Quantifying Uncertainty in Economic Growth Prediction Using the Neutrosophic Muth Distribution

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Abstract

Uncertainty, imprecision, and incomplete information are commonly found in complex economic and financial systems, and traditional probabilistic models are thus inadequate to accurately model and forecast these systems. In this work, a new extension of the Muth distribution in the neutrosophic environment is presented leading to the neutrosophic Muth distribution (NMD). This new model introduces neutrosophic parameters aiming to quantify vague and uncertain information and provides a flexible and robust approach to modeling right-skewed economic data. Some key characteristics including the density function and cumulative distribution function, moment generating function, and origin moments are obtained in the neutrosophic framework. The study of a model treated under uncertainty is described and an inferential method transforming it into neutrosophic maximum likelihood by interval-valued data is discussed. A real-world financial dataset is considered in order to prove the usefulness of the proposed distribution. The findings emphasize that the proposed distribution has the potential to be a comprehensive, flexible, and potential model for handling uncertainty in economics and finance data.

Keywords: Neurotrophic logic; Uncertainty modeling; Neutrosophic probability; Estimation; Simulation

1. Introduction

Probability theory is one of the cornerstones of the science of economic growth forecasting, a powerful tool for dealing with uncertain situations and understanding complicated phenomena in the economy [1]. Economy is a very stochastic system as it comes across numerous stochastic events, like consumer wish changes, technology innovations, policy changes (political), natural disasters and global market dynamics and so on [2]. These factors add a lot of uncertainty to the forecast of economic indicators such as GDP growth, inflation, unemployment rates and investment patterns. Probability theory allows economists and analysts to formalize this uncertainty in a clear and numerical way [3-5]. Using the language of probability distributions, economists can describe and make sense of the variability inherent in their economic data, detect underlying patterns and predict future trends more accurately. In addition, probability theory makes it possible to estimate important measures of risk, such as value at risk (VaR) or conditional value at risk (CVaR), which are crucial for financial institutions to handle portfolio risk that may have implications for wider economic stability [6]. Further, probabilistic formulas facilitate the use of Bayesian inference that is commonly employed in revising economic forecasts as fresher measurement results arrive [7]. It is this interactive learning that is key in the fast-paced business world of today where decisions have to be made instantly [8]. Game theory and economic models, by which better policies and strategies can be developed, are also based on probability distributions [9].

The Muth distribution for modeling the distribution of failure times is an important distribution in reliability analysis and has also gained widespread interest in the realm of economic and financial data modeling, due to its flexibility, and its ability to model right-skewed behavior exhibited in the kind of data usually found in this area of knowledge [10]. Economic statistics such as income distribution, time-to-bankruptcy, length of investment holding periods and financial return duration frequently show imported tails and asymmetry, which are characteristics featured in the Muth distribution [11]. Its flexible functional form enables it to accommodate

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various hazard functions, relevant to survival analysis, duration modeling in economics (e.g., unemployment spells, time-to-default in credit risk analysis). Furthermore, Muth distribution serves as an important alternative to traditional distributions where weak-form conditions, like normality are not satisfied, and provides better fit in sets of data where tail behavior is critical for decision-making in finance [12]. Its mathematical tractability and parameter interpretability also makes it a suitable solution for risk analysts, actuaries and economic forecasters when trying to explore and quantify uncertainty in dynamic financial settings. Fuzzy and neutrosophic random variables are very important in the modeling of economic and financial data when uncertainty, vagueness, and disjoint information exist [13]. Traditional probabilistic models are not always capable of representing uncertainty, fuzziness, and incompleteness in realistic economic systems, including changing market trends, indeterminate investor behaviors, and imprecise economic predictions [14]. Fuzzy random variables introduce levels of vagueness through membership functions to let the analysts deal with imprecise input data in a more realistic manner. Neutrosophic random variables take one-step further by adding the third dimension “indeterminacy” by considering not only the degree of truth and falsity of vagueness, but also the unknown and the conflicting part of the financial information [15]. This is especially critical in uncertain circumstances, including fluctuation in the financial market or emerging countries, in which available data is usually incomplete, inconsistent, or delayed. By including these advanced variables, economic and financial theorists can build more stable models for risk evaluation, portfolio diversity, and economic prediction to support better and sustainable decision making under uncertainty. The idea of neutrosophic statistics is based on the development and generalization of the classical statistical theory, including also the third component (indeterminacy) along with truth and false, initiated by Florentin Smarandache [16-17]. Contrary to classical and fuzzy statistics, neutrosophic statistics exhibits the new and broader context to analyze more general types of data, namely, not only uncertain, vague and imprecise data, but also incomplete or inconsistent data, which are often met in practice (financial, economical, medical and technical problems) [18]. In such environment, neutrosophic probability distributions were introduced to describe more appropriately the intricacy and uncertainty. These comprise the neutrosophic normal distribution, neutrosophic exponential distribution, neutrosophic Weibull distribution, neutrosophic inverse gaussian distribution, neutrosophic Maxwell distribution, and so on [18-22]. These distributions incorporate neutrosophic parameters, which are given as intervals or triplets of values depending on the truth, indeterminacy, and falsity. This structure is at the basis of a more realistic and complex way of representing the data, whenever the information is incomplete, uncertain, or open to multiple interpretations. Therefore, neutrosophic statistics can pave the way for a new perspective of uncertainty modeling, as well as for the versatility and reality of statistics in different scientific and applied fields.

In this study, we provide the neutrosophic form of the Muth distribution since it has a great significance for the modelling of several real-life problems in economy and finance. To the best of our knowledge, no such distribution has been constructed. We here propose a new neutrosophic version of Muth distribution in order to model the neutrality within them and use it to model: the uncertainty present in financial market and economic datasets, e.g., unpredictable behavior of markets, incomplete information, and changes in economic indicators. The developed neutrosophic Muth distribution mixes neutrosophic parameters to describe data characteristics in terms of incomplete information and thus provides a more flexible and powerful model for decision makers. This new combination not only extends the modeling power of the original Muth distribution but also adds and reinforces the field of neutrosophic statistics. That shall allow a more advanced study of the complex economic behavior where traditional probabilistic models give up.

The rest of the work is structured as follows. Section 2 explains the basic structure of the Muth distribution and its statistical properties. Section 3 describes the corresponding structure of the Muth distribution for imprecise information. Section 4 provides an illustration about the estimation procedure. Section 5 explains the practical utility of the model for real data set. Section 6 concludes the major findings of the work.

2. Classical Structure of Muth Distribution

A random variable $\mathcal{Y}$ follows a Muth distribution with shape parameter β if its cumulative distribution function (CDF) $F(y;\beta) = P(\mathcal{Y} \le y)$ is given by:

$$F(y;\beta) = 1 - exp\left(\beta y - \frac{1}{\beta}\left(e^{\beta y} - 1\right)\right),\ y > 0. \qquad (1)$$

where $\beta \in (0,1]$ is the precise value of shape parameter of the distribution.

