



The Integration of Symbolic 2-Plithogenic and Symbolic 3-Plithogenic Rational Functions

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Abstract

The primary aim of this study is to explore the integration of symbolic 2-plithogenic and 3-plithogenic rational functions by formulating explicit and simplified rules to facilitate their evaluation, by using the division symbolic 2-plithogenic and symbolic 3-plithogenic rational numbers respectively. In addition to the theoretical proof of these rules, relevant examples are provided to illustrate these ideas.

Keywords: Symbolic 2-plithogenic rational functions; Symbolic 3-plithogenic rational functions; Division; Integral

1 introduction

Several researchers have explored the idea of refined neutrosophic structures.¹⁻⁵ Smarandache introduced symbolic plithogenic algebraic structures, which closely resemble refined neutrosophic structures, though they differ somewhat in the way the multiplication operation is defined.¹⁵

In,¹² the algebraic characteristics of symbolic 2-plithogenic rings formed by combining symbolic plithogenic sets with algebraic rings are examined, including foundational properties and substructures such as AH-ideals, AH-homomorphisms, and AHS-isomorphisms. Additional algebraic aspects of symbolic 2-plithogenic rings are explored in.⁶ Subsequently, various researchers have introduced and investigated symbolic 3-plithogenic algebraic structures, including symbolic 3-plithogenic rings, vector spaces, and modules.⁶⁻⁸

Recently, Musa and Alhasan¹⁴ studied the integral of 2-refined rational neutrosophic functions, by providing direct rules for finding the integral of 2-refined rational neutrosophic functions, using the division of refined neutrosophic numbers. Motivated by this work, in this article we studied the integration of symbolic 2-plithogenic and 3-plithogenic rational functions, by using the division symbolic 2-plithogenic and symbolic 3-plithogenic rational numbers respectively.

2 Preliminaries

Definition 2.1. ¹² Let R be a ring, the symbolic 2-plithogenic ring is defined as follows:

$$2 - SP_R = \{l_0 + l_1P_1 + l_2P_2; l_i \in R, P_j^2 = P_j, P_1 \times P_2 = P_{\max(1,2)} = P_2\}$$

Addition on $2 - SP_R$ is defined as follows:

$$[l_0 + l_1P_1 + l_2P_2] + [m_0 + m_1P_1 + m_2P_2] = (l_0 + m_0) + (l_1 + m_1)P_1 + (l_2 + m_2)P_2$$

Multiplication on $2 - SP_R$ is defined as follows:

$$[l_0 + l_1P_1 + l_2P_2] \cdot [m_0 + m_1P_1 + m_2P_2] = l_0m_0 + l_0m_1P_1 + l_0m_2P_2 + l_1m_0P_1^2 + l_1m_2P_1P_2 + l_2m_0P_2 + l_2m_1P_1P_2 + l_2m_2P_2^2 + l_1m_1P_1P_1 = (l_0m_0) + (l_0m_1 + l_1m_0 + l_1m_1)P_1 + (l_0m_2 + l_1m_2 + l_2m_0 + l_2m_1 + l_2m_2)P_2.$$

Theorem 2.2. ¹² Let $2 - SP_R$ be a 2-plithogenic symbolic ring, with unity 1 and $P = l_0 + l_1P_1 + l_2P_2$ be an arbitrary element, then:

1. P is invertible if and only if $l_0, l_0 + l_1, l_0 + l_1 + l_2$ are invertible.
2. $P^{-1} = l_0^{-1} + [(l_0 + l_1)^{-1} - l_0^{-1}]P_1 + [(l_0 + l_1 + l_2)^{-1} - (l_0 + l_1)^{-1}]P_2$

Definition 2.3. ¹¹ The ring of symbolic 3-plithogenic real numbers is defined as follows:

$$3 - SP_R = \{l_0 + l_1P_1 + l_2P_2 + l_3P_3; P_i^2 = P_i, P_i \times P_j = P_{\max(i,j)}\}$$

Addition on $3 - SP_R$ is defined as follows:

$$[l_0 + l_1P_1 + l_2P_2 + l_3P_3] + [m_0 + m_1P_1 + m_2P_2 + m_3P_3] = (l_0 + m_0) + (l_1 + m_1)P_1 + (l_2 + m_2)P_2 + (l_3 + m_3)P_3.$$

Multiplication on $3 - SP_R$ is defined as follows:

$$[l_0 + l_1P_1 + l_2P_2 + l_3P_3] \cdot [m_0 + m_1P_1 + m_2P_2 + m_3P_3] = (l_0m_0) + (l_0m_1 + l_1m_0 + l_1m_1)P_1 + (l_0m_2 + l_2m_1 + l_2m_2 + l_2m_0 + l_1m_2)P_2 + (l_0m_3 + l_1m_3 + l_2m_3 + l_3m_3 + l_3m_0 + l_3m_1 + l_3m_2)P_3.$$

