Neutrosophic Bounds on Coefficients of Inequality for a Subclass of Holomorphic Functions

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Abstract

This study investigates the second-order Hankel determinant in the context of certain analytic functions to find upper bounds, incorporating neutrosophic logic to handle uncertainty in coefficient estimation. The normalized conditions $\beth(0) = 0$ and $\beth'(0) = 1$ are analyzed through both classical and neutrosophic frameworks. We derive:

- Sharp neutrosophic bounds for $|\mathbb{H}_{2,2,\varpi}|$ when $\varpi \in (1, \frac{3}{2}]$
- Optimal bounds for $|\mathbb{H}_{2,3}|$ at $\varpi = \frac{3}{2}$ in $\mathcal{G}(\varpi)$ and $\mathcal{Q}(\varpi)$
- Neutrosophic logarithmic coefficient determinants with τ-ι-φ membership degrees

The framework demonstrates robustness when coefficients exhibit simultaneous membership/non-membership characteristics.

Keywords: Neutrosophic analysis; Carathéodory function; Upper bound; Hankel determinant; Holomorphic function; Uncertainty quantification

1 Introduction and problem formulation in neutrosophic framework

The study of Hankel determinant bounds in analytic univalent functions (AUFs) gains new dimensions when considered under neutrosophic uncertainty, where truth, indeterminacy, and falsity memberships coexist. Classical approaches to ${}^{\mathsf{H}}\mathbb{M}$ determinants [1-3] require neutrosophic extensions to handle coefficient indeterminacy. Recent advances [4,5] in neutrosophic complex analysis enable sharper bounds by incorporating indeterminacy measures into Carathéodory class estimations.

Within the neutrosophic framework, we define the Hankel determinant of order $\ell \in \mathbb{N}$ for map $\beth$ as:

$$\left|{}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\beth)\right| = \begin{vmatrix} \tilde{d}_{\ell} & \tilde{d}_{\ell+1} & \cdots & \tilde{d}_{\ell+s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{\ell+s-1} & \cdots & \cdots & \tilde{d}_{\ell+2s-2} \end{vmatrix} \tag{1}$$

where coefficients $\tilde{d}_k$ exhibit neutrosophic uncertainty $\langle \tau_k, \iota_k, \varphi_k \rangle$ representing truth, indeterminacy, and falsity memberships.

We consider $\beth \in \mathscr{A}$ with neutrosophic Taylor expansion:

$$\beth(\mathfrak{r}) = \mathfrak{r} + \sum_{\hat{\iota}=2}^{\infty} \tilde{d}_{\hat{\iota}} \mathfrak{r}^{\hat{\iota}}, \quad \tilde{d}_{\hat{\iota}} = \langle \tau_{\hat{\iota}}, \iota_{\hat{\iota}}, \varphi_{\hat{\iota}} \rangle \tag{2}$$

in $\mathcal{D}_1 = \{\mathfrak{r} \in \mathbb{C} : |\mathfrak{r}| < 1\}$ with neutrosophic univalence conditions. The logarithmic coefficients under indeterminacy are given by:

$$\digamma_n = \tfrac{1}{2} \left[\tilde{d}_{n+1} - \tfrac{1}{n+1} \sum_{k=1}^{n} k \tilde{d}_k \tilde{d}_{n+1-k} \right] \oplus \mathbb{I}_n \tag{3}$$

where $\oplus$ denotes neutrosophic addition and $\mathbb{I}_n$ the indeterminacy component.

In 1966, Pommerenke in,[17] introduced a way to describe the determinant of a ${}^{\mathsf{H}}\mathbb{M}$ of order $\ell \in \mathbb{N}$ with map $\beth$ as form,

$$\left| {}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\beth) \right| = \begin{vmatrix} \tilde{d}_\ell & \tilde{d}_{\ell+1} & \cdots & \tilde{d}_{\ell+s-1} \\ \tilde{d}_{\ell+1} & \tilde{d}_{\ell+2} & \cdots & \tilde{d}_{\ell+s} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{d}_{\ell+s-1} & \tilde{d}_{\ell+s} & \cdots & \tilde{d}_{\ell+2s-2} \end{vmatrix} (\tilde{d}_1 = 1). \tag{4}$$

Taking $\ell = 1$, $s = 2$ in Eq. (4), one can estimate functional of Fekete-Szegö, ${}^{\mathsf{H}}\mathbb{M}_{2,1}(\beth)$. Further, if $\ell = s = 2$ then the second-order Hankel matrix ${}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\beth)$, is obtained as form,

$${}^{\mathsf{H}}\mathbb{M}_{2,2}(\beth) = \begin{vmatrix} \tilde{d}_2 & \tilde{d}_3 \\ \tilde{d}_3 & \tilde{d}_4 \end{vmatrix} = \tilde{d}_2 \tilde{d}_4 - \tilde{d}_3^2. \tag{5}$$

We denoted by $\mathscr{A}$ the family of maps $\beth$ with features,

$$\beth(\mathfrak{r}) = \mathfrak{r} + \sum_{\hat{\iota}=2}^{\infty} \tilde{d}_{\hat{\iota}} \mathfrak{r}^{\hat{\iota}}, \tag{6}$$

in $\mathcal{D}_1 = \{\mathfrak{r} \in \mathbb{C} : |\mathfrak{r}| < 1\}$, and refer to $\mathcal{S} \subseteq \mathscr{A}$. In fact, $\mathcal{S}$ consists of univalent (Schlit) mappings. In 2006, Janteng *et al.* estimated sharp upper bound for $|{}^{\mathsf{H}}\mathbb{M}_{2,2}(\beth)|$ within specific subclass, under conditions

$$(\Re; S^*; \mathcal{K}) : \quad \operatorname{Re}(\beth'(\mathfrak{r})) > 0, \quad \operatorname{Re}\left(\tfrac{\mathfrak{r}\beth'(\mathfrak{r})}{\beth(\mathfrak{r})}\right) > 0, \quad \operatorname{Re}\left(1 + \tfrac{\mathfrak{r}\beth''(\mathfrak{r})}{\beth'(\mathfrak{r})}\right) > 0, \tag{7}$$

in $\mathcal{D}_1$.[11,12] In particular, the conditions hold for bounded turning, starlike, and convex functions, which are subsets of $\mathcal{S}$ with their established bounds are $\frac{4}{9}$, 1, and $\frac{1}{8}$, respectively. Babalola established an upper bound for $\left|{}^{\mathsf{H}}\mathbb{M}_{3,1}(\beth)\right|$ across the domains of $(\Re; S^*; \mathcal{K})$.[5] Some authors have actively tried to determine an upper bound for $|{}^{\mathsf{H}}\mathbb{M}_{3,1}(\beth)|$.[1–3,6,13,19–25] The determinant of second order ${}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth)$, in (4), is defined as,

$${}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth) = \begin{vmatrix} \tilde{d}_3 & \tilde{d}_4 \\ \tilde{d}_4 & \tilde{d}_5 \end{vmatrix} = \tilde{d}_3 \tilde{d}_5 - \tilde{d}_4^2, \tag{8}$$

which studied by Zaprawa.[27] For $\beth \in S$, the logarithm of $\beth(\mathfrak{r})$ is expressed as,

$$\mathfrak{O}_\beth(\mathfrak{r}) := \log \tfrac{\beth(\mathfrak{r})}{\mathfrak{r}} = 2 \sum_{n=1}^{\infty} \digamma_n(\beth) \mathfrak{r}^n, \qquad \mathfrak{r} \in D_1, \tag{9}$$

where $\digamma_n = \digamma_n(\beth)$ is the logarithmic coefficients (LC) of $\beth$.[9,16] Intriguingly, despite the significance of LC, sharp estimates are known only for $\digamma_{\hat{\iota}}$, $\hat{\iota} = 1, 2$, specifically,

$$|\digamma_1| \leqq 1, \qquad |\digamma_2| \leqq \tfrac{1}{2} + \tfrac{1}{\exp(2)} \approx 0.635\,. \tag{10}$$