The corresponding probability density function (PDF) of $\mathcal{Y}$, $f(y;\beta)$, is the following

$$f(y;\beta) = \left(e^{\beta y} - \beta\right) \exp\left(\beta y - \frac{1}{\beta}\left(e^{\beta y} - 1\right)\right),\ y > 0, \qquad (2)$$

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In statistics, there is a need to describe the behavior of random variables using the PDF and CDF. The PDF tells you the probability of the outcome, while the CDF tells you the probability of a variable taking a value less than or equal to y. Together, they facilitate the modeling, analysis, and interpretation of the distribution of data. For different values of shape parameters, the PDF and CDF of the Muth distribution are given in Figure 1(a) and Figure 1(b) respectively.

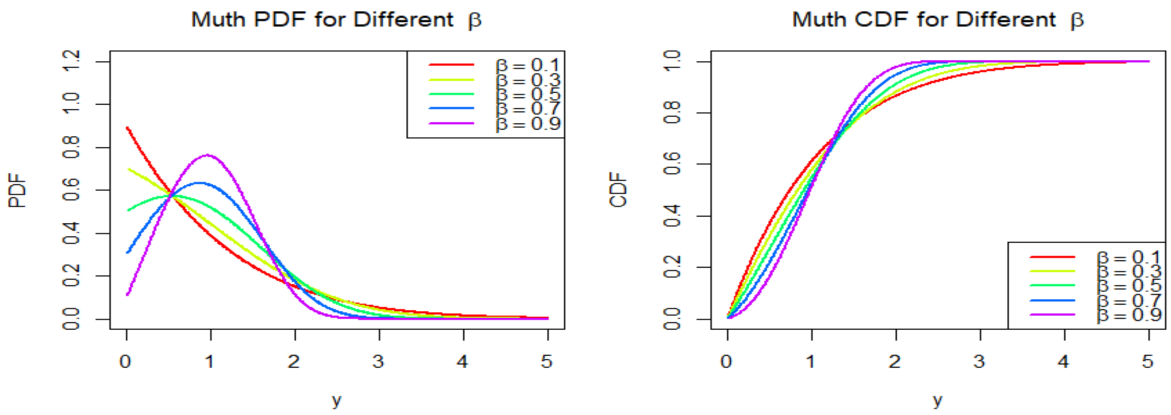

Figure 1. Basic plots of PDF and CDF curves of the Muth distribution

Figure 1 is comprised of two side by side plots that demonstrate the Muth distribution for several parameter values of shape parameters. On the left, the plot of the probability density changes, which tells us the concentration or dispersion of the values. The right plot shows the cumulative distribution (i. e., how the distribution piles up throughout the range). Each curve defines a different parameter value and allows us to compare shapes of distributions.

The moment generating function (MGF) is one of the important functions of any distribution. We first derive a closed-form expression from the MGF of the Muth distribution, expressed in terms of the exponential integral function. The MGF of the random variable $\mathcal{Y}$ is defined as $M(t;\beta) = E[e^{t\mathcal{Y}}], t \in \mathbb{R}$. Using this function, we derive the moments of $\mathcal{Y}$ in terms of the generalized function.

The gamma function $\Gamma(b,z)$ can be defined as.

$$\Gamma(b,z) = \int_z^\infty t^{b-1} e^{-t} dt, \quad b \in \mathbb{C}, z \in R. \quad (3)$$

In addition, the usual exponential integral function $E_s(z)$ can be defined as follows:

$$E_s(z) = z^{s-1}\Gamma(1-s,z), \quad s,z \in \mathbb{C}. \quad (4)$$

With the above notation established we can define the algorithm as:

Theorem 1. Let $\mathcal{Y}$ be a random variable following Muth distribution with parameter $\beta \in (0,1)$. The moment generating function of $\mathcal{Y}$ is given by:

$$M(t;\beta) = \frac{e^{1/\beta}}{\beta} t E_{-t/\beta}\left(1/\beta\right) + 1, \quad -\infty < t < \infty. \quad (5)$$

Proof: By the definition of the moment generating function $M(t;\beta) = E[e^{t\mathcal{Y}}]$ and applying equation (2), we derive for any real t:

$$M(t;\beta) = E[e^{t\mathcal{Y}}] = \int_0^\infty e^{ty} dF(y;\beta)$$

$$= \int_0^\infty e^{ty}\left(e^{\beta y} - \beta\right) \exp\left\{\beta y - \frac{1}{\beta}\left(e^{\beta y} - 1\right)\right\} dy.$$

Using transformation $v = \frac{e^{\beta y}}{\beta}$, we can write as:

$$M(t;\beta) = e^{1/\beta} \beta^{1+t/\beta} \left(\int_{1/\beta}^\infty v^{1+t/\beta} e^{-v} dv - \int_{1/\beta}^\infty v^{t/\beta} e^{-v} dv\right)$$

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$$= e^{1/\beta}\beta^{1+t/\beta}\left\{\Gamma\left(2+\frac{t}{\beta},\frac{1}{\beta}\right)-\Gamma\left(1+\frac{t}{\beta},\frac{1}{\beta}\right)\right\}. \quad (6)$$

Observe from equation (6) that the upper incomplete gamma function satisfies the recurrence relation: $\Gamma(\beta+1,z)=\beta\Gamma(\beta,z)+z^{\beta}e^{-z}$

$$M(t;\beta)=e^{1/\beta}\beta^{t/\beta}t\Gamma\left(1+\frac{t}{\beta},\frac{1}{\beta}\right)+1,\ -\infty<t<\infty. \quad (7)$$

Ultimately, we get the intended outcome from the equation above and Eq. (4).

Remark 1. We emphasize that the moment generating function of the Muth distribution admits an alternative representation in terms of the upper incomplete gamma function through equation (7).

To compute the rth moment of $\mathcal{Y}$

$$E[\mathcal{Y}^r]=\int_0^{\infty} y^r dF(y;\beta),\ r=1,2,...,$$

We shall use the well-known property that

$$E[\mathcal{Y}^r]=\frac{\partial^r}{\partial t^r}M(t;\beta)|t=0,\ r=1,2,...,$$

The following integral representation defines the generalized exponential function.

$$E_s^m(z)=\frac{1}{\Gamma(m+1)}\int_1^{\infty}(log v)^m v^{-s}e^{-zv}dv,\quad s,z\in\mathbb{C},\ m=0,1,..., \quad (8)$$

where log stands for the natural logarithm.