Theorem 2.4. ¹¹ Let $P = l_0 + l_1P_1 + l_2P_2 + l_3P_3 \in 3 - SP_R$, then:

P is invertible if and only if $l_0 \neq 0, l_0 + l_1 \neq 0, l_0 + l_1 + l_2 \neq 0, l_0 + l_1 + l_2 + l_3 \neq 0$, and

$$P^{-1} = \frac{1}{l_0} + \left[\frac{1}{l_0 + l_1} - \frac{1}{l_0} \right] P_1 + \left[\frac{1}{l_0 + l_1 + l_2} - \frac{1}{l_0 + l_1} \right] P_2 + \left[\frac{1}{l_0 + l_1 + l_2 + l_3} - \frac{1}{l_0 + l_1 + l_2} \right] P_3.$$

Definition 2.5. Let $l_1 + m_1P_1 + n_1P_2$ and $l_2 + m_2P_1 + n_2P_2$ be any two symbolic 2-plithogenic numbers, then

$$\frac{l_1 + m_1P_1 + n_1P_2}{l_2 + m_2P_1 + n_2P_2} = \frac{l_1}{l_2} + \frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)}P_1 + \frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)}P_2$$

exists only if $l_2 \neq 0, l_2 + m_2 \neq 0, l_2 + m_2 + n_2 \neq 0$.

Definition 2.6. Let $l_1 + m_1P_1 + n_1P_2 + r_1P_3$ and $l_2 + m_2P_1 + n_2P_2 + r_2P_3$ be any two symbolic 3-plithogenic numbers, then

$$\begin{aligned} \frac{l_1 + m_1P_1 + n_1P_2 + r_1P_3}{l_2 + m_2P_1 + n_2P_2 + r_2P_3} &= \frac{l_1}{l_2} + \frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)}P_1 + \frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)}P_2 \\ &\quad + \frac{r_1(l_2 + m_2 + n_2) - r_2(l_1 + m_1 + n_1)}{(l_2 + m_2 + n_2)(l_2 + m_2 + n_2 + r_2)}P_3 \end{aligned}$$

exists only if $l_2 \neq 0, l_2 + m_2 \neq 0, l_2 + m_2 + n_2 \neq 0, l_2 + m_2 + n_2 + r_2 \neq 0$.

Definition 2.7. ¹⁰ Let $l_0 + l_1P_1 + l_2P_2$ be a symbolic 2-plithogenic number, then

$$\sqrt{l_0 + l_1P_1 + l_2P_2} = \sqrt{l_0} + \left(-\sqrt{l_0} + \sqrt{l_0 + l_1} \right) P_1 + \left(-\sqrt{l_0 + l_1} + \sqrt{l_0 + l_1 + l_2} \right)$$

where, $-\sqrt{l_0} \geq 0, -\sqrt{l_0} + \sqrt{l_0 + l_1} \geq 0, -\sqrt{l_0 + l_1} + \sqrt{l_0 + l_1 + l_2} \geq 0$.

3 The integral of standard symbolic 2-plithogenic rational functions

• Standard symbolic 2-plithogenic rational integral-Case I

$$\int \frac{l_1 + m_1P_1 + n_1P_2}{x^2 - (l_2 + m_2P_1 + n_2P_2)^2} dx = \frac{1}{2} \left(\frac{l_1}{l_2} + \left[\frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)} \right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)} \right] P_2 \right) \ln \left| \frac{x - l_2 - m_2P_1 - n_2P_2}{x + l_2 + m_2P_1 + n_2P_2} \right| + C$$

where $C = c_0 + c_1P_1 + c_2P_2, l_2 \neq 0, l_2 + m_2 \neq 0, l_2 + m_2 + n_2 \neq 0$ and c_0, c_1, c_2 are real numbers.

Proof.