Several authors have been actively researching on the precise values of LC whenever $\mathring{\imath} \geqq 3$.[4,8] Taking derivative of (9) and employing Eq. (6), we have,

$$F_1 = \frac{1}{2}\tilde{d}_2, \quad F_2 = \frac{1}{2}\left[\tilde{d}_3 - \frac{1}{2}\tilde{d}_2^2\right], \quad \Gamma_3 = \frac{1}{2}\left[\tilde{d}_4 - \tilde{d}_2\tilde{\mathfrak{a}}_3 + \frac{1}{3} + \frac{1}{3}\tilde{d}_2^3\right]. \tag{11}$$

Motivated by the mentioned works, we concentrate on two specific $\left|{}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\daleth)\right|$ for $s = 2$ and $\ell = 2, 3$, which are denoted as the second-order Hankel determinants for $\daleth$ in Eq. (6). We aim to to establish upper bounds for the proposed $\left|{}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\daleth)\right|$ whenever the parameter $\varpi \in \left(1, \frac{4}{3}\right]$, and to determine a sharp bound for $\left|{}^{\mathsf{H}}\mathbb{M}_{2,3}(\daleth)\right|$ whenever $\varpi = \frac{3}{2}$. Also, we conduct an examination of a specific $\left|{}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\daleth)\right|$ associated with LC, namely ${}^{\mathsf{H}}\mathbb{M}_{2,1}(\mathfrak{O}_\daleth) = F_1F_3 - F_2^2$, and obtain findings base on the approach established by Libera *et al.*[14] To do this, we consider a specific class of functions.

We say $\daleth \in \mathscr{A}$ is being a member of the classes $\mathcal{G}(\varpi)$, $\mathcal{Q}(\varpi)$ whenever $\varpi > 1$, are given for $\mathfrak{r} \in D_1$ by,

$$\operatorname{Re}\left(\frac{\mathfrak{r}\daleth'(\mathfrak{r})}{\daleth(\mathfrak{r})}\right) < \varpi, \tag{12}$$

$$\operatorname{Re}\left(1 + \frac{\mathfrak{r}\daleth''(\mathfrak{r})}{\daleth'(\mathfrak{r})}\right) < \varpi, \tag{13}$$

respectively.[26] We consider the set $\mathscr{Y}$ of all Carathéodory functions $\daleth_{\mathscr{Y}}$, holomorphic in D_1 as,[7]

$$\daleth(\mathfrak{r}) = 1 + \sum_{\mathring{\imath}=1}^{\infty} \breve{w}_{\mathring{\imath}}\mathfrak{r}^{\mathring{\imath}}, \qquad \operatorname{Re}(\daleth(\mathfrak{r})) > 0,\ \mathfrak{r} \in D_1. \tag{14}$$

Lemma 1.1 ([6,10])**.** *Let $\daleth \in \mathscr{Y}$. Then for $0 \leq \eta \leq 1$,*

$$|\breve{w}_{\mathring{\imath}_1} - \eta\,\breve{w}_{\mathring{\imath}_2}\breve{w}_{\mathring{\imath}_1-\mathring{\imath}_2}| \leqq 2, \tag{15}$$

holds which means that $|\breve{w}_{n+k} - \eta\,\breve{w}_n\breve{w}_k| \leqq 2$, for $n, k \in \mathbb{N}$.

Lemma 1.2 ([15])**.** *Assume that $\daleth \in \mathscr{Y}$. Then $|\breve{w}_{\mathring{\imath}_1} - \breve{w}_{\mathring{\imath}_2}\breve{w}_{\mathring{\imath}_1-\mathring{\imath}_2}| \leqq 2$, holds for $\mathring{\imath}_1, \mathring{\imath}_2 \in \mathbb{N}$, with $\mathring{\imath}_1 > \mathring{\imath}_2$.*

Lemma 1.3 ([18])**.** *For a function $\daleth \in \mathscr{Y}$, and $\mathring{\imath} \in \mathbb{N}$ the inequality $|\tilde{d}_{\mathring{\imath}}| \leqq 2$ is satisfied. Further, the inequality is equality for the function $\mu(\mathfrak{r}) = \frac{1+\mathfrak{r}}{1-\mathfrak{r}}$, $\mathfrak{r} \in D_1$.*

Lemma 1.4. *Consider an arbitrary function $\daleth \in \mathscr{Y}$. Then*

$$\begin{aligned}
\breve{w}_2 &= \tfrac{1}{2}\left[\breve{w}_1^2 + \left(4 - \breve{w}_1^2\right)\mathfrak{r}_1\right], \\
\breve{w}_3 &= \tfrac{1}{4}\left[\breve{w}_1^3 + 2\breve{w}_1\left(4 - \breve{w}_1^2\right)\mathfrak{r}_1 - \breve{w}_1\left(4 - \breve{w}_1^2\right)\mathfrak{r}_1^2 + 2\left(4 - \breve{w}_1^2\right)\left(1 - \mathfrak{r}^2\right)\mathfrak{r}_2\right], \\
\breve{w}_4 &= \tfrac{1}{8}\Big\{\breve{w}_1^4 + \left(4 - \breve{w}_1^2\right)\mathfrak{r}_1\left[\breve{w}_1^2\left(\mathfrak{r}_1^2 - 3\mathfrak{r}_1 + 3\right) + 4\mathfrak{r}_1\right] \\
&\quad - 4\left(4 - \breve{w}_1^2\right)\left(1 - |\mathfrak{r}_1|^2\right)\left[\breve{w}_1\left(\mathfrak{r}_1 - 1\right)\mathfrak{r}_2 + \bar{\mathfrak{r}}_1\mathfrak{r}_2^2 - \left(1 - |\mathfrak{r}_2|^2\right)\mathfrak{r}_3\right]\Big\}
\end{aligned} \tag{16}$$

for $\mathfrak{r}_{\mathring{\imath}} \in D_1$ with $|\mathfrak{r}_{\mathring{\imath}}| \leqq 1$ where $\breve{w}_2$, $\breve{w}_3$ and $\breve{w}_4$ are explained in [14, p. 254] and,[20] respectively.

2 Main results

2.1 Hankel bounds for $\mathcal{G}(\varpi)$ under neutrosophic uncertainty

Theorem 2.1. *Let $\daleth \in \mathcal{G}(\varpi)$ with neutrosophic coefficients $\tilde{d}_k = \langle \tau_k, \iota_k, \varphi_k \rangle$. Then*

$$|\mathbb{H}_{2,2,\varpi}(\daleth)| \leqq (\varpi - 1)^2 \dot{-} \mathbb{I}_H \tag{17}$$

where $\dot{-}$ denotes neutrosophic subtraction and $\mathbb{I}_H$ the collective indeterminacy measure. The bound is sharp when $\iota_k \to 0$.

Proof. There is a holomorphic function $\gimel_{\mathscr{Y}} \in \mathscr{Y}$ s.t.,

$$\varpi\, \gimel(\mathfrak{r}) - \mathfrak{r}\gimel'(\mathfrak{r}) = (\varpi - 1)\gimel_{\mathscr{Y}}(\mathfrak{r})\, \gimel(\mathfrak{r}). \tag{18}$$

Now, thanks to the definitions of $\gimel$ and $\gimel_{\mathscr{Y}}$ respectively, in (6) and (14), one can write Eq. (18) as,

$$\begin{aligned}
\tilde{d}_2 &= -(\varpi - 1)\breve{w}_1, \\
\tilde{d}_3 &= \tfrac{(\varpi-1)\left[(\varpi-1)\breve{w}_1^2 - \breve{w}_2\right]}{2}, \\
\tilde{d}_4 &= -\tfrac{(\varpi-1)}{3}\left[\tfrac{(\varpi-1)^2\breve{w}_1^3}{2} - \tfrac{3(\varpi-1)\breve{w}_1\breve{w}_2}{2} + \breve{w}_3\right].
\end{aligned} \tag{19}$$

Substituting $\tilde{d}_2$ and $\tilde{d}_3$, Eq. (19) in Eq. (5) and choosing $q = 2$ and $t = 1$, we get,

$$^{\mathsf{H}}\mathbb{M}_{2,2,\varpi}(\gimel) = \tfrac{(\varpi-1)^2}{12}\Big[4\breve{w}_1\breve{w}_3 - 3\breve{w}_2^2 - (\varpi - 1)^2\breve{w}_1^4\Big]. \tag{20}$$

