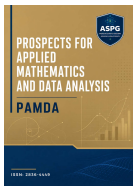
Prospects for Applied Mathematics and Data Analysis

♦ ISSN: 2836-4449 Vol: 05, No. 01, 2025 ♦ PP. 08–19

♦ Article DOI: https://doi.org/10.54216/PAMDA.050102



FULL LENGTH ARTICLE

On the Properties and Illustrative Examples of Soft SuperHypergraphs and Rough SuperHypergraphs

Takaaki Fujita[1,*] **Atiqe Ur Rahman**[2] **Arkan A. Ghaib**[3] **Talal Ali Al-Hawary**[4] **Arif Mehmood Khattak**[5]

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan

[2] Department of Mathematics, University of Management and Technology, Lahore, 54000, Pakistan

[3] Department of Information Technology, Management Technical College, Southern Technical University, Basrah, 61004, Iraq

[4] Department of Mathematics, Yarmouk University, Irbid, Jordan

[5] Department of Mathematics, Institute of Numerical Sciences, Gomal University, Dera Ismail Khan 29050, KPK, Pakistan

Emails: Takaaki.fujita060@gmail.com; aurkhb@gmail.com; arkan.ghaib@stu.edu.iq; talalhawary@yahoo.com; mehdaniyal@gmail.com

Received: December 04, 2024 Revised: January 01, 2025 Accepted: January 27, 2025 ⋆ Corresponding author

ABSTRACT

In graph theory, a hypergraph generalizes a classical graph by allowing each hyperedge to join any number of vertices, thereby modeling relationships beyond simple pairwise connections [1]. A superhypergraph takes this further by applying recursive powerset constructions to its hyperedge set, creating hierarchical and self-referential network layers [2, 3]. A soft graph defines a family of subgraphs parameterized over a fixed universe of vertices and edges, while a rough graph uses lower and upper approximations to capture uncertainty in graph structure. In this paper, we revisit Soft SuperHypergraphs and Rough SuperHypergraphs, integrating established soft and rough hypergraph frameworks [18, 25] with the layered complexity of superhypergraphs. We provide precise definitions, illustrative examples, and a detailed analysis of their fundamental properties, demonstrating their potential for modeling hierarchical and uncertain network systems.

Keywords: Superhypergraph ▪ Hypergraph ▪ Soft Graph ▪ Rough Graph ▪ Soft HyperGraph ▪ Rough HyperGraph

1. INTRODUCTION

In mathematics, a hypergraph [1] is a generalization of the classical graph structure [4]. While a graph consists of a set of vertices (or nodes) and edges that connect pairs of vertices, a hypergraph allows for more flexible relationships. Specifically, a hyperedge can join any number of vertices, not just two; it may connect three, four, or even all of the vertices in the hypergraph. Smarandache [2, 3] extended this concept to n-superhypergraphs and plithogenic n-superhypergraphs. These notions are believed to offer advantages, such as providing an intuitive way to represent hierarchical and complex networks in real-world systems.

Smarandache [3] presented the foundations and real-world interpretation of n-superhypergraphs, while Hamidi et al. [14] studied their spectra through flows. Wang et al. [5] and Feng et al. [6] examined the implementation of hypergraph energy functions in hypergraph neural networks. Razzaque et al. [7] developed intuitionistic fuzzy graph models with applications to complex decision-making problems. Garrett [8] investigated structural properties of superhypergraphs and neutrosophic superhypergraphs, and Ghods et al. [9] introduced restricted neutrosophic superhypergraphs and superhypertrees. Smarandache [10] further explored superhyperstructures and neutrosophic superhyperstructures. Extensions

© 2025 The Author(s). Published by ASPG. This article is licensed under CC BY 4.0.

using fuzzy, neutrosophic, soft, and rough frameworks have been developed; like classical fuzzy graphs [11], these extensions show promise for diverse applications, especially in decision-making contexts.

From these developments, research on hypergraphs, superhypergraphs, and related concepts has proven important. Meanwhile, studies on soft superhypergraphs and rough superhypergraphs remain in their early stages. By reexamining soft superhypergraphs and rough superhypergraphs and investigating their mathematical properties, we aim to promote the usefulness of superhypergraphs. In this paper, we explore soft superhypergraphs and rough superhypergraphs, along with their core properties. These frameworks extend soft graphs and rough graphs by incorporating the hierarchical structure of superhypergraphs.

The organization of this paper is as follows. Section 2 reviews the existing definitions of superhypergraphs and the powerset. Section 3 presents the definition and properties of soft n-superhypergraphs. Section 4 introduces the definition and properties of rough n-superhypergraphs. Section 5 concludes the paper and outlines directions for future work.

2. PRELIMINARIES

In this section, we recall the basic notions and notation that will be used throughout this paper. Unless otherwise specified, all graphs are assumed to be finite, undirected, and simple.

2.1 Superhypergraphs

A hypergraph generalizes a graph by allowing each edge—called a hyperedge—to join any nonempty collection of vertices, thereby modeling higher-order relationships [5, 6]. Building on this idea, a superhypergraph further extends the concept by iterating the powerset construction over a base set, yielding nested, multi-level structures [3, 8].

Definition 2.1 (Base Set). Let S be a nonempty set. We refer to S as a base set when it serves as the fundamental domain from which all subsequent powerset and higher-order constructions are derived. Formally,

$$S = \{x : x \text{ lies in the underlying universe}\}. \tag{1}$$

All elements in iterative constructs such as $\mathrm{POWER}(S)$, $\mathrm{POWER}_n(S)$, or their nonempty variants are drawn from this S.

Definition 2.2 (Powerset). For any set S, the powerset $\mathrm{POWER}(S)$ is the collection of all subsets of S, including both the empty set and S itself [3]:

$$\mathrm{POWER}(S) = \{A : A \subseteq S\}. \tag{2}$$

This construction enumerates every possible combination of elements in S.

Example 2.3 (Powerset of Pizza Toppings). For $S = \{\text{cheese}, \text{tomato}, \text{basil}\}$, $\mathrm{POWER}(S)$ contains all $2^3 = 8$ subsets, ranging from $\varnothing$ to S.