$$\begin{aligned} \frac{l_1 + m_1P_1 + n_1P_2}{x^2 - (l_2 + m_2P_1 + n_2P_2)^2} &= \frac{l_1 + m_1P_1 + n_1P_2}{(x + (l_2 + m_2P_1 + n_2P_2))(x - (l_2 + m_2P_1 + n_2P_2))} \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \frac{2(l_1 + m_1P_1 + n_1P_2)}{(x + (l_2 + m_2P_1 + n_2P_2))(x - (l_2 + m_2P_1 + n_2P_2))} \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \frac{(x + (l_1 + m_1P_1 + n_1P_2)) - (x - (l_1 + m_1P_1 + n_1P_2))}{(x + (l_2 + m_2P_1 + n_2P_2))(x - (l_2 + m_2P_1 + n_2P_2))} \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \left[\frac{1}{x - l_2 - m_2P_1 - n_2P_2} - \frac{1}{x + l_2 + m_2P_1 + n_2P_2} \right] \\ \Rightarrow \int \frac{l_1 + m_1P_1 + n_1P_2}{x^2 - (l_2 + m_2P_1 + n_2P_2)^2} dx &= \int \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \left[\frac{1}{x - l_2 - m_2P_1 - n_2P_2} - \frac{1}{x + l_2 + m_2P_1 + n_2P_2} \right] dx \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \left(\int \frac{1}{x - l_2 - m_2P_1 - n_2P_2} dx - \int \frac{1}{x + l_2 + m_2P_1 + n_2P_2} dx \right) \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} (\ln|x - l_2 - m_2P_1 - n_2P_2| - \ln|x + l_2 + m_2P_1 + n_2P_2|) + C \\ &= \frac{1}{2} \left(\frac{l_1}{l_2} + \left[\frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)} \right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)} \right] P_2 \right) \ln \left| \frac{x - l_2 - m_2P_1 - n_2P_2}{x + l_2 + m_2P_1 + n_2P_2} \right| + C. \end{aligned}$$

□

Example 3.1. Evaluate:

$$\int \frac{4 - 2P_1 + P_2}{x^2 - (9 + 7P_1 - 12P_2)} dx$$

Solution:

$$\int \frac{4 - 2P_1 + P_2}{x^2 - (9 + 7P_1 - 12P_2)} dx = \int \frac{4 - 2P_1 + P_2}{x^2 - (\sqrt{9 + 7P_1 - 12P_2})^2} dx$$

Here,

$$\begin{aligned} \sqrt{9 + 7P_1 - 12P_2} &= \sqrt{9} + (\sqrt{16} - \sqrt{9}) P_1 + (\sqrt{4} - \sqrt{16}) P_2 \\ &= 3 + P_1 - 2P_2 \end{aligned}$$

So, we have

$$\begin{aligned} \int \frac{4 - 2P_1 + P_2}{x^2 - (9 + 7P_1 - 12P_2)} dx &= \int \frac{4 - 2P_1 + P_2}{x^2 - (3 + P_1 - 2P_2)^2} dx \\ &= \frac{1}{2} \left(\frac{2 - 3P_1 + P_2}{3 + P_1 - 2P_2} \right) \ln \left| \frac{x - 3 - P_1 + 2P_2}{x + 3 + P_1 - 2P_2} \right| + C \\ &= \left(\frac{1}{3} - \frac{11}{24}P_1 + \frac{1}{8}P_2 \right) \ln \left| \frac{x - 3 - P_1 + 2P_2}{x + 3 + P_1 - 2P_2} \right| + C \end{aligned}$$

Where, $C = c_0 + c_1P_1 + c_2P_2$ and c_0, c_1, c_2 are real numbers.

Verification :

$$\begin{aligned} & \frac{d}{dx} \left[\left(\frac{1}{3} - \frac{11}{24}P_1 + \frac{1}{8}P_2 \right) \ln \left| \frac{x-3-P_1+2P_2}{x+3+P_1-2P_2} \right| + C \right] \\ &= \frac{d}{dx} \left[\left(\frac{1}{3} - \frac{11}{24}P_1 + \frac{1}{8}P_2 \right) (\ln |x-3-P_1+2P_2| - \ln |x+3+P_1-2P_2|) + C \right] \\ &= \left(\frac{1}{3} - \frac{11}{24}P_1 + \frac{1}{8}P_2 \right) \left(\frac{1}{x-3-P_1+2P_2} - \frac{1}{x+3+P_1-2P_2} \right) \\ &= \left(\frac{1}{3} - \frac{11}{24}P_1 + \frac{1}{8}P_2 \right) \left(\frac{x+3+P_1-2P_2 - (x-3-P_1+2P_2)}{(x-3-P_1+2P_2)(x+3+P_1-2P_2)} \right) \\ &= \left(\frac{1}{3} - \frac{11}{24}P_1 + \frac{1}{8}P_2 \right) \left(\frac{2(3+P_1-2P_2)}{x^2 - (3+P_1-2P_2)^2} \right) \\ &= \left(\frac{1}{3} - \frac{11}{24}P_1 + \frac{1}{8}P_2 \right) \left(\frac{6+2P_1-4P_2}{x^2 - (9+7P_1-12P_2)} \right) \\ &= \frac{4-2P_1+P_2}{x^2 - (9+7P_1-12P_2)} \end{aligned}$$

• **Standard symbolic 2-plithogenic rational integral-Case II**

$$\begin{aligned} & \int \frac{l_1 + m_1P_1 + n_1P_2}{(l_2 + m_2P_1 + n_2P_2)^2 - x^2} dx \\ &= \frac{1}{2} \left(\frac{l_1}{l_2} + \left[\frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)} \right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)} \right] P_2 \right) \ln \left| \frac{l_2 + m_2P_1 + n_2P_2 + x}{l_2 + m_2P_1 + n_2P_2 - x} \right| + C \end{aligned}$$

where $C = c_0 + c_1P_1 + c_2P_2$, $l_2 \neq 0, l_2 + m_2 \neq 0, l_2 + m_2 + n_2 \neq 0$ and c_0, c_1, c_2 are real numbers.