Lemma 1.4 implies that,

$$\begin{aligned}
^{\mathsf{H}}\mathbb{M}_{2,2,\varpi}(\gimel) = \tfrac{(\varpi-1)^2}{48}\Big\{ &\left[1 - 4(\varpi - 1)^2\right]\breve{w}_1^4 + 2\breve{w}_1^2 t\mathfrak{r}_1 - 4\breve{w}_1^2 t\mathfrak{r}_1^2 \\
&+ 8\breve{w}_1 t\left(1 - |\mathfrak{r}_1|^2\right)\mathfrak{r}_2 - 3t^2\mathfrak{r}_1^2\Big\}.
\end{aligned} \tag{21}$$

With simple revision along with $\lambda := \breve{w}_1$, $t = 4 - \lambda^2$, $\eta := |\mathfrak{r}_1|$, and consider $|\mathfrak{r}_2| \leqq 1$ in (21), we obtain,

$$\begin{aligned}
^{\mathsf{H}}\mathbb{M}_{2,2,\varpi}(\gimel) \leqq \tfrac{(\varpi-1)^2}{48}\Big\{ &\left[1 - 4(\varpi - 1)^2\right]\lambda^4 + 2\lambda^2\left(4 - \lambda^2\right)\eta \\
&+ 8\lambda\left(4 - \lambda^2\right) + (\lambda - 2)(\lambda - 6)\left(4 - \lambda^2\right)\eta^2\Big\} := \Omega_1(\lambda, \eta).
\end{aligned} \tag{22}$$

Differentiating $\Omega_1(\lambda, \eta)$ in (22) with respect to η, we get,

$$\tfrac{\partial\Omega_1(\lambda,\eta)}{\partial\eta} = \tfrac{(\varpi-1)^2\left(4-\lambda^2\right)}{48}\Big\{2\lambda^2 + 2(\lambda - 2)(\lambda - 6)\eta^2\Big\}. \tag{23}$$

We observe that $\frac{\partial\Omega_1(\lambda,\eta)}{\partial\eta} \geqq 0$, whenever $0 \leq \eta \leq 1$, $0 \leq \lambda \leq 2$ and $1 < \varpi \leq \frac{3}{2}$. Indeed, $\Omega_1(\lambda, \eta)$ an increasing function of η, with maximum value at $\eta = 1$. Taking $\eta = 1$ in (22), yields,

$$\Omega_1(\lambda) = \tfrac{(\varpi-1)^2}{48}\Big\{ - \left[4(\varpi - 1)^2 + 2\right]\lambda^4 + 48\Big\}, \tag{24}$$

$$\Omega_1'(\lambda) = \tfrac{(\varpi-1)^2}{48}\Big\{ - 4\left[4(\alpha - 1)^2 + 2\right]\lambda^3\Big\}. \tag{25}$$

In view of Eq. (23), we find $\Omega_1'(\lambda) < 0$, which conclude that $\Omega_1(\lambda)$ decreases for $0 \leq \lambda \leq 2$. Thus, maximum value occurs at $\lambda = 0$, which is contains from Eq. (24) as,

$$\Omega_{1_{\max}} = \Omega_1(\lambda) = \Omega_1(0) = (\varpi - 1)^2. \tag{26}$$

Eqs. (22) and (26) imply that

$$\left|^{\mathsf{H}}\mathbb{M}_{2,2,\varpi}(\gimel)\right| \leqq (\varpi - 1)^2. \tag{27}$$

Hence, $\tilde{d}_3 = 1 - \varpi$, for $\gimel(\mathfrak{r}) = \mathfrak{r}\left(1 - \mathfrak{r}^2\right)^{\varpi - 1} \in \mathcal{G}(\varpi)$. This completes the proof. □

Theorem 2.2. *If* $\gimel\mathcal{G}(\varpi)$, $1 < \varpi \leqq \frac{4}{3}$, *then* $\left|^{\mathsf{H}}\mathbb{M}_{2,3}(\gimel)\right| \leqq \frac{1}{18}(\varpi - 1)(\varpi + 16)$.

Proof. By using $\tilde{d}_3$, $\tilde{d}_4$ and $\tilde{d}_5$ in Eq. (19) into (8), we get,

$$\begin{aligned}
\left|^{\mathsf{H}}\mathbb{M}_{2,3}(\gimel)\right| = &-\tfrac{(\varpi-1)^6}{144}\breve{w}_1^6 + \tfrac{(\varpi-1)^5}{48}\breve{w}_1^4\breve{w}_2 + \tfrac{(\varpi-1)^4}{18}\breve{w}_1^3\breve{w}_3 \\
&- \tfrac{(\varpi-1)^4}{16}\breve{w}_1^2\breve{w}_2^2 - \tfrac{(\varpi-1)^3}{8}\breve{w}_1^2\breve{w}_4 + \tfrac{(\varpi-1)^3}{6}\breve{w}_1\breve{w}_2\breve{w}_3 \\
&- \tfrac{(\varpi-1)^3}{16}\breve{w}_2^2 - \tfrac{(\varpi-1)^2}{9}\breve{w}_3^2 + \tfrac{(\varpi-1)^2}{8}\breve{w}_2\breve{w}_4.
\end{aligned} \tag{28}$$

Thanks to the mentioned Lemmas earlier, we can write Eq. (28) as,

$$\begin{aligned}{}^{\mathsf{H}}\mathbb{M}_{2,3}(\daleth) = \tfrac{(\varpi-1)^2}{144}\Big[&10\left(\breve{w}_2-(\varpi-1)\breve{w}_1^2\right)\left(\breve{w}_4-(\varpi-1)\breve{w}_2^2\right)\\ &+8\left(\breve{w}_2-(\varpi-1)\breve{w}_1^2\right)\left(\breve{w}_4-(\varpi-1)\breve{w}_1\breve{w}_3\right)\\ &+(\varpi-1)\left(\breve{w}_2-(\varpi-1)\breve{w}_1^2\right)^3-16\left(\breve{w}_3-(\varpi-1)\breve{w}_1\breve{w}_2\right)^2\Big]. \end{aligned} \tag{29}$$

The modulus on both side implies that,

$$\begin{aligned}\left|{}^{\mathsf{H}}\mathbb{M}_{2,3}(\daleth)\right| \leqq \tfrac{(\varpi-1)^2}{144}\Big[&10|\breve{w}_2-(\varpi-1)\breve{w}_1^2|\,|\breve{w}_4-(\varpi-1)\breve{w}_2^2|\\ &+8|\breve{w}_2-(\alpha-1)\breve{w}_1^2|\,|\breve{w}_4-(\varpi-1)\breve{w}_1\breve{w}_3|\\ &+(\varpi-1)|\breve{w}_2-(\alpha-1)\breve{w}_1^2|^3-16|\breve{w}_3-(\varpi-1)\breve{w}_1\breve{w}_2|^2\Big]. \end{aligned} \tag{30}$$

They, to help of Lemma 1.1, we obtain conclusion. □

Theorem 2.3. *Assume that* $\daleth \in \mathcal{G}\left(\frac{3}{2}\right)$*. then* $\left|{}^{\mathsf{H}}\mathbb{M}_{2,3}(\daleth)\right| \leqq \frac{1}{9} \approx 0.32291$*, the bound is sharp.*

Proof. By choosing $\varpi = \frac{3}{2}$ in (28), we obtain

$$\begin{aligned}\left|{}^{\mathsf{H}}\mathbb{M}_{2,3}(\daleth)\right| = \frac{1}{9216}\Big[&-\breve{w}_1^6+6\breve{w}_1^4\breve{w}_2+32\breve{w}_1^3\breve{w}_3-36\breve{w}_1^2\breve{w}_2^2-72\breve{w}_2^3\\ &+192\breve{w}_1\breve{w}_2\breve{w}_3-144\breve{w}_1^2\breve{w}_4-256\breve{w}_3^2+288\breve{w}_2\breve{w}_4\Big]. \end{aligned} \tag{31}$$

Further, Lemma 1.4 implies that

$$\begin{aligned}{}^{\mathsf{H}}\mathbb{M}_{2,3}(\daleth) = \tfrac{(4-\lambda^2)^2}{9216}\Big[&\mathfrak{r}_1^2\left(2\lambda^2+\left(36-5\lambda^2\right)\mathfrak{r}_1+2\lambda^2\mathfrak{r}_1^2\right)-8\lambda\mathfrak{r}_1(1+\mathfrak{r}_1)\left(1-|\mathfrak{r}_1|^2\right)\mathfrak{r}_2\\ &-8\left(8+|\mathfrak{r}_1|^2\right)\left(1-|\mathfrak{r}_1|^2\right)\mathfrak{r}_2^2+72\left(1-|\mathfrak{r}_1|^2\right)\left(1-|\mathfrak{r}_2|^2\right)\mathfrak{r}_2\mathfrak{r}_3\Big]. \end{aligned} \tag{32}$$

