed sets and strong $\mathcal{I}_g^*$-closed sets.

(c) A new strong form of continuous functions via $\mathcal{I}_g^*$-closed sets called, contra $\mathcal{I}_g^*$-continuous functions, strongly contra-e-$\mathcal{I}$-continuous functions. We also introduced and studied weaker notion of sets and functions, weakly $\mathcal{I}_g^*$-continuous.

(d) The relationships between the notions of Maximal $\mathcal{I}_g^*$-closed sets, $\mathcal{I}_g^*$-compact spaces and $\mathcal{I}_g^*$-connected spaces ideal topological space were obtained and discussed.

7 Suggestions for Further Work

In the end of this work we introduce some open problems:

- All notions and definitions will be extend in the future work via neutrosophic topological space and defined new concepts also via neutrosophic topological space such as neutrosophic g^*-closed sets, neutrosophic $\mathcal{I}_g^*$-closed, neutrosophic contra $\mathcal{I}_g^*$-continuous functions, neutrosophic strongly contra-e-$\mathcal{I}$-continuous functions and others.
- A new class of functions, called strongly faint $\mathcal{I}_g^*$-continuous function. Relationships among strongly faint $\mathcal{I}_g^*$-continuous functions and $\mathcal{I}_g^*$-connected spaces, $\mathcal{I}_g^*$-normal spaces and $\mathcal{I}_g^*$-compact spaces are investigated. Furthermore, the relationships between strongly faint $\mathcal{I}_g^*$-continuous functions and graphs are also investigated.
- We present $\mathcal{I}_g^*$-closed sets in two strong forms: $\mathcal{I}_g^*$-regular sets and $\mathcal{I}_g^*\theta$-closed sets. We also introduce a new class of functions, strongly $\mathcal{I}_g^*\theta$-continuous functions, which are a generalization of $\theta_{\mathcal{I}}$-pre continuous functions, using e-$\theta_{\mathcal{I}}$-open sets. We derive some characterizations of strongly $\mathcal{I}_g^*\theta$-continuous functions.

The research conducted in this thesis has uncovered numerous promising avenues for additional study. We plan to present new sets and operators pertaining to ideal minimal space, bitopological space, and topological groups. Several of these areas that merit further exploration can be succinctly outlined as follows:

(1) Bitopological space. One can continue to study the g^*-closed sets in bitopological space , the relatively new notion of quasi-g^*-closed sets is introduced and investigated. Hence, the notion of quasi-g^*-continuity between bitopological spaces is defined and a decomposition is provided. Moreover, we investigate a group of quasi-g^*-homeomorphisms and define several new bitopological spaces.

Our questions here are: What is the definition of g^*-closed sets for neutrosophic topological (bitopological, double fuzzy topological) space?

Can we extend the main properties in g^*-closed-sets for topological spaces to their analogical structures in g^*-closed for bitopological (bitopological , double fuzzt topological) space?

Author's contributions

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This article was written in collaboration with just one of the contributors (**Dr. Wadei Faris Al-omeri**). The final manuscript was read, written, and approved by the writers (**Dr. Wadei Faris Al-omeri**).

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