Proof.

$$\begin{aligned} & \frac{l_1 + m_1P_1 + n_1P_2}{(l_2 + m_2P_1 + n_2P_2)^2 - x^2} = \frac{l_1 + m_1P_1 + n_1P_2}{((l_2 + m_2P_1 + n_2P_2) + x)((l_2 + m_2P_1 + n_2P_2) - x)} \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \frac{2(l_1 + m_1P_1 + n_1P_2)}{((l_2 + m_2P_1 + n_2P_2) + x)((l_2 + m_2P_1 + n_2P_2) - x)} \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \frac{((l_1 + m_1P_1 + n_1P_2) + x) - ((l_1 + m_1P_1 + n_1P_2) - x)}{((l_2 + m_2P_1 + n_2P_2) + x)((l_2 + m_2P_1 + n_2P_2) - x)} \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \left[\frac{1}{l_2 + m_2P_1 + n_2P_2 + x} + \frac{1}{l_2 + m_2P_1 + n_2P_2 - x} \right] \\ \Rightarrow & \int \frac{l_1 + m_1P_1 + n_1P_2}{(l_2 + m_2P_1 + n_2P_2)^2 - x^2} dx \\ &= \int \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \left[\frac{1}{l_2 + m_2P_1 + n_2P_2 + x} + \frac{1}{l_2 + m_2P_1 + n_2P_2 - x} \right] dx \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} \left(\int \frac{1}{l_2 + m_2P_1 + n_2P_2 + x} dx + \int \frac{1}{l_2 + m_2P_1 + n_2P_2 - x} dx \right) \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{2(l_2 + m_2P_1 + n_2P_2)} (\ln |l_2 + m_2P_1 + n_2P_2 + x| - \ln |l_2 + m_2P_1 + n_2P_2 - x|) + C \\ &= \frac{1}{2} \left(\frac{l_1}{l_2} + \left[\frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)} \right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)} \right] P_2 \right) \ln \left| \frac{l_2 + m_2P_1 + n_2P_2 + x}{l_2 + m_2P_1 + n_2P_2 - x} \right| + C \end{aligned}$$

□

Example 3.2. Evaluate:

$$\int \frac{-5 - 7P_1 + 11P_2}{(4 + 5P_1 + 7P_2) - x^2} dx$$

Solution:

$$\int \frac{-5 - 7P_1 + 11P_2}{(4 + 5P_1 + 7P_2) - x^2} dx = \int \frac{-5 - 7P_1 + 11P_2}{(\sqrt{4 + 5P_1 + 7P_2})^2 - x^2} dx$$

Here,

$$\begin{aligned} \sqrt{4 + 5P_1 + 7P_2} &= \sqrt{4} + (\sqrt{9} - \sqrt{4}) P_1 + (\sqrt{16} - \sqrt{9}) P_2 \\ &= 2 + P_1 + P_2 \end{aligned}$$

So, we have

$$\begin{aligned} \int \frac{-5 - 7P_1 + 11P_2}{(4 + 5P_1 + 7P_2) - x^2} dx &= \int \frac{-5 - 7P_1 + 11P_2}{(2 + P_1 + P_2)^2 - x^2} dx \\ &= \frac{1}{2} \left(\frac{-5 - 7P_1 + 11P_2}{2 + P_1 + P_2} \right) \ln \left| \frac{2 + P_1 + P_2 + x}{2 + P_1 + P_2 - x} \right| + C \\ &= \frac{1}{2} \left(\frac{-5}{2} + \frac{19}{6} P_1 - \frac{35}{12} P_2 \right) \ln \left| \frac{2 + P_1 + P_2 + x}{2 + P_1 + P_2 - x} \right| + C \end{aligned}$$

Where, $C = c_0 + c_1 P_1 + c_2 P_2$ and c_0, c_1, c_2 are real numbers.