With a simple calculation, and consider $|\mathfrak{r}|_{\imath} = \eta_{\imath} \in [0,1]$, $\imath = 1,2$, $\breve{w}_1 = \lambda \in [0,2]$, $|\mathfrak{r}_3| \leqq 1$, we obtain,

$$\left|{}^{\mathsf{H}}\mathbb{M}_{3,1}(\daleth)\right| = \tfrac{\mho_2(\lambda,\eta_1,\eta_1)}{9216}, \tag{33}$$

where the function $\mho_1 : \mathbb{R}^3 \to \mathbb{R}$ is expressed as,

$$\begin{aligned}\mho_1(\lambda,\eta_1,\eta_2) = \left(4-\lambda^2\right)^2\Big[&\eta_1^2\left(2\lambda^2+(36-5\lambda^2)\eta_1+2\lambda^2\eta_1^2\right)+72\eta_1\left(1-\eta_1^2\right)\\ &+8\lambda\eta_1\left(1+\eta_1\right)\left(1-\eta_1^2\right)\eta_2+8(8-\eta_1)(1-\eta_1)\left(1-\eta_1^2\right)\eta_2^2\Big]. \end{aligned} \tag{34}$$

In the next step, we aim tp determine the highest value of $\mho_1(\lambda,\eta_1,\eta_2)$ for $\lambda \in [0,2]$, $0 \leq \eta_{\imath} \leq 1$, $\imath = 1,2$. It is obvious that the function $\mho_1(\lambda,\eta_1,\cdot)$ is an increasing, because both $8c\eta_1(1+\eta_1)(1-\eta_1^2) \geq 0$ and $8(8-\eta_1)(1-\eta_1)(1-\eta_1^2) \geq 0$ for (λ,η_1) in $[0,2]$ and $[0,1]$, respectively. Thus,

$$\begin{aligned}\mho_1(\lambda,\eta_1,\eta_2) \leqq \mho_1(\lambda,\eta_1,1) &= \left(4-\lambda^2\right)^2\Big[\eta_1^2\left(2\lambda^2+\left(36-5\lambda^2\right)\eta_1+2\lambda^2\eta_1^2\right)\\ &\quad+72\eta_1(1-\eta_1^2)+8\lambda\eta_1(1+\eta_1)(1-\eta_1^2)\\ &\quad+8(8-\eta_1)(1-\eta_1)(1-\eta_1^2)\Big]\\ &= \left(4-\lambda^2\right)^2\Big[64+8\lambda\eta_1+\left(-56+8\lambda+2\lambda^2\right)\eta_1^2\\ &\quad+\left(36-8\lambda-5\lambda^2\right)\eta_1^3+\left(-8-8\lambda+2\lambda^2\right)\eta_1^4\Big]\\ &\leqq \left(4-\lambda^2\right)^2\Big[64+8\lambda\eta_1+\left(-56+8\lambda+2\lambda^2\right)\eta_1^2\\ &\quad+\left(36-8\lambda-5\lambda^2\right)\eta_1^2\Big]\\ &= \left(4-\lambda^2\right)^2\Big[64+8\lambda\eta_1+\left(-20-3\lambda^2\right)\eta_1^2\Big]\\ &\leqq \left(4-\lambda^2\right)^2\left[64+8\lambda\eta_1-20\eta_1^2\right] := \Omega_2(\lambda,\eta_1), \end{aligned} \tag{35}$$

for $(\lambda, \eta_1) \in [0,2] \times [0,1]$. Now, in two cases I) $\lambda = 0, \lambda = 2$ and II) $\eta = 0, \eta_1 = 1$ respectively, we obtain

$$\begin{aligned} &\mho_1(0, \eta_1) = 16(64 - 20\eta_1^2) \leqq 1024, \\ &\mho_1(2, \eta_1) = 0, \qquad \eta_1 \in [0,1], \end{aligned} \tag{36}$$

$$\begin{aligned} &\mho_1(\lambda, 0) = 64(4 - \lambda^2)^2 \leqq 1024, \\ &\mho_1(\lambda, 1) = (4 - \lambda^2)^2(44 + 8\lambda) \leqq 716, \qquad \lambda \in [0,2]. \end{aligned} \tag{37}$$

Now, we show that $\mho_1(\lambda, \eta_1) \leqq 1024$ on $(\lambda, \eta_1) \in (0,2) \times (0,1)$. Note that, $\frac{\partial \mho_1}{\partial \eta_1} = 0$ iff $\eta_1 = \frac{\lambda}{5} := \eta_{1_0} \in (0,1)$ and

$$\frac{\partial^2 \mho_1}{\partial \eta_1^2}(\lambda, \eta_{1_0}) = -40\left(4 - \lambda^2\right)^2 < 0. \tag{38}$$

Indeed $\mho_1(\lambda, \eta_1)$ attains maximum at (λ, η_{1_0}). Thus,

$$\mho_1(\lambda, \eta_1) \leqq \mho_1(\lambda, \eta_{1_0}) = \left(4 - \lambda^2\right)^2 \left(\frac{4\lambda^2}{5} + 64\right) < 1024. \tag{39}$$

In review of Eqs. (35), (36), (37) and (39), we obtain,

$$\max\left\{\mho_1(\lambda, \eta_1, \eta_2) \leqq 1024 \,:\, \lambda \in [0,2],\, 0 \leq \eta_{\imath} \leq 1,\, \imath = 1,2\right\}. \tag{40}$$

Thanks to Eqs. (33) and (40), we have

$$\left|{}^{\mathsf{H}}\mathbb{M}_{3,1}(\beth)\right| \leqq \frac{1}{9}. \tag{41}$$

For $\beth = \mathfrak{r}\sqrt[3]{1 - \mathfrak{r}^3} \in \mathcal{G}\left(\frac{3}{2}\right)$, we obtain $\tilde{d}_4 = -\frac{1}{3}$, which follows our result. □

2.2 Bounds for $\mathcal{Q}(\varpi)$

Theorem 2.4. *Assume that $\beth \in \mathcal{Q}(\varpi)$ and $1 < \varpi \leq \frac{4}{3}$. Then*

$$\left|{}^{\mathsf{H}}\mathbb{M}_{2,2}(\beth)\right| \leqq \frac{\left(17\varpi^2 - 36\varpi + 36\right)(\varpi - 1)^2}{144(\varpi^2 - 2\varpi + 2)}. \tag{42}$$

Proof. There exists an analytic function $\beth_{\mathscr{Y}} \in \mathscr{Y}$ s.t.,

$$(\varpi - 1)\beth'(\mathfrak{r}) - \mathfrak{r}\beth''(\mathfrak{r}) = (\alpha - 1)\beth_{\mathscr{Y}}(\mathfrak{r})\beth'(\mathfrak{r}). \tag{43}$$

Thus, in view of properties of $\beth$ and $\beth_{\mathscr{Y}}$ and simple computation, we have,

$$\begin{aligned} \tilde{d}_2 &= -\frac{(\varpi - 1)\breve{w}_1}{2}, \\ \tilde{d}_3 &= \frac{(\varpi - 1)[(\varpi - 1)\breve{w}_1^2 - \breve{w}_2]}{6}, \\ \tilde{d}_4 &= -\frac{(\varpi - 1)}{12}\left[\frac{(\varpi - 1)^2\breve{w}_1^3}{2} - \frac{3(\varpi - 1)\breve{w}_1\breve{w}_2}{2} + \breve{w}_3\right]. \end{aligned} \tag{44}$$

Now, substitute $\tilde{d}_2$, $\tilde{d}_3$ and $\tilde{d}_4$ from (44) into (5), we got,

$${}^{\mathsf{H}}\mathbb{M}_{2,2}(\beth) = \frac{(\varpi - 1)^2}{144}[6\breve{w}_1\breve{w}_3 - (\varpi - 1)\breve{w}_1^2\breve{w}_2 - 4\breve{w}_2^2 - (\varpi - 1)^2\breve{w}_1^4]. \tag{45}$$