Definition 2.4 (n-th Iterated Powerset). Let H be a nonempty set. The n-th iterated powerset of H, denoted $\mathrm{POWER}_n(H)$, is defined by [2, 3]

$$\mathrm{POWER}_1(H) = \mathrm{POWER}(H), \tag{3}$$

$$\mathrm{POWER}_{n+1}(H) = \mathrm{POWER}\big(\mathrm{POWER}_n(H)\big), \qquad n \geq 1. \tag{4}$$

Dually, the n-th iterated nonempty powerset, written $\mathrm{POWER}^*_n(H)$, omits the empty set at each level:

$$\mathrm{POWER}^*_1(H) = \mathrm{POWER}(H) \setminus \{\varnothing\}, \tag{5}$$

$$\mathrm{POWER}^*_{n+1}(H) = \mathrm{POWER}\big(\mathrm{POWER}^*_n(H)\big) \setminus \{\varnothing\}. \tag{6}$$

These iterated constructions underlie the definition of superhypergraphs and their variants.

Example 2.5 (Iterated Powerset of the Japanese Government Structure). Consider the three main branches of government as the base set

$$H_0 = \{\text{Executive}, \text{Legislative}, \text{Judicial}\}. \tag{7}$$

Their first iterated powerset is

$$\begin{aligned}\mathrm{POWER}_1(H_0) &= \mathscr{P}(H_0) \\ &= \{\varnothing, \{\text{Executive}\}, \{\text{Legislative}\}, \\ &\quad \{\text{Judicial}\}, \{\text{Executive}, \text{Legislative}\}, \\ &\quad \{\text{Executive}, \text{Judicial}\}, \{\text{Legislative}, \text{Judicial}\}, \\ &\quad \{\text{Executive}, \text{Legislative}, \text{Judicial}\}\}.\end{aligned} \tag{8}$$

Omitting the empty set gives the nonempty first powerset $\mathrm{POWER}^*_1(H_0)$.

At the second level, we form all subsets of $\mathrm{POWER}_1(H_0)$:

$$\mathrm{POWER}_2(H_0) = \mathscr{P}\big(\mathrm{POWER}_1(H_0)\big), \tag{9}$$

which includes, for instance,

$$\begin{aligned}&\big\{\{\text{Executive}\}, \{\text{Legislative}, \text{Judicial}\}\big\}, \\ &\big\{\{\text{Executive}, \text{Judicial}\}, \{\text{Legislative}\}, \\ &\qquad \{\text{Executive}, \text{Legislative}, \text{Judicial}\}\big\}, \ldots\end{aligned}$$

One can interpret these second-level elements as "coalitions of coalitions," such as a joint committee of the Executive branch acting together with the combined Legislative–Judicial oversight body.

Proceeding further, the third iterated powerset is

$$\mathrm{POWER}_3(H_0) = \mathscr{P}\big(\mathrm{POWER}_2(H_0)\big), \tag{10}$$

whose members are collections of such second-level coalitions, modeling meta-governance structures (e.g., councils of inter-branch committees).

These nested constructions illustrate how the n-th iterated powerset can capture multi-scale, hierarchical groupings—from individual branches to committees of committees and beyond—providing the foundation for an n-superhypergraph representation of complex governmental networks.

Definition 2.6 (Graph). A graph G is an ordered pair $G = (V, E)$, where [4]:

- V is a nonempty set of vertices (or nodes),
- $E \subseteq \{\{u, v\} : u, v \in V,\ u \neq v\}$ is a set of edges, each con-

necting two distinct vertices.

Example 2.7 (Real-World Graph). Vertices represent users in a social network; edges represent friendships between users.

Definition 2.8 (Hypergraph). A hypergraph $H = (V(H), E(H))$ consists of [1, 12]:

- a nonempty set $V(H)$ of vertices;
- a set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Example 2.9 (Culinary Recipe Hypergraph). A culinary recipe is a set of instructions detailing ingredients and steps to prepare a specific dish or food item [13]. We model a collection of recipes as a hypergraph on the set of basic ingredients.

Vertices (ingredients):

$$V = \{\text{Flour}, \text{Sugar}, \text{Eggs}, \text{Milk}, \text{Butter}, \text{Chocolate}, \text{Baking Powder}, \text{Vanilla}\}. \quad (11)$$

Hyperedges (recipes):

$$e_{\text{Pancakes}} = \{\text{Flour}, \text{Milk}, \text{Eggs}, \text{Baking Powder}, \text{Butter}\}, \quad (12)$$

$$e_{\text{Chocolate Cake}} = \{\text{Flour}, \text{Sugar}, \text{Eggs}, \text{Butter}, \text{Chocolate}, \text{Vanilla}, \text{Baking Powder}\}, \quad (13)$$

$$e_{\text{Cookies}} = \{\text{Flour}, \text{Sugar}, \text{Eggs}, \text{Butter}, \text{Vanilla}\}. \quad (14)$$

Thus

$$E = \{e_{\text{Pancakes}}, e_{\text{Chocolate Cake}}, e_{\text{Cookies}}\}, \qquad H = (V, E). \quad (15)$$

Each recipe corresponds to a hyperedge connecting exactly the set of ingredients it requires. This representation captures higher-order relationships among ingredients—beyond pairwise co-occurrence—facilitating analysis of recipe similarity, substitution patterns, and ingredient networks in culinary data.

Definition 2.10 (n-SuperHyperGraph). Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by [2, 14]

$$\text{POWER}_0(V_0) = V_0, \quad (16)$$

$$\text{POWER}_{k+1}(V_0) = \text{POWER}\big(\text{POWER}_k(V_0)\big), \quad (17)$$

where $\text{POWER}(\cdot)$ denotes the usual powerset operation. An n-SuperHyperGraph is then a pair

$$\text{SuperHyG}(n) = (V, E), \quad (18)$$

with

$$V \subseteq \text{POWER}_n(V_0), \qquad E \subseteq \text{POWER}_n(V_0). \quad (19)$$

Each element of V is called an n-supervertex and each element of E an n-superedge.