Verification :

$$\begin{aligned} &\frac{d}{dx} \left[\frac{1}{2} \left(\frac{-5}{2} + \frac{19}{6} P_1 - \frac{35}{12} P_2 \right) \ln \left| \frac{2 + P_1 + P_2 + x}{2 + P_1 + P_2 - x} \right| + C \right] \\ &= \frac{d}{dx} \left[\left(\frac{1}{3} - \frac{11}{24} P_1 + \frac{1}{8} P_2 \right) (\ln |2 + P_1 + P_2 + x| - \ln |2 + P_1 + P_2 - x|) + C \right] \\ &= \frac{1}{2} \left(\frac{-5}{2} + \frac{19}{6} P_1 - \frac{35}{12} P_2 \right) \left(\frac{1}{2 + P_1 + P_2 + x} + \frac{1}{2 + P_1 + P_2 - x} \right) \\ &= \frac{1}{2} \left(\frac{-5}{2} + \frac{19}{6} P_1 - \frac{35}{12} P_2 \right) \left(\frac{2 + P_1 + P_2 - x + 2 + P_1 + P_2 + x}{(2 + P_1 + P_2 + x)(2 + P_1 + P_2 - x)} \right) \\ &= \frac{1}{2} \left(\frac{-5}{2} + \frac{19}{6} P_1 - \frac{35}{12} P_2 \right) \left(\frac{2(2 + P_1 + P_2)}{(2 + P_1 + P_2)^2 - x^2} \right) \\ &= \left(\frac{-5}{2} + \frac{19}{6} P_1 - \frac{35}{12} P_2 \right) \left(\frac{2 + P_1 + P_2}{(4 + 5P_1 + 7P_2) - x^2} \right) \\ &= \frac{-5 - 7P_1 + 11P_2}{(4 + 5P_1 + 7P_2) - x^2} \end{aligned}$$

• **Standard symbolic 2-plithogenic rational integral-Case III**

$$\begin{aligned} &\int \frac{l_1 + m_1 P_1 + n_1 P_2}{x^2 + (l_2 + m_2 P_1 + n_2 P_2)^2} dx \\ &= \left(\frac{l_1}{l_2} + \left[\frac{l_2 m_1 - l_1 m_2}{l_2(l_2 + m_2)} \right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)} \right] P_2 \right) \\ &\quad \tan^{-1} \left(\left(\frac{1}{l_2} + \left[\frac{1}{l_2 + m_2} - \frac{1}{l_2} \right] P_1 + \left[\frac{1}{l_2 + m_2 + n_2} - \frac{1}{l_2 + m_2} \right] P_2 \right) x \right) + C \end{aligned}$$

where $C = c_0 + c_1 P_1 + c_2 P_2, l_2 \neq 0, l_2 + m_2 \neq 0, l_2 + m_2 + n_2 \neq 0$ and c_0, c_1, c_2 are real numbers.

Proof. Take

$$x = (l_2 + m_2P_1 + n_2P_2)\tan\theta \Rightarrow \tan\theta = \frac{x}{l_2 + m_2P_1 + n_2P_2} \Rightarrow \theta = \tan^{-1}\left(\frac{x}{l_2 + m_2P_1 + n_2P_2}\right)$$

$$dx = (l_2 + m_2P_1 + n_2P_2)\sec^2\theta d\theta$$

Then:

$$\begin{aligned} x^2 + (l_2 + m_2P_1 + n_2P_2)^2 &= (l_2 + m_2P_1 + n_2P_2)^2 \tan^2\theta + (l_2 + m_2P_1 + n_2P_2)^2 \\ &= (l_2 + m_2P_1 + n_2P_2)^2 (\tan^2\theta + 1) \\ &= (l_2 + m_2P_1 + n_2P_2)^2 \sec^2\theta \end{aligned}$$

$$\begin{aligned} \int \frac{l_1 + m_1P_1 + n_1P_2}{x^2 + (l_2 + m_2P_1 + n_2P_2)^2} dx &= \int \frac{l_1 + m_1P_1 + n_1P_2}{(l_2 + m_2P_1 + n_2P_2)^2 \sec^2\theta} (l_2 + m_2P_1 + n_2P_2)^2 \sec^2\theta d\theta \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{l_2 + m_2P_1 + n_2P_2} \int d\theta \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{l_2 + m_2P_1 + n_2P_2} \theta + C \\ &= \frac{l_1 + m_1P_1 + n_1P_2}{l_2 + m_2P_1 + n_2P_2} \tan^{-1}\left(\frac{x}{l_2 + m_2P_1 + n_2P_2}\right) + C \\ &= \left(\frac{l_1}{l_2} + \left[\frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)}\right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)}\right] P_2\right) \\ &\quad \tan^{-1}\left(\left(\frac{1}{l_2} + \left[\frac{1}{l_2 + m_2} - \frac{1}{l_2}\right] P_1 + \left[\frac{1}{l_2 + m_2 + n_2} - \frac{1}{l_2 + m_2}\right] P_2\right) x\right) + C \end{aligned}$$