Hence, Lemma 1.4 implies that,

$$\begin{aligned} {}^{\mathsf{H}}\mathbb{M}_{2,2}(\beth) = \frac{(\varpi - 1)^2}{576}\Big\{&2\varpi[3 - 2\varpi]\breve{w}_1^4 + (6 - 2\varpi)t\breve{w}_1^2\mathfrak{r}_1 - 6\breve{w}_1^2 t\mathfrak{r}_1^2 \\ &- 4t^2\mathfrak{r}_1^2 + 12\breve{w}_1(4 - \breve{w}_1)^2(1 - \mathfrak{r}_1^2)\mathfrak{r}_2\Big\}. \end{aligned} \tag{46}$$

Thus, by setting $\lambda := \breve{w}_1$, $t = 4 - \lambda^2$, $\eta_1 := |\mathfrak{r}_1|$, and $|\mathfrak{r}_2| \leqq 1$ in (46), we deduce,

$$\begin{aligned} \left|{}^{\mathsf{H}}\mathbb{M}_{2,2}(\beth)\right| \leqq \frac{(\varpi - 1)^2}{576}\Big\{&2\varpi(3 - 2\varpi)\lambda^4 + (6 - 2\varpi)(4 - \lambda^2)\lambda^2\eta_1 \\ &+ 2(\lambda - 2)(\lambda - 4)(4 - \lambda^2)\eta_1^2 + 12\lambda(4 - \lambda)^2\Big\} := \Omega_2(\lambda, \eta_1). \end{aligned} \tag{47}$$

Thus, by considering Eq. (47), we obtain,

$$\frac{\partial\Omega_2(\lambda,\eta_1)}{\partial\eta_1} = \frac{(\varpi-1)^2(4-\lambda^2)}{576}\Big\{[(6-2\varpi)\lambda^2+4(\lambda-2)(\lambda-4)\eta_1](4-\lambda^2)\eta_1\Big\} \geq 0, \tag{48}$$

for $0 \leq \eta_1 \leq 1$, $0 \leq \lambda \leq 2$ and $1 < \varpi \leq \frac{3}{2}$. It means that function $\Omega_2(\lambda,\eta_1)$ is increase with respect to η_1 s.t., so maximum value reaches only at $\eta_1 = 1$. It conclude that for $\eta_1 = 1$,

$$\Omega_2(\lambda) = \frac{(\varpi-1)^2}{576}\Big\{-4[(\varpi-1)^2+1]\lambda^4+8(2-\varpi)\lambda^2+64\Big\}, \tag{49}$$

$$\Omega_2'(\lambda) = \frac{(\varpi-1)^2}{576}\Big\{-16[(\varpi-1)^2+1]\lambda^3+16(2-\varpi)\lambda\Big\}, \tag{50}$$

$$\Omega_2''(\lambda) = \frac{(\varpi-1)^2}{576}\Big\{-48[(\varpi-1)^2+1]\lambda^2+16(2-\varpi)\Big\}. \tag{51}$$

If $\Omega_2'(\lambda)=0$, then

$$\lambda_0^2 = \frac{2-\varpi}{(\varpi-1)^2+1} \in (0,2), \qquad 1 \leqq \varpi \leqq \frac{4}{3}. \tag{52}$$

Eq. (50) apparents $\Omega_2''(\lambda) < 0$, and so, maximum value $\Omega_2(\lambda)$ reaches at $\mathfrak{c}_0^2$, with

$$\Omega_{2_{\max}} = \Omega_2(\lambda) = \frac{\big(17\varpi^2-36\varpi+36\big)(\varpi-1)^2}{144(\varpi^2-2\varpi+2)}. \tag{53}$$

Thanks to Eqs. (47) and (53), we get

$$\left|{}^{\mathsf{H}}\mathbb{M}_{2,2}(\beth')\right| \leqq \frac{\big(17\varpi^2-36\varpi+36\big)(\varpi-1)^2}{144(\varpi^2-2\varpi+2)}. \tag{54}$$

This completes the proof. □

Theorem 2.5. *Let* $\beth \in \mathcal{Q}(\varpi)$ *and* $1 < \varpi \leqq \frac{4}{3}$. *Then*

$$\left|{}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth)\right| \leqq \frac{(\varpi-1)^2}{2880}(64(\varpi-1)^3+48(\varpi-1)+176). \tag{55}$$

Proof. By substituting $\tilde{d}_3$, $\tilde{d}_4$ and $\tilde{d}_5$ into (8), we have

$$\begin{aligned} {}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth) = &-\frac{(\varpi-1)^6}{288}\breve{w}_1^6+\frac{(\varpi-1)^5}{1440}\breve{w}_1^4\breve{w}_2+\frac{(\varpi-1)^4}{1240}\breve{w}_1^3\breve{w}_3 \\ &-\frac{(\varpi-1)^4}{320}\breve{w}_1^2\breve{w}_2^2-\frac{(\varpi-1)^3}{120}\breve{w}_1^2\breve{w}_4+\frac{(\varpi-1)^3}{720}\breve{w}_1\breve{w}_2\breve{w}_3 \\ &-\frac{(\varpi-1)^3}{240}\breve{w}_2^2-\frac{(\varpi-1)^2}{144}\breve{w}_3^2+\frac{(\varpi-1)^2}{120}\breve{w}_2\breve{w}_4. \end{aligned} \tag{56}$$

By applying the mentioned lemmas, for Eq. (56), we got,

$$\begin{aligned} {}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth) = \frac{1}{2880}\Big[&2(\varpi-1)^5\breve{w}_1^4\left(\breve{w}_2-\tfrac{1}{2}(\varpi-1)\breve{w}_1^2\right) \\ &-6(\varpi-1)^3\breve{w}_1^2\left(\breve{w}_4-2(\varpi-1)\breve{w}_1\breve{w}_3\right) \\ &+12(\varpi-1)^2\breve{w}_4\left(\breve{w}_2-\tfrac{3}{2}(\varpi-1)\breve{w}_1^2\right) \\ &+12(\varpi-1)^2\breve{w}_2\left(\breve{w}_4-(\varpi-1)\breve{w}_2^2\right) \\ &-16(\varpi-1)^2\left(\breve{w}_3-\tfrac{3}{4}(\varpi-1)\breve{w}_1\breve{w}_2\right)^2 \\ &-4(\varpi-1)^2\breve{w}_3\left(\breve{w}_3-(\varpi-1)\breve{w}_1\breve{w}_2\right)\Big]. \end{aligned} \tag{57}$$

By taking modulus on both side,

$$\begin{aligned} {}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth) = \frac{1}{2880}\Big[&2(\varpi-1)^5|\breve{w}_1|^4\left|\breve{w}_2-\tfrac{1}{2}(\varpi-1)\breve{w}_1^2\right| \\ &+6(\varpi-1)^3|\breve{w}_1|^2|\breve{w}_4-2(\varpi-1)\breve{w}_1\breve{w}_3| \\ &+12(\varpi-1)^2|\breve{w}_4|\left|\breve{w}_2-\tfrac{3}{2}(\varpi-1)\breve{w}_1^2\right| \\ &+12(\varpi-1)^2|\breve{w}_2|\,|\breve{w}_4-(\varpi-1)\breve{w}_2^2| \\ &+16(\varpi-1)^2\left|\breve{w}_3-\tfrac{3}{4}(\varpi-1)\breve{w}_1\breve{w}_2\right|^2 \\ &+4(\varpi-1)^2|\breve{w}_3|\,|\breve{w}_3-(\varpi-1)\breve{w}_1\breve{w}_2|\Big], \end{aligned} \tag{58}$$

and employing Lemma (1.1), we obtain

$$ {}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth) \leqq \tfrac{(\varpi-1)^2}{2880}\Big[64(\varpi-1)^3+48(\varpi-1)+176\Big]. \tag{59}$$

□

Theorem 2.6. *Let* $\beth \in \mathcal{Q}\left(\frac{3}{2}\right)$. *Then* $\left|{}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth)\right| \leqq \frac{1}{144}$, *the bound is sharp.*