Let X be a random variable following a Muth distribution with parameter $\beta\in(0,1)$. The derivatives of the moment generating function $M(t;\beta)$ are given by:

$$\frac{\partial^r}{\partial t^r}M(t;\beta)=\frac{e^{1/\beta}\Gamma(r+1)}{\beta^r}\left(E_{-t/\beta}^{r-1}\left(1/\beta\right)+\frac{t}{\beta}E_{-t/\beta}^{r}\left(1/\beta\right)\right),\qquad r=1,2,....$$

From Eq. (5), after some computations, we get the following:

$$\frac{\partial^r}{\partial t^r}M(t;\beta)=\frac{e^{1/\beta}}{\beta}\left(r\frac{\partial^{r-1}}{\partial t^{r-1}}E_{-t/\beta}\left(1/\beta\right)+t\frac{\partial^r}{\partial t^r}E_{-t/\beta}\left(1/\beta\right)\right),\ r=1,2,....$$

Then, the conclusion derives from the following equation, taking into consideration that

$$(-1)^r\frac{\partial^r}{\partial s^r}E_s(z)=\Gamma(r+1)E_s^r(z),\ \ r=1,2,...,$$

Assuming that $\frac{\partial^0}{\partial s^0}E_s(z)=E_s(z)$.

Let $\mathcal{Y}$ follow a Muth distribution with parameter $\beta\in(0,1)$. Then the rth moment of $\mathcal{Y}$ is given by:

$$E[\mathcal{Y}^r]=\frac{e^{1/\beta}\Gamma(r+1)}{\beta^r}E_0^{r-1}\left(1/\beta\right),\ \ r=1,2,.... \quad (9)$$

The results follow from Lemma 1 because $E[\mathcal{Y}^r]=\frac{\partial^r}{\partial t^r}M(t;\beta)|t=0$

As an immediate consequence of Proposition 1, we can see that closed-form expressions for the moments of the Muth distribution cannot be expressed in terms of elementary functions, except for the first moment ($r=1$) which we will explicitly derive. Furthermore, we observe the second moment $E[\mathcal{Y}^2]$ as follows:

Let $\mathcal{Y}\sim$Muth(β) be a random variable with shape parameter $\beta\in(0,1)$. *Then:*

$$(i)\ E[\mathcal{Y}]=1,\qquad (ii)\ E[\mathcal{Y}^2]=\frac{2e^{1/\beta}}{\beta}\Gamma\left(0,1/\beta\right).$$

The result follows from eq. (9) since $E_0^0\left(1/\beta\right)=\beta e^{-1/\beta}$. (ii) from eq. (8) and (9), we have

$$E[\mathcal{Y}^2]=\frac{2e^{1/\beta}}{\beta^2}E_0^1\left(1/\beta\right)=\frac{2e^{1/\beta}}{\beta^2}\int_1^{\infty}\log(v)e^{-v/\beta}dv=\frac{2e^{1/\beta}}{\beta}\int_1^{\infty}\frac{e^{-v/\beta}}{v}dv,$$

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The Muth distribution exhibits interesting properties beyond its shape and spread, particularly concerning its mode and a special constant known as the golden ratio.

The golden ratio, denoted by ϕ, is defined as the unique positive solution to the quadratic equation: $y^2 - y - 1 = 0$

$$\phi = {(1+\sqrt{5})}/{2} \approx 1.618033988749.$$

We state the following proposition.

Let $\mathcal{Y} \sim Muth(\beta)$ be a random variable following a *Muth distribution with shape parameter* $\beta \in (0,1)$. Then:

$$mode(\mathcal{Y}) = \begin{cases} 0, & 0 < \beta \leq {1}/{\phi^2} \\ \frac{\log(\beta\phi^2)}{\beta}, & {1}/{\phi^2} < \beta \leq 1. \end{cases} \quad (10)$$

The first derivative of Eq (1) can be written as

$$\frac{\partial}{\partial y} f(y;\beta) = \left(\beta e^{\beta y} - \left(e^{\beta y} - \beta\right)^2\right) \exp\left(\beta y - \frac{1}{\beta}\left(e^{\beta y} - 1\right)\right).$$

To determine the mode of the random variable $\mathcal{Y}$, we solve the following optimization problem: $\frac{\partial}{\partial y} f(y;\beta) = 0$ is the probability density function of the Muth distribution. This first-order condition leads to the equation:

$$\beta e^{\beta y} - \left(e^{\beta y} - \beta\right)^2 = 0. \quad (11)$$

Using the fundamental identity of the golden ratio, $\phi^2 = 1 + \phi$, we can show that $y = \log(\beta\phi^2)/\beta$ is the unique positive solution to equation (11) when $\beta\phi^2 > 1$. Otherwise, the left-hand side of (11) remains negative for all $y > 0$. Furthermore, through direct verification we establish that:

$$\frac{\partial}{\partial y} f(y;\beta)|_{y=\log(\beta\phi^2)/\beta} = e^{{(1-\beta\phi^2)}/{\beta}} \beta^4\phi^2(\phi^6 - 6\phi^4 + 7\phi^2 - 1)$$

The inequality $\phi^6 - 6\phi^4 + 7\phi^2 - 1 < 0$ implies: $mode(\mathcal{Y}) = \log(\beta\phi^2)/\beta$ for $\beta \in (0, {1}/{\phi^2}]$.

$mode(\mathcal{Y}) = 0$ for $\beta \in (0, {1}/{\phi^2}]$ completing the proof.

The golden ratio is used to be discovered when the mode changes from zero to positive in the distribution. By examining the slope of the distribution, there is a threshold for the parameter where it starts to change. Below a breaking point related to the golden ratio the peak of the distribution stays at its origin. But the peak is shifted to the even positive side (although only the region above the threshold matters). It is the transition between these two that is connected to the behavior of the probability density function, and the golden ratio characterizes this boundary. This concept of golden ratio and mode is demonstrated in Figure 2.

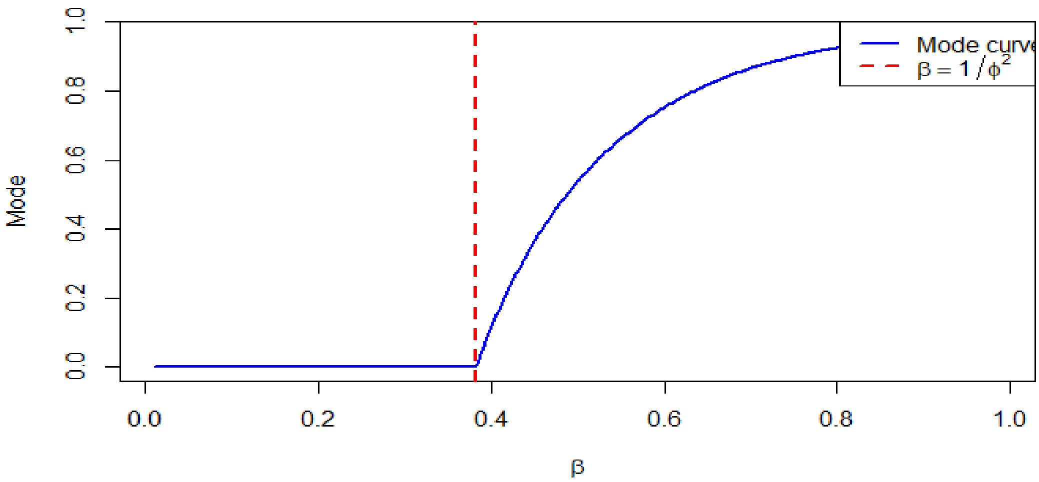

Figure 2. Mode of Muth distribution and golden ratio threshold

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Figure 2 shows an example of how the peak of this distribution moves as the shape parameter increases. Initially the peak is stationary at the origin implying most values are concentrated around 0. Above a threshold (dashed line), the center of the peak starts shifting away in the negative direction. This change represents a change in the behavior of the distribution, and the red line is the position beyond which the change occurs.