Example 2.11 (Cloud Infrastructure as a 2-SuperHyperGraph). Consider a simple cloud deployment whose physical servers form the base set

$$V_0 = \{S_1, S_2, S_3\}. \quad (20)$$

At the first level, we model virtual machines (VMs) as subsets of servers:

$$\text{POWER}_1(V_0) = \{\{S_1\}, \{S_2\}, \{S_3\}, \{S_1, S_2\}\}. \quad (21)$$

Select the VM-set

$$V_1 = \{v_1 = \{S_1\}, v_2 = \{S_2\}, v_3 = \{S_3\}, v_4 = \{S_1, S_2\}\}. \quad (22)$$

At the second level, we view container clusters as subsets of these VMs:

$$\text{POWER}_2(V_0) = \mathscr{P}\big(\text{POWER}_1(V_0)\big), \quad (23)$$

and choose two clusters:

$$V_2 = \{C_1, C_2\}, \qquad C_1 = \{v_1, v_2\}, \quad C_2 = \{v_2, v_3\}. \quad (24)$$

Finally, suppose we have a microservice interaction that spans both clusters, giving the single 2-superedge

$$E_2 = \{C_1, C_2\}. \quad (25)$$

Thus

$$\text{SuperHyG}(2) = \big(V_2, \{E_2\}\big) \quad (26)$$

is a 2-SuperHyperGraph capturing the hierarchy

$$\underbrace{\{S_i\}}_{\text{physical servers}} \longrightarrow \underbrace{\{v_j\}}_{\text{VMs}} \longrightarrow \underbrace{\{C_k\}}_{\text{container clusters}} \longrightarrow \underbrace{E_2}_{\text{microservice interaction}}. \quad (27)$$

In this model:

- Level-0 vertices S_i represent the physical hosts.
- Level-1 supervertices v_j are individual VMs (or small VM clusters).
- Level-2 supervertices C_k group VMs into container clusters.
- The 2-superedge E_2 encodes an application service that communicates across both clusters.

Example 2.12 (3-SuperHyperGraph: Multinational Company Structure). A multinational company operates in multiple countries, managing production or delivering services across international borders with centralized control [15]. We illustrate a 3-SuperHyperGraph modeling the hierarchy of a multinational company.

Base employees:

$$V_0 = \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}\}. \quad (28)$$

Level-1 supervertices (project teams):

$$T_1 = \{\text{Ayano}, \text{Masahiro}\}, \quad (29)$$

$$T_2 = \{\text{Tae}, \text{Shinya}\} \subseteq \text{POWER}_1(V_0). \quad (30)$$

Level-2 supervertices (functional units):

$$U_1 = \{T_1\}, \quad U_2 = \{T_2\}, \quad U_3 = \{T_1, T_2\} \subseteq \mathrm{POWER}_2(V_0). \tag{31}$$

Level-3 supervertices (corporate divisions):

$$D_1 = \{U_1, U_3\}, \qquad D_2 = \{U_2, U_3\}, \tag{32}$$
$$D_3 = \{U_1, U_2\} \subseteq \mathrm{POWER}_3(V_0). \tag{33}$$

Define

$$V = \{D_1, D_2, D_3\}, \tag{34}$$
$$E = \big\{\{D_1, D_3\}, \{D_2, D_3\}\big\} \subseteq \mathrm{POWER}_3(V_0). \tag{35}$$

Then

$$\mathrm{SuperHyG}(3) = (V, E) \tag{36}$$

is a 3-SuperHyperGraph capturing both the hierarchical grouping (employees → teams → units → divisions) and the inter-division collaborations within the company.

3. SOFT n-SUPERHYPERGRAPHS

Soft-set theory provides a parameterized framework for modeling uncertainty without relying on membership functions or probabilities [16]. In graph theory, this idea gives rise to soft graphs, where each parameter selects a subgraph from a fixed host graph [17]. By analogy, soft hypergraphs associate each parameter with a collection of vertices and the induced hyperedges [18, 19]. We now extend these notions to the hierarchical setting of n-superhypergraphs.

Definition 3.1 (Soft Graph). Let $G = (V, E)$ be a simple undirected graph and C a nonempty set of parameters. A soft graph over G with parameter set C is a quadruple

$$(G, C, A, B), \tag{37}$$

where

$$A : C \longrightarrow \mathscr{P}(V), \qquad B : C \longrightarrow \mathscr{P}(E), \tag{38}$$

and for each $c \in C$,

$$B(c) \subseteq \{\{u, v\} \in E : u \in A(c),\ v \in A(c)\}. \tag{39}$$

The pair $(A(c), B(c))$ is called the soft subgraph at parameter c.

Example 3.2 (Soft Graph: Communication Modes in a Company). Let $G = (V, E)$ represent the undirected communication network of a small company:

$$V = \{\mathrm{Ayano}, \mathrm{Masahiro}, \mathrm{Tae}, \mathrm{Shinya}\}, \tag{40}$$

and

$$E = \big\{\{\mathrm{Ayano}, \mathrm{Masahiro}\}, \{\mathrm{Ayano}, \mathrm{Tae}\}, \{\mathrm{Masahiro}, \mathrm{Shinya}\}, \{\mathrm{Tae}, \mathrm{Shinya}\}\big\}, \tag{41}$$

where each edge indicates that the two employees can communicate. Choose the set of communication media as parameters:

$$C = \{\mathrm{Email}, \mathrm{Slack}, \mathrm{Meeting}\}. \tag{42}$$

Define the mappings $A : C \to \mathscr{P}(V)$ and $B : C \to \mathscr{P}(E)$ by

$$A(\mathrm{Email}) = \{\mathrm{Ayano}, \mathrm{Masahiro}, \mathrm{Tae}, \mathrm{Shinya}\}, \tag{43}$$
$$A(\mathrm{Slack}) = \{\mathrm{Ayano}, \mathrm{Masahiro}, \mathrm{Tae}\}, \tag{44}$$
$$A(\mathrm{Meeting}) = \{\mathrm{Masahiro}, \mathrm{Tae}, \mathrm{Shinya}\}, \tag{45}$$

and

$$B(\mathrm{Email}) = \big\{\{\mathrm{Ayano}, \mathrm{Masahiro}\}, \{\mathrm{Ayano}, \mathrm{Tae}\}, \{\mathrm{Masahiro}, \mathrm{Shinya}\}\big\}, \tag{46}$$
$$B(\mathrm{Slack}) = \big\{\{\mathrm{Ayano}, \mathrm{Masahiro}\}, \{\mathrm{Masahiro}, \mathrm{Tae}\}\big\}, \tag{47}$$
$$B(\mathrm{Meeting}) = \big\{\{\mathrm{Masahiro}, \mathrm{Shinya}\}, \{\mathrm{Tae}, \mathrm{Shinya}\}\big\}. \tag{48}$$

One checks that for each medium $c \in C$, every edge in $B(c)$ indeed connects two vertices in $A(c)$. Thus (G, C, A, B) is a valid soft graph modeling how different communication channels induce subgraphs of the overall employee network.

Definition 3.3 (Soft Hypergraph). Let $H = (V, E)$ be a hypergraph with $E \subseteq \mathscr{P}(V)$, and let C be a nonempty parameter set. A soft hypergraph over H with parameters C is a quadruple

$$(H, C, A, B), \tag{49}$$

where

$$A : C \longrightarrow \mathscr{P}(V), \qquad B : C \longrightarrow \mathscr{P}(E), \tag{50}$$

and for every $c \in C$,

$$B(c) \subseteq \{e \in E : e \subseteq A(c)\}. \tag{51}$$

The substructure $(A(c), B(c))$ is called the soft subhypergraph at c.