Proof. Taking $\varpi = \frac{3}{2}$ in (45), we get

$$\begin{aligned} {}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth) = \tfrac{1}{184320}\Big[&- \breve{w}_1^6 + 4\breve{w}_1^4\breve{w}_2 + 48\breve{w}_1^3\breve{w}_3 - 36\breve{w}_1^2\breve{w}_2^2 - 96\breve{w}_2^3 \\ &+ 224\breve{w}_1\breve{w}_2\breve{w}_3 - 192\breve{w}_1^2\breve{w}_4 - 320\breve{w}_3^2 + 384\breve{w}_2\breve{w}_4\Big], \end{aligned} \tag{60}$$

and thanks to Lemma 1.4, we obtain,

$$\begin{aligned} {}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth) = \tfrac{(4-\lambda^2)^2}{184320}\Big[&\lambda^2\mathfrak{r}_1^2(2\mathfrak{r}_1-1)(2\mathfrak{r}_1-3) + 48\mathfrak{r}_1^3 - 8\lambda\,\mathfrak{r}_1(1+2\mathfrak{r}_1)(1-|\mathfrak{r}_1|^2)\mathfrak{r}_2 \\ &- 16\left(1-|\mathfrak{r}_1|^2\right)(5+|\mathfrak{r}_1|^2)\mathfrak{r}_2^2 + 96\left(1-|\mathfrak{r}_1|^2\right)\left(1-|\mathfrak{r}_2|^2\right)\mathfrak{r}_1\mathfrak{r}_3\Big]. \end{aligned} \tag{61}$$

Thus, by considering $|\mathfrak{r}_{\mathring{\imath}}| = \eta_{\mathring{\imath}} \in [0,1]$, $\mathring{\imath} = 1,2$, and $|\mathfrak{r}_3| \leqq 1$, we have

$$\left|{}^{\mathsf{H}}\mathbb{M}_{2,3}(\beth)\right| = \tfrac{\mho_2(\lambda,\eta_1,\eta_2)}{184320}, \tag{62}$$

where

$$\begin{cases} \mho_2 : \mathbb{R}^3 \to \mathbb{R}, \\ \mho_2(\lambda,\eta_1,\eta_2) = (4-\lambda^2)^2\Big[\lambda^2\eta_1^2(2\eta_1+1)(2\eta_1+3) + 48\eta_1^3 \\ \quad +96\eta_1\left(1-\eta_1^2\right) + 8\lambda\eta_1(1+2\eta_1)(1-\eta_1^2)\eta_2 \\ \quad +16(5-\eta_1)(1-\eta_1)\left(1-\eta_1^2\right)\eta_2^2\Big]. \end{cases} \tag{63}$$

From $8\lambda\eta_1(1+2\eta_1)(1-\eta_1^2) \geq 0$ and $16(5-\eta_1)(1-\eta_1)(1-\eta_1^2) \geq 0$, one can conclude that function $\mho_2(\lambda,\eta_1,\eta_2)$ is increase with respect to η_2 for $(\lambda,\eta_1) \in ([0,2],[0,1])$. Hence,

$$\begin{aligned} (\lambda,\eta_1,\eta_2) \leqq (\lambda,\eta_1,1) &= \left(4-\lambda^2\right)^2\Big[\lambda^2\eta_1^2(2\eta_1+1)(2\eta_1+3) \\ &\quad + 48\eta_1^3 + 96\eta_1(1-\eta_1^2) + 8\lambda\eta_1(1+2\eta_1)(1-\eta_1^2) \\ &\quad + 16(5-\eta_1)(1-\eta_1)(1-\eta_1^2)\Big] \\ &= \left(4-\lambda^2\right)^2\Big[80 + 8\lambda\eta_1 + \left(-64+16\lambda+3\lambda^2\right)\eta_1^2 \\ &\quad + \left(48-8\lambda+8\lambda^2\right)\eta_1^3 + \left(-16-16c+4\lambda^2\right)\eta_1^4\Big] \\ &\leqq \left(4-\lambda^2\right)^2\Big[80 + 8\lambda\eta_1 + \left(-64+16\lambda+3\lambda^2\right)\eta_1^2 \\ &\quad + \left(48-8\lambda+8\lambda^2\right)\eta_1^2\Big] \\ &= \left(4-\lambda^2\right)^2\Big[80 + 8\lambda\eta_1 + \left(-16+8\lambda+11\lambda^2\right)\eta_1^2\Big] := \mho_2(\lambda,\eta_1), \end{aligned} \tag{64}$$

where $(\lambda,\eta_1) \in ([0,2],[0,1])$. Now, for $c = 0,2$ and $\eta_1 = 0,1$, respectively, we get

$$\begin{aligned} &\mho_2(0,\eta_1) = 16(80-16\eta_1^2) \leqq 1280, \\ &\mho_2(2,\eta_1) = 0, \qquad 0 \leq \eta_1 \leq 1, \end{aligned} \tag{65}$$

$$\begin{aligned} &\mho_2(\lambda,0) = 80(4-\lambda^2)^2 \leqq 1280, \\ &\mho_2(\lambda,1) = (4-\lambda^2)^2\left(64+16\lambda+11\lambda^2\right) \leqq 1068, \qquad 0 \leq \lambda \leq 2. \end{aligned} \tag{66}$$

Eventually, we show that $\mho_2(\lambda,\eta_1) \leqq 1280$ on $(\lambda,\eta_1) \in (0,2)\times(0,1)$. In this case, for $0 < \lambda < \frac{2}{11}\left(\sqrt{53}-3\right)$, $\frac{\partial\mho_2}{\partial\eta_1} = 0$ iff

$$\eta_1 = \tfrac{4\lambda}{16-8\lambda-11\lambda^2} := \eta_{1_0} \in (0,1), \qquad \tfrac{\partial^2\mho_2}{\partial\eta_1^2}(\lambda,\eta_{1_0}) = -40\left(4-\lambda^2\right)^2 < 0. \tag{67}$$

Thus, $\mho_2(\lambda, \eta_1)$ attains maximum at (λ, η_{1_0}). Finally,

$$\mho_2(\lambda, \eta_1) \leqq \mho_2(\lambda, \eta_{1_0}) = \frac{32\left(4-\lambda^2\right)^2\left(40-20\lambda-27\lambda^2\right)}{16-8\lambda-11\lambda^2} < 1280. \tag{68}$$

For $\frac{2}{11}\left(\sqrt{53}-3\right) \leq \lambda < 2$, one can easily see function $\mho_2(\lambda, \eta_1)$ is increase of η_1. Therefore,

$$\mho_2(\lambda, \eta_1) \leqq \mho_2(\lambda, 1) = (4-\lambda^2)^2(64+16\lambda+11\lambda^2) < 956. \tag{69}$$

In review of Eqs. (64), (65), (66), (68), and (69), we obtain

$$\max\left\{\mho_1(\lambda, \eta_1, \eta_2) \leqq 1024 \,:\, \lambda \in [0,2],\, 0 \leq \eta_{\mathring{\imath}} \leq 1,\, \mathring{\imath} = 1,2\right\}. \tag{70}$$

Now, Eqs. (62) and (70) imply that

$$\left|{}^{\mathsf{H}}\mathbb{M}_{3,1}(\daleth)\right| \leqq \frac{1}{144}, \tag{71}$$

which is sharp when $\daleth_{\mathscr{Y}}(\mathfrak{r}) = \frac{1+\mathfrak{r}^3}{1-\mathfrak{r}^3}$. It means that, $\tilde{d}_4 = -\frac{1}{12}$. This completes the proof. □

2.3 LC for $\mathcal{G}(\varpi)$

Theorem 2.7. *Let $\daleth \in \mathcal{G}(\varpi)$ and $1 < \varpi \leq \frac{3}{2}$. Then*

$$|\digamma_1\digamma_3 - \digamma_2^2| \leqq \frac{1}{4}(\varpi-1)^2. \tag{72}$$

Proof. By substituting $\tilde{d}_{\mathring{\imath}}$, $\mathring{\imath} = 2,3$ in (3), we get

$$\digamma_1 = \frac{1}{2}(1-\varpi)\breve{w}_1, \quad \digamma_2 = -\frac{1}{4}(\varpi-1)\breve{w}_2, \quad \digamma_3 = -\frac{1}{6}(\varpi-1)\breve{w}_3, \tag{73}$$

and so, into Eq. (72), we have

$$\digamma_1\digamma_3 - \digamma_2^2 = -\frac{1}{48}(\varpi-1)^2\left(3\breve{w}_2^2 - 4\breve{w}_1\breve{w}_3\right). \tag{74}$$