This section also presents a fundamental aspect of the Muth distribution that mirrors the normal behavior of right-skewed distribution: the mode of the value is small, the middle value or median is intermediate, and the mean is large. This ordered relationship is useful in explaining the way that the distribution disburses values, particularly in situations, which mean that most of the outputs are small, but a few large values drag the average up. The argument is illustrated by corresponding results for the median and mode, both of which are given in terms of well-known special functions. Crucially, the same relationship also holds for the entire valid range of the distribution's shape parameter, implying that the Muth distribution's right-skewedness is consistently practical in this setup.

Let median $(\mathcal{Y})$ denote the median of the random variable $\mathcal{Y}$, defined through the quantile function $\mathcal{Q}$ from Proposition 2 as: $Median(\mathcal{Y}) = \mathcal{Q}\left(\frac{1}{2};\beta\right)$

Let $\mathcal{Y}\sim Muth(\beta)$ be a random variable following a Muth distribution with shape parameter $\beta \in (0,1)$. Then:

$$Mode(\mathcal{Y}) < Median(\mathcal{Y}) < E[\mathcal{Y}] = 1$$

The Lambert W function is defined as:

$$\frac{\partial}{\partial z}\mathcal{W}(z) = \frac{\mathcal{W}(z)}{z(1+\mathcal{W}(z))},\ z \neq 0. \quad (12)$$

Additionally, the median of $\mathcal{Y}$ is

$$Median(\mathcal{Y}) = \frac{1}{\beta}\log\left(-\beta\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\right). \quad (13)$$

An important consequence of the above expression is that the argument of the logarithmic function satisfies $argument > 1$. This inequality holds for all $\beta \in (0,1)$.

$$-\beta\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right) > 1\ \ \forall\beta \in (0,1],$$

since the median of a strictly positive random variable is a positive number.

$$\frac{\partial}{\partial\beta}\left\{-\beta\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\right\} = -\frac{\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\left(1+\beta\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\right)}{\beta\left(1+\beta\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\right)} > 0,$$

where the last inequity follows from the fact that $-\beta\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right) < -1$ for any $\beta \in (0,1)$.

$$Mode(\mathcal{Y}) - Median(\mathcal{Y}) = \frac{1}{\beta}\left\{2\log(\varphi) - \log\left(-\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\right)\right\}. \quad (14)$$

Hence,

$$\frac{\partial}{\partial\beta}\left\{-\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\right\} = \frac{(\beta-1)\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)}{\beta^2\left(1+\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\right)} < 0,$$

Further simplification yielded:

$$2\log(\varphi) < \log\left(-\mathcal{W}_{-1}\left(\frac{-1}{2\beta e^{1/\beta}}\right)\right), \beta \in \left({}^{1}/_{\varphi^2}, 1\right), \quad (15)$$

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In this section, we have also discussed the estimation procedure of classical Muth distribution using maximum likelihood method. We use method called maximum likelihood estimation, in which the actual value of the parameter is sought that will make the observed data most likely to have arisen under the assumed distribution. It does so by creating a function that tells us how probable the data is, depending on the value of the parameter, and then tries to figure out when this function reaches its maximum. Because the position of this maximum cannot be found by using simple formulas, numerical techniques are employed to find it. When this value is discovered, then an additional search ensures that this point is indeed a maximum, as opposed to a minimum or an inflection. This operation will yield the maximum likelihood estimate of the distribution's parameter from the data provided. Consider a random sample $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_n$ drawn from a Muth distribution with unknown parameter $\beta \in (0,1)$, and let $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_n$ represent the corresponding observed values. The likelihood function is given by: $L(\beta) = \prod_{i=1}^{n} f(y_i;\beta)$, where $f(y;\beta)$ denotes the Muth probability density function. The log-likelihood function is then:

$$logL(\beta) = \sum_{i=1}^{n} \log\left(e^{\beta y_i} - \beta\right) - \frac{1}{\beta}\sum_{i=1}^{n}\left(e^{\beta y_i} - 1\right) + \beta \sum_{i=1}^{n} y_i. \quad (16)$$

The ML estimate of β is the value, say $\hat{\beta}$, that maximizes eq. (6.1). then, to get $\hat{\beta}$ we must numerically solve

$$\frac{\partial}{\partial\beta}\log\mathcal{L}(\beta) = \sum_{i=1}^{n} \frac{y_i e^{\beta y_i} - 1}{e^{\beta y_i} - \beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} e^{\beta y_i} - \frac{1}{\beta}\sum_{i=1}^{n} y_i e^{\beta y_i} + \sum_{i=1}^{n} y_i - \frac{n}{\beta^2} = 0,$$

And check that the solution $\hat{\beta}$ satisfies $(\partial^2 logL(\beta)/\partial\beta^2)|_{\beta=\hat{\beta}} < 0$.

3. Proposed Model

A random variable $\mathcal{Y}$ follows a Neutrosophic Muth distribution (MD_N) with shape parameter β_n if its probability density function (PDF) is given by:

$$f_N(y;\, \beta_n) = \left(e^{\beta_n y} - \beta_n\right) \exp\left(\beta_n y - \frac{1}{\beta_n}\left(e^{\beta_n y} - 1\right)\right), \quad y > 0, \quad (17)$$

where $\beta_n \in (0,1]$.

The visual display of the PDF is given in Figure 3 with different values of β_n.