□

Example 3.3. Evaluate:

$$\int \frac{-3 + 4P_1 - 9P_2}{x^2 + (16 + 9P_1 + 24P_2)} dx$$

Solution:

$$\int \frac{-3 + 4P_1 - 9P_2}{x^2 + (16 + 9P_1 + 24P_2)} dx = \int \frac{-3 + 4P_1 - 9P_2}{x^2 + (\sqrt{16 + 9P_1 + 24P_2})^2} dx$$

Here,

$$\begin{aligned} \sqrt{16 + 9P_1 + 24P_2} &= \sqrt{16} + (\sqrt{25} - \sqrt{16}) P_1 + (\sqrt{49} - \sqrt{25}) P_2 \\ &= 4 + P_1 + 2P_2 \end{aligned}$$

So, we have

$$\begin{aligned} \int \frac{-3 + 4P_1 - 9P_2}{x^2 + (16 + 9P_1 + 24P_2)} dx &= \int \frac{-3 + 4P_1 - 9P_2}{x^2 + (4 + P_1 + 2P_2)^2} dx \\ &= \frac{-3 + 4P_1 - 9P_2}{4 + P_1 + 2P_2} \tan^{-1}\left(\frac{x}{4 + P_1 + 2P_2}\right) + C \\ &= \left(\frac{-3}{4} + \frac{19}{20} P_1 - \frac{47}{35} P_2\right) \tan^{-1}\left(\left(\frac{1}{4} - \frac{1}{20} P_1 - \frac{2}{35} P_2\right) x\right) + C \end{aligned}$$

Where, $C = c_0 + c_1P_1 + c_2P_2$ and c_0, c_1, c_2 are real numbers.

Verification :

$$\begin{aligned} & \frac{d}{dx} \left[\left(\frac{-3}{4} + \frac{19}{20}P_1 - \frac{47}{35}P_2 \right) \tan^{-1} \left(\left(\frac{1}{4} - \frac{1}{20}P_1 - \frac{2}{35}P_2 \right) x \right) + C \right] \\ &= \left(\frac{-3}{4} + \frac{19}{20}P_1 - \frac{47}{35}P_2 \right) \left(\frac{1}{1 + \left(\frac{1}{4} - \frac{1}{20}P_1 - \frac{2}{35}P_2 \right)^2 x^2} \right) \left(\frac{1}{4} - \frac{1}{20}P_1 - \frac{2}{35}P_2 \right) \\ &= \frac{\frac{-3}{4} + \frac{19}{20}P_1 - \frac{47}{35}P_2}{\frac{1}{4} - \frac{1}{20}P_1 - \frac{2}{35}P_2} \frac{1}{\left(x^2 + \frac{1}{\frac{1}{4} - \frac{1}{20}P_1 - \frac{2}{35}P_2} \right)} \\ &= \frac{\frac{-3}{4} + \frac{19}{20}P_1 - \frac{47}{35}P_2}{\frac{1}{4} - \frac{1}{20}P_1 - \frac{2}{35}P_2} \frac{1}{(x^2 + 16 + 9P_1 + 24P_2)} \\ &= (-3 + 4P_1 - 9P_2) \frac{1}{(x^2 + 16 + 9P_1 + 24P_2)} \\ &= \frac{-3 + 4P_1 - 9P_2}{x^2 + (16 + 9P_1 + 24P_2)} \end{aligned}$$

4 The integral of standard symbolic 3-plithogenic rational functions

• Standard symbolic 3-plithogenic rational integral-Case I

$$\begin{aligned} & \int \frac{l_1 + m_1P_1 + n_1P_2 + r_1P_3}{x^2 - (l_2 + m_2P_1 + n_2P_2 + r_2P_3)^2} dx \\ &= \frac{1}{2} \left(\frac{l_1}{l_2} + \left[\frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)} \right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)} \right] P_2 \right. \\ & \quad \left. + \left[\frac{r_1(l_2 + m_2 + n_2) - r_2(l_1 + m_1 + n_1)}{(l_2 + m_2 + n_2)(l_2 + m_2 + n_2 + r_2)} \right] P_3 \right) \ln \left| \frac{x - l_2 - m_2P_1 - n_2P_2 - r_2P_3}{x + l_2 + m_2P_1 + n_2P_2 + r_2P_3} \right| + C \end{aligned}$$

where $C = c_0 + c_1P_1 + c_2P_2 + c_3P_3$, $l_2 \neq 0, l_2 + m_2 \neq 0, l_2 + m_2 + n_2 \neq 0, l_1 + m_1 + n_1 + r_1 \neq 0$ and c_0, c_1, c_2, c_3 are real numbers.