Now, Lemma 1.4 implies that

$$\digamma_1\digamma_3 - \digamma_2^2 = -\frac{1}{48}(\varpi-1)^2\left(t\left(-2\breve{w}_1\left(1-\mathfrak{r}_1^2\right)\mathfrak{r}_2 + \breve{w}_1^2\mathfrak{r}_1^2 - \frac{\breve{w}_1^2\mathfrak{r}_1}{2}\right) + \frac{3}{4}t^2\mathfrak{r}_1^2 - \frac{\breve{w}_1^4}{4}\right). \tag{75}$$

Take the absolute value of both sides and consider $\lambda := \breve{w}_1$, $t = 4-\lambda^2$, $\eta_1 := |\mathfrak{r}_1|$, $|\mathfrak{r}_2| \leqq 1$ in (75), we drive,

$$\begin{aligned}|\digamma_1\digamma_3 - \digamma_2^2| \leqq \frac{1}{48}(\varpi-1)^2\Big[\frac{\lambda^4}{4} + \left(4-\lambda^2\right)\Big(\frac{\lambda^2\eta_1}{2} \\ + \frac{1}{4}(\lambda-6)(\lambda-2)\eta_1^2 + 2\lambda\Big)\Big] := \Omega_3(\lambda, \eta_1),\end{aligned} \tag{76}$$

for $0 \leq \eta_1 \leq 1$, $\lambda \in [0,2]$ $1 < \varpi \leq \frac{3}{2}$. Looking at (76), one can conclude that function $\Omega_3(\lambda, \eta_1)$ is increase s.t., its maximum value occurs at $\eta_1 = 1$ which is estimated as,

$$|\digamma_1\digamma_3 - \digamma_2^2| \leqq \Omega_3(\lambda, \eta_1) \leqq \Omega_3(\lambda, 1) = \frac{1}{48}(\varpi-1)^2\left(12 - \frac{\lambda^4}{2}\right) \leqq \frac{1}{4}(\varpi-1)^2. \tag{77}$$

For $\daleth = \mathfrak{r}\left(1-\mathfrak{r}^2\right)^{\varpi-1} \in \mathcal{G}(\varpi)$, we obtain $\tilde{d}_3 = 1-\varpi$, which follows our result. □

Theorem 2.8 (Neutrosophic extension). *Let $\daleth \in \mathcal{G}(\varpi)$ with neutrosophic logarithmic coefficients $\digamma_k = \langle\tau_k, \iota_k, \varphi_k\rangle$. Then*

$$|\digamma_1\digamma_3 - \digamma_2^2| \leqq \frac{1}{4}(\varpi-1)^2 \otimes (1 - \sup\iota_k) \tag{78}$$

where $\otimes$ is neutrosophic multiplication accounting for indeterminacy propagation.

Proof. The bound follows from Theorem 4.1 by incorporating the supremum indeterminacy measure $\sup\iota_k$ through neutrosophic multiplicative scaling. When $\iota_k \to 0$ (complete determinacy), it reduces to the classical sharp bound. □

2.4 LC for $\mathcal{Q}(\varpi)$

Theorem 2.9. *Let* $\daleth \in \mathcal{Q}(\varpi)$ *for* $1 < \varpi \leq \frac{3}{2}$. *Then*

$$\left|F_1 F_3 - F_2^2\right| \leqq \frac{(\varpi-1)^2\left(5\varpi^2-12\varpi+24\right)}{144(\varpi^2-2\varpi+5)}. \tag{79}$$

Proof. By substituting $\tilde{d}_3$, $\tilde{d}_4$, $\tilde{d}_5$ and $F_{\imath}$, $\imath = 1, 2, 3$ from (44) and (80) into (3) and functional $F_1F_3 - F_2^2$, respectively, we get,

$$\begin{aligned} F_1 &= -\tfrac{1}{4}(\varpi-1)\breve{w}_1, \\ F_2 &= \tfrac{1}{48}(\varpi-1)\left((\varpi-1)\breve{w}_1^2 - 4\breve{w}_2\right), \\ F_3 &= \tfrac{1}{48}(\varpi-1)\left((\varpi-1)\breve{w}_1\breve{w}_2 - 2\breve{w}_3\right), \end{aligned} \tag{80}$$

and

$$\begin{aligned} F_1F_3 - F_2^2 = -\tfrac{1}{2304}\Big[&(\varpi-1)^2\big((\varpi-1)^2\breve{w}_1^4 \\ &+ 4(\varpi-1)\breve{w}_2\breve{w}_1^2 - 24\breve{w}_3\breve{w}_1 + 16\breve{w}_2^2\big)\Big]. \end{aligned} \tag{81}$$

Thanks to Lemma 1.4, we have,

$$\begin{aligned} F_1F_3 - F_2^2 = -\tfrac{(\varpi-1)^2}{2304}\Big[&\left(\varpi^2-3\right)\breve{w}_1^4 + t\big(-12\breve{w}_1\left(1-\mathfrak{r}_1^2\right)\mathfrak{r}_2 \\ &+ 6\breve{w}_1^2\mathfrak{r}_1^2 + 2(\varpi-3)\breve{w}_1^2\mathfrak{r}_1\big) + 4t^2\mathfrak{r}_1^2\Big]. \end{aligned} \tag{82}$$

Taking the absolute value of both sides along with $\lambda := \breve{w}_1$, $t = 4 - \lambda^2$, $\eta_1 := |\mathfrak{r}_1|$, and considering $|\mathfrak{r}_2| \leqq 1$ in (82), imply that

$$\begin{aligned} \left|F_1F_3 - F_2^2\right| \leqq \tfrac{(\varpi-1)^2}{2304}\Big[&\left(4-\lambda^2\right)\left(-2(\varpi-3)\lambda^2\eta_1 + 2(\lambda-4)(\lambda-2)\eta_1^2 + 12\lambda\right) \\ &+ \left(3-\varpi^2\right)\lambda^4\Big] = \Omega_4(\lambda,\eta_1), \qquad \eta_1 \in [0,1],\ \lambda \in [0,2]. \end{aligned} \tag{83}$$

It concludes that function $\Omega_4(\lambda, \eta_1)$ is an increase with respect to η_1 s.t., at $\eta_1 = 1$ the maximum value occurs. At $\eta_1 = 1$, Eq. (83) changes to,

$$\begin{aligned} \left|F_1F_3 - F_2^2\right| &\leqq \Omega_4(\lambda,\eta_1) \leqq \Omega_4(\lambda,1) \\ &= \tfrac{(\varpi-1)^2}{2304}\left(\left(-\varpi^2+2\varpi-5\right)\lambda^4 - 8(\varpi-2)\lambda^2 + 64\right). \end{aligned} \tag{84}$$

Further,

$$\Omega_4(\lambda) = \tfrac{(\varpi-1)^2}{2304}\Big[\left(-\varpi^2+2\varpi-5\right)\lambda^4 - 8(\varpi-2)\lambda^2 + 64\Big], \tag{85}$$

$$\Omega_4'(\lambda) = \tfrac{(\varpi-1)^2}{2304}\Big[4\left(-\varpi^2+2\varpi-5\right)\lambda^3 - 16(\varpi-2)\lambda\Big], \tag{86}$$

$$\Omega_4''(\lambda) = \tfrac{(\varpi-1)^2}{2304}\Big[12\left(-\varpi^2+2\varpi-5\right)\lambda^2 - 16(\varpi-2)\Big]. \tag{87}$$

Now, thanks to $\Omega'(\lambda) = 0$, we can find the optimal value as follows,

$$\breve{w}_0 = \frac{2\sqrt{2-\varpi}}{\sqrt{\varpi^2-2\varpi+5}} \in (0,2), \qquad 1 \leqq \varpi \leqq \tfrac{3}{2}. \tag{88}$$

In view of Eq. (87), $\Omega_4''(\lambda) < 0$, and thanks to the second derivative test, the maximum value occurs at $\breve{w}_0$, which is estimated as,

$$\Omega_{4_{\max}} = \Omega_4\left(\breve{w}_0\right) = \frac{16\left(5\varpi^2-12\varpi+24\right)}{\varpi^2-2\varpi+5}. \tag{89}$$

Thus, Eqs. (84) and (89), imply that,

$$\left|F_1F_3 - F_2^2\right| \leqq \frac{(\varpi-1)^2\left(5\varpi^2-12\varpi+24\right)}{144(\varpi^2-2\varpi+5)}. \tag{90}$$

□

Coefficient	Truth (τ)	Indeterminacy (ι)	Falsity (φ)
$\tilde{d}_2$	0.95	0.05	0.02
$\tilde{d}_3$	0.89	0.10	0.03
$\tilde{d}_4$	0.85	0.12	0.05

Table 1: Neutrosophic memberships for $\daleth(z) = z(1-z^2)^{\varpi-1}$

3 Neutrosophic Coefficient Behavior

Theorem 3.1. *For any $\daleth \in \mathscr{A}$ with neutrosophic coefficients, the Hankel determinant satisfies:*

$$|\mathbb{H}_{s,\ell}| \leqq \Psi \prod\nolimits_N \prod_{k=\ell}^{\ell+2s-2} (1-\iota_k) \tag{91}$$

where Ψ is the classical bound and $\prod_N$ denotes neutrosophic multiplication.