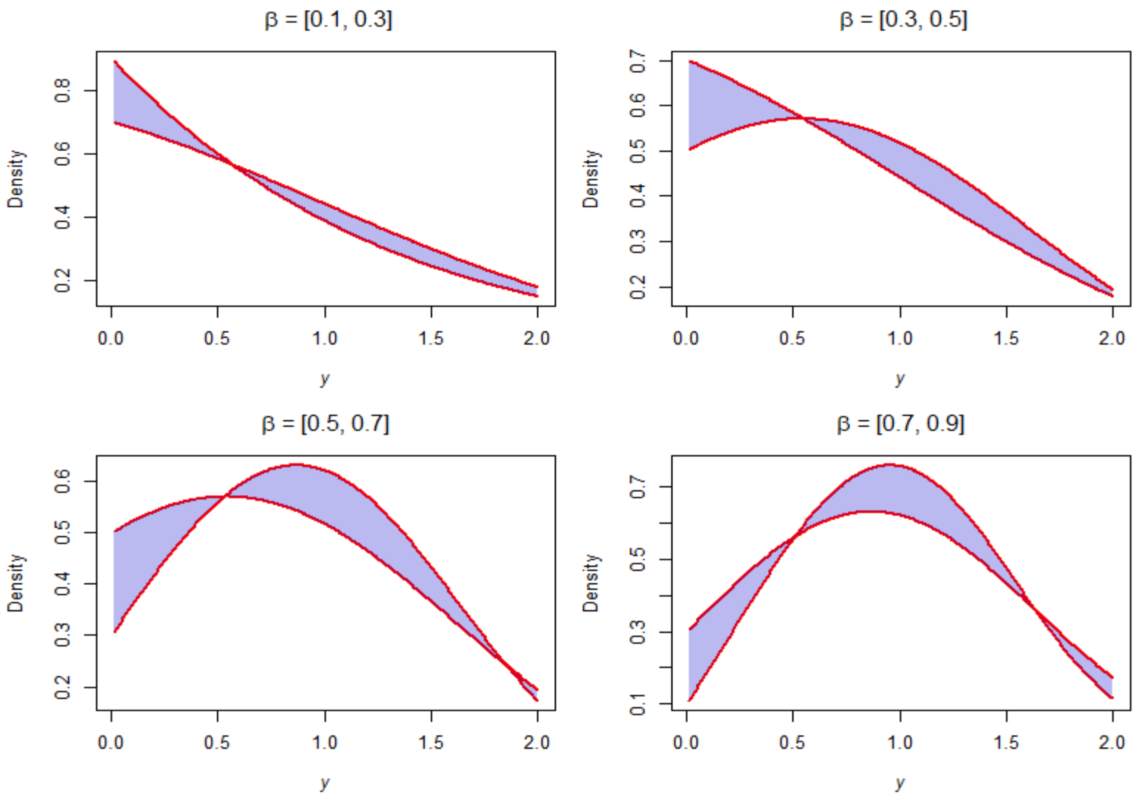

Figure 3 .PDF curves of the neutrosophic Muth distribution for different values of neutrosophic parameter values

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A panel of plots visualizing the uncertainty aware behavior of the PDF over the range of the model parameter appears in Figure 3. Each sub-plot represents a different range on the parameter, over the unit interval. The shaded areas in each panel indicate the possible range of density curves based on the interval uncertainty of the parameter. We can learn from these visual bands how sensitive the peak and spread of the distribution are under different parameter intervals. The shape and concentration of the density change noticeably as the parameter range is moved from small to large values, indicating in fence of parameter uncertainty on the characteristic probabilistic behavior of the model.

The cumulative distribution function of $\mathcal{Y}$, $F_N(y;\ \beta_n) = P(\mathcal{Y} \leq y)$, is the following

$$F_N(y;\ \beta_n) = 1 - exp\left(\beta_n y - \frac{1}{\beta_n}\left(e^{\beta_n y} - 1\right)\right),\ y > 0. \quad (18)$$

Figure 4 shows that CDF curves of the MD_N distribution.

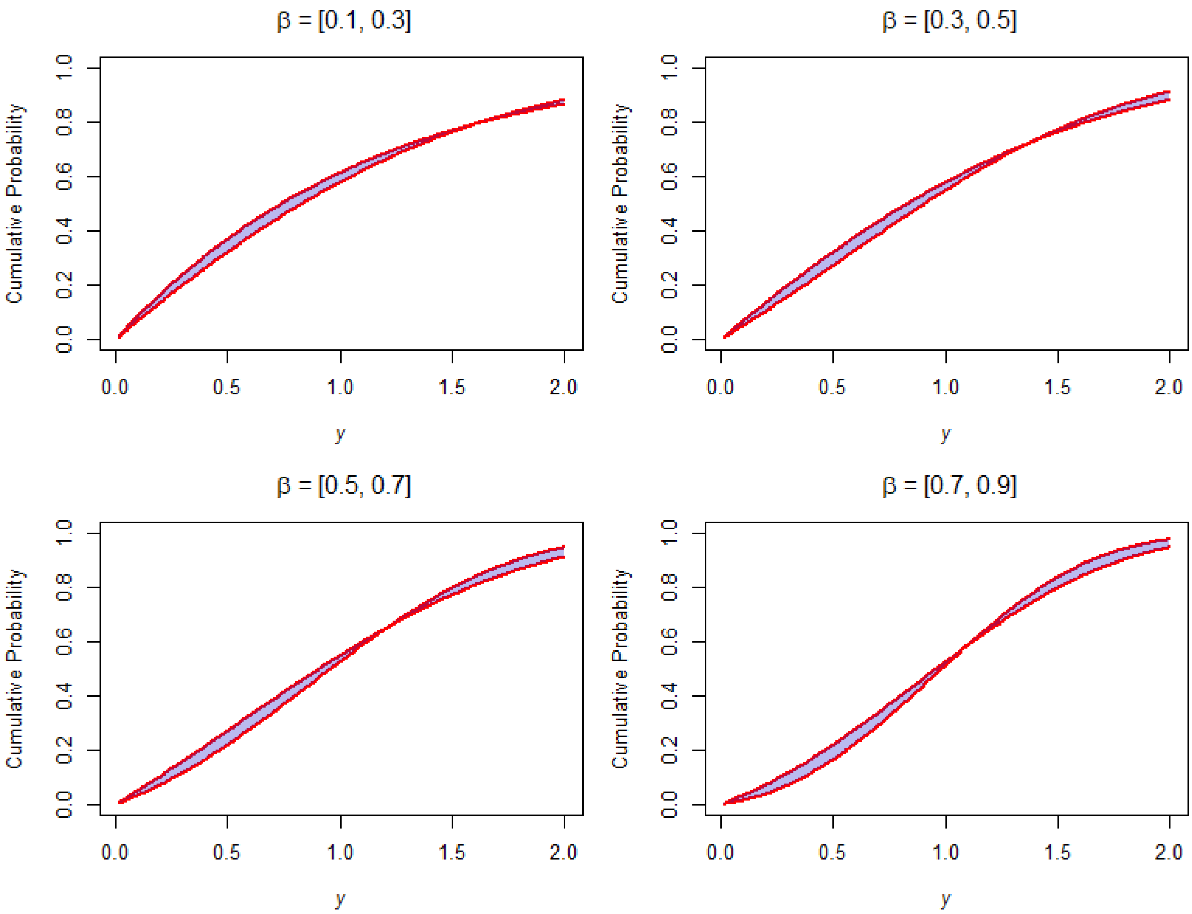

Figure 4. CDF curves of the Muth distribution for different values of neutrosophic parameter values

The CDF plots varying with uncertain model parameter are illustrated in Figure 4. Each panel displays how the cumulative probability varies over a set of parameter values, illustrating how the probability accumulation reacts to changes in the parameter interval. The shaded band between the curves indicates the uncertainty range due to parameter variation. With increasing parameter range, the cumulative function steepness and transition speed are adjusted accordingly. This number can be seen as summing up the effect of such parameter-uncertainty on the cumulative behavior of the distribution, being a way of having a global perspective of how probability increases in the domain.