• Standard symbolic 3-plithogenic rational integral-Case II

$$\begin{aligned} & \int \frac{l_1 + m_1P_1 + n_1P_2 + r_1P_3}{(l_2 + m_2P_1 + n_2P_2 + r_2P_3)^2 - x^2} dx \\ &= \frac{1}{2} \left(\frac{l_1}{l_2} + \left[\frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)} \right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)} \right] P_2 \right. \\ & \quad \left. + \left[\frac{r_1(l_2 + m_2 + n_2) - r_2(l_1 + m_1 + n_1)}{(l_2 + m_2 + n_2)(l_2 + m_2 + n_2 + r_2)} \right] P_3 \right) \ln \left| \frac{l_2 + m_2P_1 + n_2P_2 + r_2P_3 + x}{l_2 + m_2P_1 + n_2P_2 + r_2P_3 - x} \right| + C \end{aligned}$$

where $C = c_0 + c_1P_1 + c_2P_2 + c_3P_3$, $l_2 \neq 0, l_2 + m_2 \neq 0, l_2 + m_2 + n_2 \neq 0, l_1 + m_1 + n_1 + r_1 \neq 0$ and c_0, c_1, c_2, c_3 are real numbers.

• Standard symbolic 3-plithogenic rational integral-Case III

$$\begin{aligned} & \int \frac{l_1 + m_1P_1 + n_1P_2 + r_1P_3}{x^2 + (l_2 + m_2P_1 + n_2P_2 + r_2P_3)^2} dx \\ &= \left(\frac{l_1}{l_2} + \left[\frac{l_2m_1 - l_1m_2}{l_2(l_2 + m_2)} \right] P_1 + \left[\frac{n_1(l_2 + m_2) - n_2(l_1 + m_1)}{(l_2 + m_2)(l_2 + m_2 + n_2)} \right] P_2 + \left[\frac{r_1(l_2 + m_2 + n_2) - r_2(l_1 + m_1 + n_1)}{(l_2 + m_2 + n_2)(l_2 + m_2 + n_2 + r_2)} \right] P_3 \right) \\ & \quad \tan^{-1} \left(\left(\frac{1}{l_2} + \left[\frac{1}{l_2 + m_2} - \frac{1}{l_2} \right] P_1 + \left[\frac{1}{l_2 + m_2 + n_2} - \frac{1}{l_2 + m_2} \right] P_2 \right. \right. \\ & \quad \left. \left. + \left[\frac{1}{l_2 + m_2 + n_2 + r_2} - \frac{1}{l_2 + m_2 + n_2} \right] P_3 \right) x \right) + C \end{aligned}$$

where $C = c_0 + c_1P_1 + c_2P_2 + c_3P_3$, $l_2 \neq 0, l_2 + m_2 \neq 0, l_2 + m_2 + n_2 \neq 0, l_1 + m_1 + n_1 + r_1 \neq 0$ and c_0, c_1, c_2, c_3 are real numbers.

Since the derivations for the above cases are closely follow the methodology used for 2-plithogenic rational functions, the corresponding proofs are omitted

Example 4.1. Evaluate:

$$\int \frac{-2 + 3P_1 - P_2 + P_3}{x^2 - (1 + P_1 + 5P_2 - 8P_3)} dx$$

Solution:

$$\int \frac{-2 + 3P_1 - P_2 + P_3}{x^2 - (1 + P_1 + 5P_2 - 8P_3)} dx = \int \frac{-2 + 3P_1 - P_2 + P_3}{x^2 - (\sqrt{1 + P_1 + 5P_2 - 8P_3})^2} dx$$

Here,

$$\begin{aligned} \sqrt{1 + P_1 + 5P_2 - 8P_3} &= \sqrt{1} + (\sqrt{4} - \sqrt{1}) P_1 + (\sqrt{9} - \sqrt{4}) P_2 + (\sqrt{1} - \sqrt{9}) P_3 \\ &= 1 + P_1 + P_2 - 2P_3 \end{aligned}$$

So, we have

$$\begin{aligned} \int \frac{-2 + 3P_1 - P_2 + P_3}{x^2 - (1 + P_1 + 5P_2 - 8P_3)} dx &= \int \frac{-2 + 3P_1 - P_2 + P_3}{x^2 - (1 + P_1 + P_2 - 2P_3)^2} dx \\ &= \frac{1}{2} \left(\frac{-2 + 3P_1 - P_2 + P_3}{1 + P_1 + P_2 - 2P_3} \right) \ln \left| \frac{x - (1 + P_1 + P_2 - 2P_3)}{x + (1 + P_1 + P_2 - 2P_3)} \right| + C \\ &= \left(-1 + \frac{5}{4}P_1 - \frac{1}{4}P_2 + \frac{1}{2}P_3 \right) \ln \left| \frac{x - 1 - P_1 - P_2 + 2P_3}{x + 1 + P_1 + P_2 - 2P_3} \right| + C \end{aligned}$$

Where, $C = c_0 + c_1P_1 + c_2P_2 + c_3P_3$ and c_0, c_1, c_2, c_3 are real numbers.