Example 3.4 (Soft Hypergraph: Community Book Clubs). Community book clubs are local groups where members meet regularly to discuss shared readings, promoting literacy, dialogue, and social interaction. Consider a neighborhood of five readers:

$$V = \{\mathrm{Ayano}, \mathrm{Masahiro}, \mathrm{Tae}, \mathrm{Shinya}, \mathrm{Eve}\}. \tag{52}$$

They form three informal book clubs:

$$e_1 = \{\mathrm{Ayano}, \mathrm{Masahiro}\} \quad (\text{Science Fiction Club}), \tag{53}$$
$$e_2 = \{\mathrm{Masahiro}, \mathrm{Tae}, \mathrm{Shinya}\} \quad (\text{History Club}), \tag{54}$$
$$e_3 = \{\mathrm{Tae}, \mathrm{Eve}\} \quad (\text{Poetry Club}). \tag{55}$$

Thus $E = \{e_1, e_2, e_3\}$ and $H = (V, E)$. Define the parameter set of preferred genres:

$$C = \{\mathrm{SciFi}, \mathrm{History}, \mathrm{Poetry}\}. \tag{56}$$

For each genre $c \in C$, let

$$A(c) = \{\text{readers who enjoy } c\}, \tag{57}$$
$$B(c) = \{\text{clubs whose membership is entirely within } A(c)\}. \tag{58}$$

Concretely:

c	$A(c)$	$B(c)$
SciFi	{Ayano, Masahiro, Tae}	$\{e_1\}$
History	{Masahiro, Tae, Shinya}	$\{e_2\}$
Poetry	{Tae, Eve}	$\{e_3\}$

Here, for example, $B(\text{SciFi}) = \{e_1\}$ since only the Science Fiction Club e_1 has all members (Ayano, Masahiro) enjoying Science Fiction. Hence the quadruple (H, C, A, B) is a soft hypergraph: each $(A(c), B(c))$ is a subhypergraph of H capturing the community's club structure filtered by genre preference.

Definition 3.5 (Soft n-Superhypergraph). Let $\text{SuperHyG}(n) = (V, E)$ be an n-superhypergraph, and let C be a nonempty set of parameters [3, 18]. A soft n-superhypergraph is a quintuple

$$(V, E, C, A, B), \tag{59}$$

where

$$A : C \longrightarrow \text{POWER}(V), \qquad B : C \longrightarrow \text{POWER}(E), \tag{60}$$

such that for each $c \in C$, the pair $(A(c), B(c))$ is a sub-superhypergraph of $\text{SuperHyG}(n)$; namely,

$$A(c) \subseteq V, \qquad B(c) \subseteq \{e \in E : e \subseteq A(c)\}. \tag{61}$$

In other words, $(A(c), B(c))$ inherits the superhypergraph structure on the vertex set $A(c)$ and its induced edges $B(c)$.

Example 3.6 (Soft 2-Superhypergraph: Cross-Department Projects). Cross-department projects involve collaboration between multiple departments, combining diverse expertise to achieve shared organizational goals efficiently and innovatively. We illustrate a soft 2-superhypergraph for a company with four employees:

$$V_0 = \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}\}. \tag{62}$$

Level-1 teams are

$$T_1 = \{\text{Ayano}, \text{Masahiro}\}, \quad T_2 = \{\text{Masahiro}, \text{Tae}\}, \quad T_3 = \{\text{Tae}, \text{Shinya}\} \subseteq \text{POWER}(V_0). \tag{63}$$

Level-2 departments (2-supervertices) are

$$D_1 = \{T_1, T_2\}, \qquad D_2 = \{T_2, T_3\}, \tag{64}$$
$$D_3 = \{T_1, T_3\} \subseteq \text{POWER}_2(V_0), \tag{65}$$

with $V = \{D_1, D_2, D_3\}$. We declare three 2-superedges representing inter-department collaborations:

$$e_1 = \{D_1, D_2\}, \quad e_2 = \{D_2, D_3\}, \quad e_3 = \{D_1, D_3\}, \tag{66}$$
$$E = \{e_1, e_2, e_3\}, \tag{67}$$

so that $\text{SuperHyG}(2) = (V, E)$. Parameters (project names) are

$$C = \{\text{Alpha}, \text{Beta}, \text{Gamma}\}. \tag{68}$$

The soft vertex mapping is

$$A(\text{Alpha}) = \{D_1, D_2\}, \tag{69}$$
$$A(\text{Beta}) = \{D_2, D_3\}, \tag{70}$$
$$A(\text{Gamma}) = \{D_1, D_3\}, \tag{71}$$

and the soft edge mapping is

$$B(\text{Alpha}) = \{e_1\}, \quad B(\text{Beta}) = \{e_2\}, \quad B(\text{Gamma}) = \{e_3\}. \tag{72}$$

Here each $(A(c), B(c))$ is a sub-superhypergraph of $\text{SuperHyG}(2)$; for example,

$$B(\text{Alpha}) = \{e_1\} \subseteq \{e \in E : e \subseteq \{D_1, D_2\}\} = \{e_1\}. \tag{73}$$

Thus (V, E, C, A, B) is a soft 2-superhypergraph, modeling how different cross-department projects select subsets of departments and their collaborations.

Example 3.7 (Soft 2-Superhypergraph: Community Skill-Sharing). Skill-sharing is the collaborative exchange of expertise, where individuals teach and learn skills to enhance mutual growth and productivity [20]. We model a neighborhood of five participants:

$$V_0 = \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}, \text{Eve}\}. \tag{74}$$

Level-1 skill groups are

$$SG_1 = \{\text{Ayano}, \text{Masahiro}\} \quad \text{(Gardening)}, \tag{75}$$
$$SG_2 = \{\text{Tae}, \text{Shinya}\} \quad \text{(Cooking)}, \tag{76}$$
$$SG_3 = \{\text{Masahiro}, \text{Tae}, \text{Eve}\} \quad \text{(Coding)}. \tag{77}$$

Level-2 community centers (2-supervertices) are

$$C_1 = \{SG_1, SG_3\}, \qquad C_2 = \{SG_1, SG_2\}, \tag{78}$$
$$C_3 = \{SG_2, SG_3\} \subseteq \text{POWER}_2(V_0), \tag{79}$$

with $V = \{C_1, C_2, C_3\}$. Community events (2-superedges) are

$$e_1 = \{C_1, C_2\}, \quad e_2 = \{C_2, C_3\}, \quad e_3 = \{C_1, C_3\}, \tag{80}$$
$$E = \{e_1, e_2, e_3\}. \tag{81}$$