Now we define its moment generating function**.** Let $\mathcal{Y}$ be a random variable following a Neutrosophic Muth distribution (MD_N) with parameter $\beta_n \in (0,1)$. The moment generating function of $\mathcal{Y}$ is given by:

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$$M_N(t;\beta_n) = \frac{e^{1/\beta_n}}{\beta_n} t E_{-t/\beta_n}\left({}^1/_{\beta_n}\right) + 1, \quad -\infty < t < \infty. \tag{19}$$

The proof can be established as follows:

By the definition of the moment generating function $M_N(t;\beta_n) = E[e^{t\mathcal{Y}}]$ and applying equation (2), we derive for any real t:

$$M_N(t;\beta_n) = E[e^{t\mathcal{Y}}] = \int_0^\infty e^{ty} dF(y;\beta_n)$$
$$= \int_0^\infty e^{ty}\left(e^{\beta_n y} - \beta_n\right) \exp\left\{\beta_n y - \frac{1}{\beta_n}\left(e^{\beta_n y} - 1\right)\right\} dy.$$

By making the chang variable $v = \frac{e^{\beta_n y}}{\beta_n}$, we obtain

$$M_N(t;\beta_n) = e^{1/\beta_n}\beta_n^{\,1+t/\beta_n}\left(\int_{1/\beta_n}^\infty v^{\,1+t/\beta_n} e^{-v} dv - \int_{1/\beta_n}^\infty v^{\,t/\beta_n} e^{-v} dv\right)$$

$$= e^{1/\beta_n}\beta_n^{\,1+t/\beta_n}\left\{\Gamma\left(2 + \frac{t}{\beta_n}, \frac{1}{\beta_n}\right) - \Gamma\left(1 + \frac{t}{\beta_n}, \frac{1}{\beta_n}\right)\right\}. \tag{20}$$

Observe from equation (6) that the upper incomplete gamma function satisfies the recurrence relation: $\Gamma(\beta_n + 1, z) = \beta_n\Gamma(\beta_n, z) + z^{\beta_n}e^{-z}$

$$M_N(t;\beta_n) = e^{1/\beta_n}\beta_n^{\,t/\beta_n} t\Gamma\left(1 + \frac{t}{\beta_n}, \frac{1}{\beta_n}\right) + 1, \ -\infty < t < \infty. \tag{21}$$

In order to compute the rth moment of $\mathcal{Y}$

$$E[\mathcal{Y}^r] = \int_0^\infty y^r dF(y;\beta_n), \ r = 1,2, \ldots,$$

We shall use the well-known property that

$$E[\mathcal{Y}^r] = \frac{\partial^r}{\partial t^r} M_N(t;\beta_n)|t = 0, \ r = 1,2, \ldots,$$

The following integral representation defines the generalized integro-exponential function.

$$E_s^m(z) = \frac{1}{\Gamma(m+1)}\int_1^\infty (log v)^m v^{-s} e^{-zv} dv, \quad s, z \in \mathbb{C}, \ m = 0,1, \ldots, \tag{22}$$

where log stands for the natural logarithm.

Let X be a random variable following a Neutrosophic Muth distribution (MD_N) with parameter $\beta_n \in (0,1)$. The derivatives of the moment generating function $M_N(t;\beta)$ are given by:

$$\frac{\partial^r}{\partial t^r} M_N(t;\beta_n) = \frac{e^{1/\beta_n}\Gamma(r+1)}{\beta_n^{\,r}}\left(E_{-t/\beta_n}^{r-1}\left({}^1/_{\beta_n}\right) + \frac{t}{\beta_n} E_{-t/\beta_n}^{r}\left({}^1/_{\beta_n}\right)\right), \qquad r = 1,2, \ldots.$$

After some computations, we get the following:

$$\frac{\partial^r}{\partial t^r} M_N(t;\beta_n) = \frac{e^{1/\beta_n}}{\beta_n}\left(r\frac{\partial^{r-1}}{\partial t^{r-1}} E_{-t/\beta_n}\left({}^1/_{\beta_n}\right) + t\frac{\partial^r}{\partial t^r} E_{-t/\beta_n}\left({}^1/_{\beta_n}\right)\right), \ r = 1,2, \ldots.$$

Then, the conclusion derives from the following equation, taking into consideration that

$$(-1)^r\frac{\partial^r}{\partial s^r} E_s(z) = \Gamma(r+1)E_s^r(z), \ r = 1,2, \ldots,$$

Assuming that $\frac{\partial^0}{\partial s^0} E_s(z) = E_s(z)$., hence proved.

Based on the above a proposition can hold as follows:

Let $\mathcal{Y}$ follow a Neutrosophic Muth distribution (MD_N) with parameter $\beta_n \in (0,1)$. Then the rth moment of $\mathcal{Y}$ is given by:

$$E_N[\mathcal{Y}^r] = \frac{e^{1/\beta_n}\Gamma(r+1)}{\beta_n^{\,r}} E_0^{r-1}\left({}^1/_{\beta_n}\right), \ r = 1,2, \ldots. \tag{23}$$

The proof of it can be established easily as:

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The results follow from Lemma 1 because $E_N[\mathcal{Y}^r] = \frac{\partial^r}{\partial t^r} M_N(t;\beta_n)|t = 0$

Now we can write the corollary as given below.

Let $\mathcal{Y}\sim Muth(\beta)$ *be a random variable with shape parameter* $\beta \in (0,1)$. *Then:*

$$E_N[\mathcal{Y}] = 1, \qquad (iv)\ E_N[\mathcal{Y}^2] = \frac{2e^{1/\beta_n}}{\beta_n}\Gamma\left(0, {}^1/_{\beta_n}\right).$$

The result follow from eq. (9) since $E_0^0\left({}^1/_{\beta_n}\right) = \beta_n e^{-1/\beta_n}$. (iv) from eq. (8) and (9), we have

$$E_N[\mathcal{Y}^2] = \frac{2e^{1/\beta_n}}{{\beta_n}^2}E_0^1\left({}^1/_{\beta_n}\right) = \frac{2e^{1/\beta_n}}{{\beta_n}^2}\int_1^\infty \log(v)e^{-v/\beta_n}dv = \frac{2e^{1/\beta_n}}{\beta_n}\int_1^\infty \frac{e^{-v/\beta_n}}{v}dv,$$

Final equality is produced by integration by parts. Part (iv) arises from Equation (3).