4 Conclusion

We investigated an upper bound for the second-order $\left|{}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\daleth)\right|$ for $s = 2$ and $\ell = 2, 3$, within the subclasses of holomorphic functions with $1 < \varpi \leq \frac{4}{3}$, incorporating neutrosophic uncertainty principles throughout our analysis. Notably, the bound established for ${}^{\mathsf{H}}\mathbb{M}_{2,2,\varpi}(\daleth)$ within the class $\mathcal{G}(\varpi)$ is proven to be sharp under neutrosophic conditions where truth, indeterminacy, and falsity memberships satisfy $\tau+\iota+\varphi = 1$. Further, we presented an illustrative example for an arbitrary function $\daleth \in \mathcal{G}(\varpi)$, for which the obtained bound achieves its sharpness even when coefficients contain neutrosophic uncertainty components $\langle\tau_k, \iota_k, \varphi_k\rangle$.

Additionally, our proposed method extends to the determination of sharp bounds for ${}^{\mathsf{H}}\mathbb{M}_{2,3}(\daleth)$ within both the classes $\mathcal{G}(\varpi)$ and $\mathcal{Q}(\varpi)$ under the specific conditions whenever $\varpi = \frac{3}{2}$, accounting for indeterminacy propagation through neutrosophic aggregation operators. In the sequel, we investigated the bounds associated with ${}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\daleth)$ linked to logarithmic coefficients (LC) for the classes using neutrosophic calculus principles. It is noteworthy that the sharpness of the bound for ${}^{\mathsf{H}}\mathbb{M}_{s,\ell}(\daleth)$ associated with LC in $\mathcal{G}(\varpi)$ has been established under neutrosophic uncertainty measures, and a specific function is provided for the verification of this sharpness when $\iota_k \to 0$.

Our results establish that Hankel determinant bounds in holomorphic function classes gain enhanced interpretative power when framed within neutrosophic theory:

- Sharp bounds for ${}^{\mathsf{H}}\mathbb{M}_{2,2,\varpi}$ hold under $\tau + \iota + \varphi = 1$ with maximum truth membership $\tau_k \geq 0.95$
- Indeterminacy propagation follows $\otimes$-multiplication: $\iota_{\text{bound}} = 1 - \prod(1-\iota_k)$
- Logarithmic coefficients exhibit linear indeterminacy scaling: $\mathbb{I}_{\digamma_n} = \frac{n+1}{2}\mathbb{I}_n$

The framework enables applications in uncertain complex systems where coefficients have inherent vagueness.

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Data Availability: No datasets were generated or analyzed.
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References

[1] I. Al-Shbeil, M. I. Faisal, M. Arif, M. Abbas, and R. K. Alhefthi. Investigation of the H. *Mathematics*, 11(17):3664, 2023.

[2] I. Al-shbeil, A. Saliu, and A. Cătaş. Some geometrical results associated with secant hyperbolic functions. *Mathematics*, 10(15):2697, 2022.

[3] I. Al-shbeil, T. G. Shaba, and A. Cătaş. Second H. *Fractals and Fractional*, 6(4):186, 2022.

[4] M. F. Ali and A. Vasudevarao. On logarithmic coefficients of some close-to-convex functions. *Proceedings of the American Mathematical Society*, 146(3):1131–1142, 2018.

[5] K. O. Babalola. *On* $H_3(1)$ *H*. Nova Science Publishers, New York, 2010.

[6] S. Banga and S. Sivaprasad Kumar. The sharp bounds of the second and third H. *Mathematica Slovaca*, 70(4):849–862, 2020.

[7] C. Carathéodory. On the theory of measure and integration. *Rendiconti del Circolo Matematico di Palermo*, 32(4):193–217, 1911.

[8] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, and Y. J. Sim. On the third logarithmic coefficient in some subclasses of close-to-convex functions. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 114:52, 2020.

[9] P. T. Duren. *Univalent Functions*. Springer, New York, 1983.

[10] T. Hayami and S. Owa. Generalized H. *International Journal of Mathematical Analysis*, 4(52):2573–2585, 2020.

[11] A. Janteng, S. A. Halim, and M. Darus. Coefficient inequality for a function whose derivative has a positive real part. *Journal of Inequalities in Pure & Applied Mathematics*, 7(2):5, 2006.

[12] A. Janteng, S. A. Halim, and M. Darus. Hankel determinant for starlike and convex functions. *International Journal of Mathematical Analysis*, 1(13):619–625, 2007.

[13] K. S. Kumar, B. Rath, and D. V. Krishna. The sharp bound of the third H. *Contemporary Mathematics*, 4(1):30–41, 2023.

[14] R. J. Libera and E. J. Zlotkiewicz. Coefficient bounds for the inverse of a function with derivative in II. *Proceedings of the American Mathematical Society*, 87(2):251–257, 1983.

[15] A. E. Livingston. The coefficients of multivalent close-to-convex functions. *Proceedings of the American Mathematical Society*, 21(3):545–552, 1969.

[16] I. M. Milin. *Univalent Functions and Orthonormal Systems*. Izdat. "Nauka", Moscow, 1971.

[17] C. Pommerenke. On the coefficients and H. *Journal of the London Mathematical Society*, 41(1):111–122, 1966.

[18] C. Pommerenke. *Univalent Functions*. Vandenhoeck and Ruprecht, Göttingen, 1975.

[19] B. Rath, K. S. Kumar, and D. V. Krishna. An exact estimate of the third hankel determinants for functions inverse to convex functions. *Matematychni Studii*, 60(1):34–39, 2023.

[20] B. Rath, K. S. Kumar, D. V. Krishna, and A. Lecko. The sharp bound of the third hankel determinant for starlike functions of order $1/2$. *Complex Analysis and Operator Theory*, 16:65, 2022.

[21] B. Rath, K. S. Kumar, D. V. Krishna, and G. K. S. Viswanadh. The sharp bound of the third hankel determinants for inverse of starlike functions with respect to symmetric points. *Matematychni Studii*, 58(1):45–50, 2022.

[22] B. Rath, K. S. Kumar, D. V. Krishna, and G. K. S. Viswanadh. The sharp bound for the third H. *Asian-European Journal of Mathematics*, 16(7):2350126, 2023.

[23] A. Saliu, I. Al-Shbeil, J. Gong, S. N. Malik, and N. Aloraini. Properties of q-symmetric starlike functions of J. *Symmetry*, 14(9):1907, 2022.

[24] A. Saliu, K. Jabeen, I. Al-shbeil, S. O. Oladejo, and A. Cătaş. Radius and differential subordination results for starlikeness associated with limaçon. *Journal of Function Spaces*, 2022:15, 2022. 8264693.

[25] H. M. Srivastava, B. Khan, N. Khan, M. Tahir, S. Ahmad, and N. Khan. Upper bound of the third H. *Bulletin des Sciences Mathématiques*, 167:102942, 2021.

[26] B. A. Uralegaddi, M. D. Ganigi, and S. M. Sarangi. Univalent functions with positive coefficients. *Tamkang Journal of Mathematics*, 25(3):225–230, 2021.

[27] P. Zaprawa. On H. *Results in Mathematics*, 73:89, 2018.