Thus $\text{SuperHyG}(2) = (V, E)$ represents the skill-sharing network. Parameters (workshop types) are

$$C = \{\text{Gardening}, \text{Cooking}, \text{Coding}\}. \tag{82}$$

We define the soft mappings $A : C \to \text{POWER}(V)$ and $B : C \to \text{POWER}(E)$ by:

Parameter c	$A(c) \subseteq V$	$B(c) \subseteq E$
Gardening	$\{C_1, C_2\}$	$\{e_1\}$
Cooking	$\{C_2, C_3\}$	$\{e_2\}$
Coding	$\{C_1, C_3\}$	$\{e_3\}$

Here, for example, $A(\text{Gardening}) = \{C_1, C_2\}$ since both centers C_1 and C_2 host gardening groups, and $B(\text{Gardening}) = \{e_1\}$ is the event linking those two centers. Hence (V, E, C, A, B) is a soft 2-superhypergraph, capturing which community centers and events are active under each workshop type.

Example 3.8 (Soft 3-Superhypergraph: University Program Intakes). A university program is a structured curriculum offering academic instruction, training, and certification in a specific field or discipline [21]. We model a small uni-

versity's degree programs built over three base courses:

$$V_0 = \{\text{Math}, \text{Physics}, \text{History}\}. \tag{83}$$

Level-1 supervertices (modules) are

$$M_1 = \{\text{Math}, \text{Physics}\}, \tag{84}$$
$$M_2 = \{\text{Physics}, \text{History}\} \subseteq \text{POWER}(V_0). \tag{85}$$

Level-2 supervertices (curricula) are

$$C_1 = \{M_1\}, \qquad C_2 = \{M_2\}, \tag{86}$$
$$C_3 = \{M_1, M_2\} \subseteq \text{POWER}_2(V_0). \tag{87}$$

Level-3 supervertices (degree programs) are

$$P_1 = \{C_1\}, \quad P_2 = \{C_2\}, \quad P_3 = \{C_1, C_2\} \subseteq \text{POWER}_3(V_0), \tag{88}$$

with $V = \{P_1, P_2, P_3\}$. We declare two 3-superedges representing program overlaps:

$$e_1 = \{P_1, P_3\}, \qquad e_2 = \{P_2, P_3\}, \qquad E = \{e_1, e_2\}. \tag{89}$$

Thus $\text{SuperHyG}(3) = (V, E)$. Intake periods (parameters) are

$$C = \{\text{Fall}, \text{Spring}, \text{Summer}\}. \tag{90}$$

Define $A : C \to \text{POWER}(V)$ and $B : C \to \text{POWER}(E)$ by the table below:

Intake	$A(\text{Intake}) \subseteq V$	$B(\text{Intake}) \subseteq E$
Fall	$\{P_1, P_3\}$	$\{e_1\}$
Spring	$\{P_2, P_3\}$	$\{e_2\}$
Summer	$\{P_1, P_2, P_3\}$	$\{e_1, e_2\}$

For example, in the Fall intake only programs P_1 and P_3 are offered, and the overlap e_1 between them occurs. Hence (V, E, C, A, B) is a soft 3-superhypergraph, modeling which degree programs and their overlaps are active in each intake period.

Example 3.9 (Soft 3-Superhypergraph: Global Research Consortia). Research consortia are collaborative alliances of institutions or researchers working jointly on shared scientific goals, projects, or innovations [22]. We model a global research network built over five researchers:

$$V_0 = \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}, \text{Eve}\}. \tag{91}$$

Level-1 supervertices (research groups) are

$$G_1 = \{\text{Ayano}, \text{Masahiro}\}, \quad G_2 = \{\text{Tae}, \text{Shinya}\},$$
$$G_3 = \{\text{Masahiro}, \text{Tae}, \text{Eve}\} \subseteq \text{POWER}(V_0). \tag{92}$$

Level-2 supervertices (institutes) are

$$I_1 = \{G_1\}, \quad I_2 = \{G_2\}, \quad I_3 = \{G_1, G_3\} \subseteq \text{POWER}_2(V_0). \tag{93}$$

Level-3 supervertices (consortia) are

$$C_1 = \{I_1, I_3\}, \qquad C_2 = \{I_2, I_3\}, \tag{94}$$
$$C_3 = \{I_1, I_2\} \subseteq \text{POWER}_3(V_0), \tag{95}$$

with $V = \{C_1, C_2, C_3\}$. We declare three 3-superedges representing consortium collaborations:

$$e_1 = \{C_1, C_2\}, \quad e_2 = \{C_2, C_3\}, \quad e_3 = \{C_1, C_3\}, \tag{96}$$
$$E = \{e_1, e_2, e_3\}, \tag{97}$$

so that $\text{SuperHyG}(3) = (V, E)$. Funding calls (parameters) are

$$C = \{\text{AI}, \text{Bio}, \text{Energy}\}. \tag{98}$$

Define $A : C \to \text{POWER}(V)$ and $B : C \to \text{POWER}(E)$ by

Call	$A(\text{Call}) \subseteq V$	$B(\text{Call}) \subseteq E$
AI	$\{C_1, C_2\}$	$\{e_1\}$
Bio	$\{C_2, C_3\}$	$\{e_2\}$
Energy	$\{C_1, C_2, C_3\}$	$\{e_1, e_2, e_3\}$

For instance, under the AI call only consortia C_1 and C_2 apply, and the collaboration e_1 between them is active. Thus (V, E, C, A, B) is a soft 3-superhypergraph, modeling which research consortia and their collaborations participate in each funding call.

Theorem 3.10 (Closure under Union). For any $c_1, c_2 \in C$, define a "union" parameter $c_\cup$ by

$$A(c_\cup) = A(c_1) \cup A(c_2), \qquad B(c_\cup) = B(c_1) \cup B(c_2). \tag{99}$$

Then $(A(c_\cup), B(c_\cup))$ is a sub-superhypergraph of $\text{SuperHyG}(n)$. Consequently, extending C by $c_\cup$ yields a new soft n-superhypergraph.