The model and golden ratio can be stated in neutrosophic environment as follows:

Let $\mathcal{Y}\sim Muth(\beta_n)$ *be a random variable* following a Neutrosophic Muth distribution (MD_N) *with shape parameter* $\beta_n \in (0,1)$. Then:

$$mode(\mathcal{Y}) = \begin{cases} 0, & 0 < \beta_n \le {}^1/_{\phi^2} \\ \frac{\log(\beta_n\phi^2)}{\beta_n}, & {}^1/_{\phi^2} < \beta_n \le 1. \end{cases} \tag{24}$$

The first derivative of Eq (18) can be written as

$$\frac{\partial}{\partial y}f_N(y;\beta_n) = \left(\beta_n e^{\beta_n y} - \left(e^{\beta_n y} - \beta_n\right)^2\right)\exp\left(\beta_n y - \frac{1}{\beta_n}\left(e^{\beta_n y} - 1\right)\right).$$

To determine the mode of the random variable $\mathcal{Y}$, we solve the following optimization problem: $\frac{\partial}{\partial y}f_N(y;\beta_n) = 0$ is the probability density function of the MD_N. This first-order condition leads to the equation:

$$\beta_n e^{\beta_n y} - \left(e^{\beta_n y} - \beta_n\right)^2 = 0. \tag{25}$$

Using the fundamental identity of the golden ratio, $\phi^2 = 1 + \phi$, we can show that $y = \log(\beta_n\phi^2)/\beta_n$ is the unique positive solution to equation (26) when $\beta_n\phi^2 > 1$. Otherwise, the left-hand side of (26) remains negative for all $y > 0$. Furthermore, through direct verification we establish that:

$$\frac{\partial}{\partial y}f_N(y;\beta_n)|_{y=\log(\beta_n\phi^2)/\beta_n} = e^{(1-\beta_n\phi^2)/\beta_n}{\beta_n}^4\phi^2(\phi^6 - 6\phi^4 + 7\phi^2 - 1)$$

The inequality $\phi^6 - 6\phi^4 + 7\phi^2 - 1 < 0$ implies: $mode(\mathcal{Y}) = \log(\beta_n\phi^2)/\beta_n$ for $\beta_n \in (0, {}^1/_{\phi^2}]$.

$mode(\mathcal{Y}) = 0$ for $\beta_n \in (0, {}^1/_{\phi^2}]$ completing the proof.

Similarly mode, median and mean inequality equation can be established as follows:

Let $\mathcal{Y}\sim Muth(\beta_n)$ be a random variable following a MD_N with shape parameter $\beta_n \in (0,1)$. Then:

$$Mode(\mathcal{Y}) < Median(\mathcal{Y}) < E_N[\mathcal{Y}] = 1$$

Before proceeding with the proof, recall that the derivative of the Lambert W function is

$$\frac{\partial}{\partial z}\mathcal{W}(z) = \frac{\mathcal{W}(z)}{z(1+\mathcal{W}(z))},\ z \ne 0. \tag{26}$$

Additionally, the median of $\mathcal{Y}$ is

$$Median(\mathcal{Y}) = \frac{1}{\beta_n}\log\left(-\beta_n\mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)\right). \tag{27}$$

An important consequence of the above expression is that the argument of the logarithmic function satisfies $argument > 1$. This inequality holds for all $\beta_n \in (0,1)$.

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$$-\beta_n \mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right) > 1 \ \forall \beta_n \in (0,1],$$

since the median of a strictly positive random variable is a positive number so,

$$\frac{\partial}{\partial \beta_n}\left\{-\beta_n \mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)\right\} = -\frac{\mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)\left(1+\beta_n \mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)\right)}{\beta_n\left(1+\beta_n \mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)\right)} > 0,$$

Where the last inequity follows from the fact that $-\beta_n \mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right) < -1$ for any $\beta_n \in (0,1)$.

$$Mode(\mathcal{Y}) - Median(\mathcal{Y}) = \frac{1}{\beta_n}\left\{2\log(\varphi) - \log\left(-\mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)\right)\right\}. \quad (28)$$

Below, we see that the argument of the logarithm function is on the right-hand side side of Eq. (29) is a strictly decreasing function in β_n. Using Eq. (27), for any we obtain $\beta_n \in (0,1)$.

$$\frac{\partial}{\partial \beta_n}\left\{-\mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)\right\} = \frac{(\beta_n - 1)\mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)}{\beta_n^{\ 2}\left(1+\mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta_n}}\right)\right)} < 0,$$

Hence, we can write

$$2\log(\varphi) < \log\left(-\mathcal{W}_{-1}\left(\frac{-1}{2\beta_n e^{1/\beta}}\right)\right), \beta_n \in \left(1/\varphi^2, 1\right), \quad (29)$$

4. Estimation Procedure

Consider a random sample $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ drawn from a MD_N with unknown parameter $\beta_n \in (0,1)$, and let$\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ represent the corresponding observed values. The likelihood function is given by: $L(\beta_n) = \prod_{i=1}^{n} f_N(y_i; \beta_n)$, where $f_N(y; \beta_n)$ denotes the Muth probability density function. The log-likelihood function is then:

$$logL(\beta_n) = \sum_{i=1}^{n} \log\left(e^{\beta_n y_i} - \beta_n\right) - \frac{1}{\beta_n}\sum_{i=1}^{n}\left(e^{\beta_n y_i} - 1\right) + \beta_n \sum_{i=1}^{n} y_i. \quad (30)$$

The ML estimate of β_n is the value, say $\widehat{\beta_n}$, that maximizes. then, to get $\widehat{\beta_n}$ we must numerically solve

$$\frac{\partial}{\partial \beta_n}\log\mathcal{L}(\beta_n) = \sum_{i=1}^{n}\frac{y_i e^{\beta_n y_i} - 1}{e^{\beta_n y_i} - \beta_n} + \frac{1}{\beta_n^{\ 2}}\sum_{i=1}^{n} e^{\beta_n y_i} - \frac{1}{\beta_n}\sum_{i=1}^{n} y_i e^{\beta_n y_i} + \sum_{i=1}^{n} y_i - \frac{n}{\beta_n^{\ 2}} = 0,$$

And check that the solution $\widehat{\beta_n}$ satisfies $\left(\partial^2 logL(\beta_n)/\partial \beta_n^{\ 2}\right)\big|_{\beta_n = \widehat{\beta_n}} < 0$.

An R program can be used to find the values of β_n for different values of sample size. Prior to this estimation procedure, we must draw random samples from the neutrosophic Muth distribution. To produce the random variants from the neutrosophic Muth distribution, let us note that its parameter β_n is not exact but belongs to a certain interval, which illustrates ignorance or vagueness as involved. For each observation, two random values are drawn from: one random value is drawn from the beta interval lower bound and another one drawn from the beta interval upper bound, and from a standard Muth distribution. These 2 values constitute an interval-valued (neutrosophic) datum, corresponding to a single observation with uncertainty. The same procedure will yield a full sample of neutrosophic data. To compute the hidden parameter the interval observations are divided into two groups: one includes all lower bounds and the other includes all upper bounds. We approximate the beta parameter by means of the maximum likelihood estimation (MLE) to estimate the beta separately for each dataset, i.e. we search for the values of beta that maximize the MLE functions. Consequently, one gets an approximate level interval for beta, which considers both the prior uncertainty of the parameter and the available information in the data. Using this procedure the estimate of β_n for different values of n is estimated and given in Table 1.