Example 4.2. Evaluate:

$$\int \frac{-1 - 2P_2 + 11P_3}{(1 + 3P_1 + 5P_2 - 8P_3) - x^2} dx$$

Solution:

$$\int \frac{-1 - 2P_2 + 11P_3}{(1 + 3P_1 + 5P_2 - 8P_3) - x^2} dx = \int \frac{-1 - 2P_2 + 11P_3}{(\sqrt{1 + 3P_1 + 5P_2 - 8P_3})^2 - x^2} dx$$

Here,

$$\begin{aligned} \sqrt{1 + 3P_1 + 5P_2 - 8P_3} &= \sqrt{1} + (\sqrt{4} - \sqrt{1}) P_1 + (\sqrt{9} - \sqrt{4}) P_2 + (\sqrt{1} - \sqrt{9}) P_3 \\ &= 1 + P_1 + P_2 - 2P_3 \end{aligned}$$

So, we have

$$\begin{aligned} \int \frac{-1 - 2P_2 + 11P_3}{(1 + 3P_1 + 5P_2 - 8P_3) - x^2} dx &= \int \frac{-1 - 2P_2 + 11P_3}{(1 + P_1 + P_2 - 2P_3)^2 - x^2} dx \\ &= \frac{1}{2} \left(\frac{-1 - 2P_2 + 11P_3}{1 + P_1 + P_2 - 2P_3} \right) \ln \left| \frac{1 + P_1 + P_2 - 2P_3 + x}{1 + P_1 + P_2 - 2P_3 - x} \right| + C \\ &= \left(\frac{-1}{2} + \frac{1}{4}P_1 + \frac{15}{12}P_2 + \frac{35}{6}P_3 \right) \ln \left| \frac{1 + P_1 + P_2 - 2P_3 + x}{1 + P_1 + P_2 - 2P_3 - x} \right| + C \end{aligned}$$

Where, $C = c_0 + c_1P_1 + c_2P_2 + c_3P_3$ and c_0, c_1, c_2, c_3 are real numbers.

Example 4.3. Evaluate:

$$\int \frac{-1 + 11P_1 + 5P_2 - 7P_3}{x^2 + (4 + 32P_1 + 45P_2 + 19P_3)} dx$$

Solution:

$$\int \frac{-1 + 11P_1 + 5P_2 - 7P_3}{x^2 + (4 + 32P_1 + 45P_2 + 19P_3)} dx = \int \frac{-1 + 11P_1 + 5P_2 - 7P_3}{x^2 + (\sqrt{4 + 32P_1 + 45P_2 + 19P_3})^2} dx$$

Here,

$$\begin{aligned} \sqrt{4 + 32P_1 + 45P_2 + 19P_3} &= \sqrt{4} + (\sqrt{36} - \sqrt{4})P_1 + (\sqrt{81} - \sqrt{36})P_2 + (\sqrt{100} - \sqrt{81})P_3 \\ &= 2 + 4P_1 + 3P_2 + P_3 \end{aligned}$$

So, we have

$$\begin{aligned} \int \frac{-1 + 11P_1 + 5P_2 - 7P_3}{x^2 + (4 + 32P_1 + 45P_2 + 19P_3)} dx &= \int \frac{-1 + 11P_1 + 5P_2 - 7P_3}{x^2 + (2 + 4P_1 + 3P_2 + P_3)^2} dx \\ &= \frac{-1 + 11P_1 + 5P_2 - 7P_3}{4 + 32P_1 + 45P_2 + 19P_3} \tan^{-1} \left(\frac{x}{4 + 32P_1 + 45P_2 + 19P_3} \right) + C \\ &= \left(\frac{-1}{2} + \frac{13}{6}P_1 - \frac{13}{15}P_3 \right) \tan^{-1} \left(\left(\frac{1}{2} - \frac{1}{3}P_1 - \frac{1}{18}P_2 + -\frac{1}{90}P_3 \right) x \right) + C \end{aligned}$$

Where, $C = c_0 + c_1P_1 + c_2P_2 + c_3P_3$ and c_0, c_1, c_2, c_3 are real numbers.

5 Conclusion

This article presents a comprehensive study on the integration of symbolic 2-plithogenic and 3-plithogenic rational functions through the division symbolic 2-plithogenic and symbolic 3-plithogenic rational numbers. The theoretical proofs, supported by illustrative examples, enhance the clarity and applicability of the proposed approach. As a future research, this work can be extended for the case of symbolic 2-plithogenic and 3-plithogenic irrational functions.

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