Proof. Since $A(c_1), A(c_2) \subseteq V$, clearly $A(c_\cup) \subseteq V$. Moreover, every edge in $B(c_\cup)$ lies in either $B(c_1)$ or $B(c_2)$, so if $e \in B(c_\cup)$ then $e \subseteq A(c_i) \subseteq A(c_1) \cup A(c_2) = A(c_\cup)$. Hence $B(c_\cup) \subseteq \{e \in E : e \subseteq A(c_\cup)\}$, and $(A(c_\cup), B(c_\cup))$ is an induced sub-superhypergraph. □

Theorem 3.11 (Closure under Intersection). For any $c_1, c_2 \in C$, define an "intersection" parameter $c_\cap$ by

$$A(c_\cap) = A(c_1) \cap A(c_2), \qquad B(c_\cap) = B(c_1) \cap B(c_2). \tag{100}$$

Then $(A(c_\cap), B(c_\cap))$ is a sub-superhypergraph of $\text{SuperHyG}(n)$.

Proof. Clearly $A(c_\cap) \subseteq V$. If $e \in B(c_\cap)$, then $e \in B(c_1)$ and $e \in B(c_2)$, so $e \subseteq A(c_1)$ and $e \subseteq A(c_2)$. Hence $e \subseteq A(c_1) \cap A(c_2) = A(c_\cap)$, showing $B(c_\cap) \subseteq \{e : e \subseteq A(c_\cap)\}$. □

Theorem 3.12 (Induced Soft Sub-n-Superhypergraph). If $C' \subseteq C$ is any nonempty subset of parameters, then

$$(V, E, C', A|_{C'}, B|_{C'}) \tag{101}$$

is also a soft n-superhypergraph.

Proof. Restriction of A and B to C' preserves the property that for each $c \in C'$, $(A(c), B(c))$ is a sub-superhypergraph, because no new parameters are introduced. □

Theorem 3.13 (Crisp Parameter Reduction). If there exists a parameter $c^* \in C$ with

$$A(c^*) = V, \qquad B(c^*) = E, \tag{102}$$

then the soft n-superhypergraph contains the exact superhypergraph SuperHyG(n) as its c^*-slice.

Proof. By $A(c^*) = V$ and $B(c^*) = E$, the pair $(A(c^*), B(c^*))$ equals (V, E), so it recovers the full n-superhypergraph. □

Theorem 3.14 (Complement Closure). For any parameter $c \in C$, define its complement by

$$A(\bar{c}) = V \setminus A(c), \qquad B(\bar{c}) = \{e \in E : e \subseteq V \setminus A(c)\}. \quad (103)$$

Then $(A(\bar{c}), B(\bar{c}))$ is a sub-superhypergraph of SuperHyG(n).

Proof. Since $A(\bar{c}) \subseteq V$ by construction, we need only check that every edge $e \in B(\bar{c})$ satisfies $e \subseteq A(\bar{c})$. But $B(\bar{c})$ was defined exactly as those edges lying entirely in $V \setminus A(c) = A(\bar{c})$. Hence $(A(\bar{c}), B(\bar{c}))$ inherits the induced superhypergraph structure. □

Theorem 3.15 (Difference Closure). For any $c_1, c_2 \in C$, define the difference parameter $c_1 \setminus c_2$ by

$$A(c_1 \setminus c_2) = A(c_1) \setminus A(c_2), \quad (104)$$
$$B(c_1 \setminus c_2) = \{e \in B(c_1) : e \subseteq A(c_1) \setminus A(c_2)\}. \quad (105)$$

Then $(A(c_1 \setminus c_2), B(c_1 \setminus c_2))$ is a sub-superhypergraph of SuperHyG(n).

Proof. Clearly $A(c_1 \setminus c_2) \subseteq A(c_1) \subseteq V$. If $e \in B(c_1 \setminus c_2)$, then $e \in B(c_1)$ and $e \subseteq A(c_1) \setminus A(c_2)$. In particular $e \subseteq A(c_1 \setminus c_2)$, so $(A(c_1 \setminus c_2), B(c_1 \setminus c_2))$ is an induced sub-superhypergraph. □

Theorem 3.16 (Boolean-Algebra Structure). Extend the set of parameters C by adjoining complement symbols $\bar{c}$ for all $c \in C$, and close under union $\cup$ and intersection $\cap$. Then the collection $\{A(d) : d \in C\}$ of vertex sets, together with the corresponding edge sets $\{B(d) : d \in C\}$, forms a Boolean algebra of sub-superhypergraphs:

$$A(d_1 \cup d_2) = A(d_1) \cup A(d_2), \quad (106)$$
$$B(d_1 \cup d_2) = B(d_1) \cup B(d_2), \quad (107)$$
$$A(d_1 \cap d_2) = A(d_1) \cap A(d_2), \quad (108)$$
$$B(d_1 \cap d_2) = B(d_1) \cap B(d_2), \quad (109)$$
$$A(\bar{d}) = V \setminus A(d), \quad (110)$$
$$B(\bar{d}) = \{e : e \subseteq V \setminus A(d)\}, \quad (111)$$

and all De Morgan and distributive laws hold.

Proof. De Morgan's laws $\overline{d_1 \cup d_2} = \bar{d}_1 \cap \bar{d}_2$ and $\overline{d_1 \cap d_2} = \bar{d}_1 \cup \bar{d}_2$ follow directly from the set-theoretic counterparts on V. Distributivity of $\cup$ over $\cap$ (and vice versa) also holds because these operations on $A(\cdot)$ and $B(\cdot)$ reduce to the usual set-theoretic union, intersection, and complement on subsets of V and E. Therefore the parameterized sub-superhypergraphs form a Boolean algebra. □

4. ROUGH n-SUPERHYPERGRAPHS

A rough graph models uncertainty in graph structures by using lower and upper approximations on vertices and edges. A rough graph can also be viewed as a graphical representation of a rough set [23]. A rough set models uncertainty by approximating a set using lower and upper bounds based on an equivalence relation [24]. A rough hypergraph extends rough set theory to hypergraphs, capturing imprecise relationships among groups of vertices via approximations [25]. A rough n-superhypergraph incorporates rough set approximations at multiple hierarchical levels of supervertices and superedges for uncertainty modeling.

Definition 4.1 (Approximation Space). Let U be a nonempty set and R an equivalence relation on U [26]. The pair (U, R) is called an approximation space. For any $X \subseteq U$, the lower and upper approximations of X are

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}, \quad (112)$$
$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \varnothing\}, \quad (113)$$

where $[x]_R = \{y \in U : (x, y) \in R\}$ is the equivalence class of x.

Example 4.2 (Real-World Illustration of an Approximation Space). Let U be the set of patients in a clinic and let R relate two patients if they share the same combination of symptoms (e.g., fever, cough, fatigue). For a target concept $X \subseteq U$ defined as "patients diagnosed with influenza,"

$$\underline{R}(X) = \{p \in U : [p]_R \subseteq X\} \quad (114)$$

is the set of patients whose entire symptom-equivalence class is confirmed influenza, while

$$\overline{R}(X) = \{p \in U : [p]_R \cap X \neq \varnothing\} \quad (115)$$

is the set of patients whose symptom group contains at least one confirmed influenza case.