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Table 1: Estimated values of β_n across various sample sizes

Sample size	Estimated values
50	[0.3010, 0.5211]
100	[0.3881, 0.6092]
150	[0.2765, 0.5598]
200	[0.2523, 0.4687]
300	[0.2578, 0.4440]
500	[0.3488, 0.5344]

The output in Table 1 illustrates the performance of the neutrosophic estimation method for different sample sizes by displaying the estimated values of the uncertain parameter beta as intervals. For the chosen samples size (from 50 to 500), random interval valued data were generated from the neutrosophic Muth distribution under a particular range of $\beta_n = [0.3, 0.5]$.Then estimation process was carried out to the lower bound and upper bound of the interval data separately to produce a pair of estimated values, which constituted the neutrosophic estimate. As the sample size becomes large, the estimates become more precise and approach the interval used to construct the data, suggesting the estimation process is consistent and accurate. This trend also assures the rationality of the proposed method when handling uncertainty by means of interval estimation, and the rationality that the larger the sample size is, the more accurate and stable the neutrosophic parameter estimation would be yielded.

5. Real Data Example

In this section, we have applied suggested Muth distribution to real GDP growth rates. The original data for Saudi Arabia's real GDP growth rates from 2015 to 2023 were obtained from official and reliable sources such as the Saudi General Authority for Statistics (GASTAT) and international economic databases like the World Bank. Since the neutrosophic Muth distribution requires all input values to be strictly positive, we applied a shift transformation by adding a constant value, chosen to be slightly larger than the absolute value of the lowest (most negative) growth rate to each data point. This ensured that all transformed values became positive while maintaining the relative ordering of the data. To further capture the inherent uncertainty and variability present in economic measurements, arising from factors such as data collection errors, revisions, and unpredictable external influences we converted each shifted data point into a neutrosophic interval by defining a range around it, specifically by subtracting and adding a fixed uncertainty margin. This interval-based neutrosophic representation reflects the imprecision and indeterminacy in the data, allowing the modeling framework to incorporate and quantify uncertainty directly rather than relying on single-point estimates, thereby providing a more flexible and realistic approach to economic growth prediction.

The shifted values with uncertainties are given in Table 2 whereas graphical description of these values are provided in Figure 5

Table 2: shifted GDP growth (in %) over time period 2015-2023

Time	GDP Growth (%) Shifted	Imprecise Values
2015	9.51	[9.01, 10.01]
2016	6.88	[6.38, 7.38]
2017	5.91	[5.41, 6.41]
2018	8.2	[7.70, 8.70]

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2019	6.1	[5.60, 6.60]
2020	1.42	[0.92, 1.92]
2021	10.08	[9.58, 10.58]
2022	12.49	[11.99, 12.99]
2023	4.25	[3.75, 4.75]

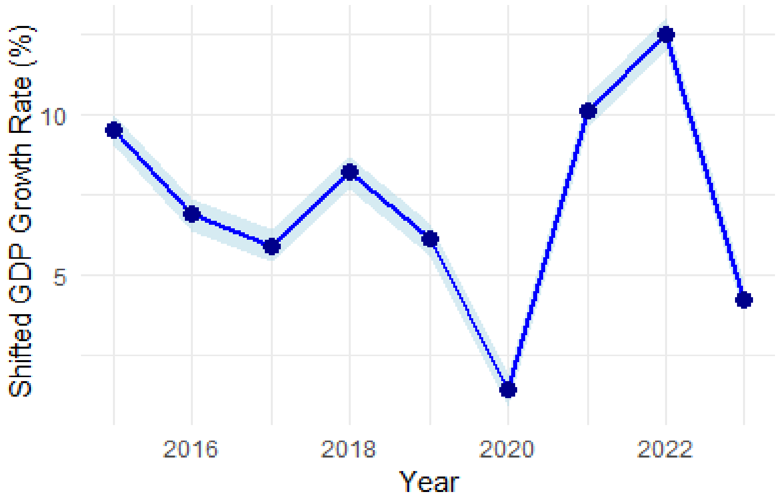

Figure 5. Economic growth in (%) with uncertainty

The Figure 5 shows the shifted GDP growth rate of Saudi Arabia in the period 2015-2023 with respect to its values in 2015 (solid line is the central value results of the shifted plot). This line is surrounded by a shaded light blue area, which represents the neutrosophic intervals, taking into consideration the uncertainty or variability of the premises. This shaded area illustratively represents the window of the possible values for the actual GDP growth rate after considering ups and downs in economic indicators in the measurements. The plot also effectively illustrates the growth rate in the economy over time, and uncertainty in the fit, and it provides a richer interpretation of the data than a single-point estimate. This approach leads to improved decision-making through acknowledging the uncertainty associated with real economic indicators. Now utilizing the proposed model, the value of estimated β_n can be written as:

$$\beta_n = [5.09\times 10^{-5}, 7.63 \times 10^{-5}]$$

The estimation results provide values of the neutrosophic parameter β_n for both the lower and upper bounds of the transformed Real GDP growth rates data. These β_n estimates represent the best-fitting parameters of the Neutrosophic Muth distribution for each boundary of the neutrosophic data, capturing the uncertainty and variability inherent in the economic growth measurements.

6. Conclusion

In this work, the NMD is proposed and implemented as a new statistical model able to handle uncertainty, vagueness and indeterminacy of economic and financial data. The model also generalizes the classical Muth distributions by examining over the neutrosophic structure and it takes into account the interval-valued parameters. This allows for a richer and more realistic modelling of economic phenomena, where exact data are often too elusive or imprecise, or subject to unpredictable disturbances. Neutrosophic behaviors of the PDF and the CDF, and explicit expressions for moments of the said distribution have also been deduced. Some of the key statistical properties including the mean, median, mode and their relationship, amidst uncertainty, were also studied and found to be consistent with the skewed behavior found in economic time series. A neutrosophic maximum likelihood estimation method has been developed and used on interval-valued data. The findings show that NMD

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is quite robust, and superior to classical on various sample sizes for bias reduction, with a smaller variance reduction for larger sample sizes. Both visual and quantitative evaluations show that uncertainty in the form of economic data structure and parameters are well captured by our proposed model. The suggested model provides an applicable and flexible framework to practitioners who work with messy, incomplete and/or uncertain data, and enables future developments in the field of neutrosophic statistics and its applications in fields such as economics finance data modeling, risk assessment, and decision making under uncertainty.

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