Definition 4.3 (Rough Graph). Let $G = (V, E)$ be a simple graph and let R_E be an equivalence relation on the edge set E [27]. For any $X \subseteq E$, define

$$\underline{R_E}(X) = \{e \in E : [e]_{R_E} \subseteq X\}, \quad (116)$$
$$\overline{R_E}(X) = \{e \in E : [e]_{R_E} \cap X \neq \varnothing\}. \quad (117)$$

The pair $(\underline{G}, \overline{G})$, where

$$\underline{G} = \big(V, \underline{R_E}(X)\big), \qquad \overline{G} = \big(V, \overline{R_E}(X)\big), \quad (118)$$

is called the rough graph approximation of the subedge-set $X \subseteq E$. If X is not a union of equivalence classes under R_E, then G is said to be a rough graph with respect to R_E.

Example 4.4 (Rough Graph: Regional Road Network). Consider a simplified road network connecting four towns:

$$V = \{T_1, T_2, T_3, T_4\}, \quad (119)$$

with undirected roads (edges)

$$E = \{e_1 = \{T_1, T_2\}, e_2 = \{T_2, T_3\}, e_3 = \{T_3, T_4\}, e_4 = \{T_4, T_1\}, e_5 = \{T_2, T_4\}\}. \quad (120)$$

Define an equivalence relation R_E on E by grouping roads

according to region:

$$[e]_{R_E} = \begin{cases} \{e_1, e_3\}, & e \in \{e_1, e_3\} \quad \text{(Northern roads)}, \\ \{e_2, e_4, e_5\}, & e \in \{e_2, e_4, e_5\} \quad \text{(Southern roads)}. \end{cases} \tag{121}$$

We wish to approximate the subset

$$X = \{e_1, e_2, e_4\} \tag{122}$$

of roads currently designated under maintenance. The lower approximation of X is

$$\underline{R_E}(X) = \{e \in E : [e]_{R_E} \subseteq X\}, \tag{123}$$

but neither equivalence class lies entirely in X, so

$$\underline{R_E}(X) = \varnothing. \tag{124}$$

The upper approximation of X is

$$\overline{R_E}(X) = \{e \in E : [e]_{R_E} \cap X \neq \varnothing\}, \tag{125}$$

and since both Northern and Southern classes meet X, we get

$$\overline{R_E}(X) = \{e_1, e_2, e_3, e_4, e_5\} = E. \tag{126}$$

Hence the rough-graph approximation of X is the pair $(\underline{G}, \overline{G})$ with $\underline{G} = (V, \varnothing)$ and $\overline{G} = (V, E)$. This models the uncertainty in which entire regional road classes are under maintenance: no region is entirely contained in the maintenance set (empty lower approximation), yet both regions are partially affected (full upper approximation).

Definition 4.5 (Rough Hypergraph). Let $H = (V, E)$ be a hypergraph, R_V an equivalence relation on V, and R_E an equivalence relation on E [25]. For any $A \subseteq V$ and $D \subseteq E$, define

$$\underline{R_V}(A) = \{v \in V : [v]_{R_V} \subseteq A\}, \tag{127}$$
$$\overline{R_V}(A) = \{v \in V : [v]_{R_V} \cap A \neq \varnothing\}, \tag{128}$$
$$\underline{R_E}(D) = \{e \in E : [e]_{R_E} \subseteq D\}, \tag{129}$$
$$\overline{R_E}(D) = \{e \in E : [e]_{R_E} \cap D \neq \varnothing\}. \tag{130}$$

The tuple

$$\left(V, E, \underline{R_V}, \overline{R_V}, \underline{R_E}, \overline{R_E}\right) \tag{131}$$

is called a rough hypergraph, capturing uncertainty on both vertices and hyperedges via rough set approximations.

Example 4.6 (Rough Hypergraph: E-Commerce Customer–Product Network). E-commerce is the buying and selling of goods or services through electronic platforms, especially the internet, enabling global transactions. We model an online store with six customers and four products.

Vertices (customers):

$$V = \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}, \text{Eve}, \text{Frank}\}. \tag{132}$$

Hyperedges (product purchasers):

$$e_1 = \{\text{Ayano}, \text{Masahiro}, \text{Tae}\}, \tag{133}$$
$$e_2 = \{\text{Masahiro}, \text{Shinya}, \text{Eve}\}, \tag{134}$$
$$e_3 = \{\text{Tae}, \text{Frank}\}, \tag{135}$$
$$e_4 = \{\text{Ayano}, \text{Eve}, \text{Frank}\}, \qquad E = \{e_1, e_2, e_3, e_4\}. \tag{136}$$

Equivalence on customers (R_V): Customers are grouped by age:

Class	Members
18–24	Ayano, Masahiro, Eve
25–34	Tae, Shinya
35+	Frank

Equivalence on products (R_E): Products are grouped by category:

Category	Hyperedges
Electronics	e_1, e_2
Books	e_3, e_4

Vertex approximations: Let $A = \{\text{Ayano}, \text{Masahiro}, \text{Eve}\}$. Then

$$\underline{R_V}(A) = \{v \in V : [v]_{R_V} \subseteq A\} \tag{137}$$
$$= \{\text{Ayano}, \text{Masahiro}, \text{Eve}\}, \tag{138}$$
$$\overline{R_V}(A) = \{v \in V : [v]_{R_V} \cap A \neq \varnothing\} \tag{139}$$
$$= \{\text{Ayano}, \text{Masahiro}, \text{Eve}\}. \tag{140}$$

Edge approximations: First, for $D_1 = \{e_3, e_4\}$ (Books category),

$$\underline{R_E}(D_1) = \{e \in E : [e]_{R_E} \subseteq D_1\} = \{e_3, e_4\}, \tag{141}$$
$$\overline{R_E}(D_1) = \{e \in E : [e]_{R_E} \cap D_1 \neq \varnothing\} = \{e_3, e_4\}. \tag{142}$$

For $D_2 = \{e_1\}$ (single Electronics product),

$$\underline{R_E}(D_2) = \varnothing, \qquad \overline{R_E}(D_2) = \{e_1, e_2\}. \tag{143}$$

Hence the tuple $(V, E, \underline{R_V}, \overline{R_V}, \underline{R_E}, \overline{R_E})$ is a rough hypergraph capturing uncertainty in both customer segmentation and product-category membership.

Definition 4.7 (Rough n-Superhypergraph). Let $\text{SuperHyG}(n) = (V, E)$ be an n-superhypergraph [3, 25]. Suppose φ is an equivalence relation on V and ψ an equivalence relation on E. For any $X \subseteq V$, define the lower and upper φ-approximations by

$$\underline{\varphi}(X) = \{v \in V : [v]_\varphi \subseteq X\}, \tag{144}$$
$$\overline{\varphi}(X) = \{v \in V : [v]_\varphi \cap X \neq \varnothing\}, \tag{145}$$

where $[v]_\varphi = \{u \in V : (u, v) \in \varphi\}$. Similarly, for any $D \subseteq E$ set

$$\underline{\psi}(D) = \{e \in E : [e]_\psi \subseteq D\}, \tag{146}$$
$$\overline{\psi}(D) = \{e \in E : [e]_\psi \cap D \neq \varnothing\}, \tag{147}$$

where $[e]_\psi = \{f \in E : (f, e) \in \psi\}$. The rough n-superhypergraph is the sextuple

$$\left(V, E, \underbrace{\underline{\varphi}, \overline{\varphi}}_{\varphi\text{-approximations on } V}, \underbrace{\underline{\psi}, \overline{\psi}}_{\psi\text{-approximations on } E}\right). \tag{148}$$

This structure captures uncertainty by replacing each exact vertex or edge set with its rough approximations under φ and ψ.

Example 4.8 (Rough 2-Superhypergraph: Corporate Division Collaborations). Corporate division collaborations involve multiple business units within a company working together to achieve shared objectives and strategic goals. Consider a company with four employees:

$$V_0 = \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}\}. \tag{149}$$

Level-1 supervertices (teams) are

$$T_1 = \{\text{Ayano}, \text{Masahiro}\}, \qquad T_2 = \{\text{Tae}, \text{Shinya}\}, \tag{150}$$

so that $\{T_1, T_2\} \subseteq \text{POWER}(V_0)$. Level-2 supervertices (divisions) are

$$D_1 = \{T_1\}, \qquad D_2 = \{T_2\}, \qquad D_3 = \{T_1, T_2\}, \tag{151}$$

hence $\{D_1, D_2, D_3\} \subseteq \text{POWER}_2(V_0)$ and we set $V = \{D_1, D_2, D_3\}$. Define three collaborations (2-superedges):

$$e_1 = \{D_1, D_3\}, \quad e_2 = \{D_2, D_3\}, \quad e_3 = \{D_1, D_2\}, \tag{152}$$
$$E = \{e_1, e_2, e_3\}. \tag{153}$$

Equivalence on divisions (φ)—by size:

$$\varphi\text{-classes: } \{D_1, D_2\} \text{ (single-team)}, \qquad \{D_3\} \text{ (two-team)}. \tag{154}$$

Equivalence on collaborations (ψ)—by project type:

$$\psi\text{-classes: } \{e_1, e_2\} \text{ (R\&D)}, \qquad \{e_3\} \text{ (Administrative)}. \tag{155}$$

Vertex approximations. Let $X = \{D_1\}$. Then

$$\underline{\varphi}(X) = \{v \in V : [v]_\varphi \subseteq X\} = \varnothing, \tag{156}$$
$$\overline{\varphi}(X) = \{v \in V : [v]_\varphi \cap X \neq \varnothing\} = \{D_1, D_2\}. \tag{157}$$

Edge approximations. Let $D = \{e_1\}$. Then

$$\underline{\psi}(D) = \{e \in E : [e]_\psi \subseteq D\} = \varnothing, \tag{158}$$
$$\overline{\psi}(D) = \{e \in E : [e]_\psi \cap D \neq \varnothing\} = \{e_1, e_2\}. \tag{159}$$

Thus $(V, E, \underline{\varphi}, \overline{\varphi}, \underline{\psi}, \overline{\psi})$ is a rough 2-superhypergraph capturing uncertain division membership and project classification in the company.

Example 4.9 (Rough 2-Superhypergraph: Neighborhood Resource Sharing). We model a city with six residents:

$$V_0 = \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}, \text{Eve}, \text{Frank}\}. \tag{160}$$

Level-1 supervertices (households) are

$$H_1 = \{\text{Ayano}, \text{Masahiro}\}, \quad H_2 = \{\text{Tae}, \text{Shinya}\}, \quad H_3 = \{\text{Eve}, \text{Frank}\} \subseteq \text{POWER}(V_0). \tag{161}$$

Level-2 supervertices (neighborhoods) are

$$N_1 = \{H_1, H_2\}, \qquad N_2 = \{H_2, H_3\}, \tag{162}$$
$$N_3 = \{H_1, H_3\} \subseteq \text{POWER}_2(V_0), \tag{163}$$

with $V = \{N_1, N_2, N_3\}$. We define three neighborhood-collaboration superedges:

$$e_1 = \{N_1, N_2\}, \quad e_2 = \{N_2, N_3\}, \quad e_3 = \{N_1, N_3\}, \tag{164}$$
$$E = \{e_1, e_2, e_3\}. \tag{165}$$

Equivalence on neighborhoods (φ): Group by socio-economic status:

$$\varphi\text{-classes: } \{N_1, N_2\} \text{ (residential)}, \qquad \{N_3\} \text{ (commercial)}. \tag{166}$$

Equivalence on collaborations (ψ): Group by event type:

$$\psi\text{-classes: } \{e_1, e_2\} \quad \text{(food distribution)}, \tag{167}$$
$$\{e_3\} \quad \text{(emergency response)}. \tag{168}$$

Vertex approximations. Let $X = \{N_2\}$. Then

$$\underline{\varphi}(X) = \{v \in V : [v]_\varphi \subseteq X\} = \varnothing, \tag{169}$$
$$\overline{\varphi}(X) = \{v \in V : [v]_\varphi \cap X \neq \varnothing\} = \{N_1, N_2\}. \tag{170}$$

Edge approximations. First, for $D_1 = \{e_1\}$ (food distribution),

$$\underline{\psi}(D_1) = \{e \in E : [e]_\psi \subseteq D_1\} = \varnothing, \tag{171}$$
$$\overline{\psi}(D_1) = \{e \in E : [e]_\psi \cap D_1 \neq \varnothing\} = \{e_1, e_2\}. \tag{172}$$

For $D_2 = \{e_3\}$ (emergency response),

$$\underline{\psi}(D_2) = \{e \in E : [e]_\psi \subseteq D_2\} = \{e_3\}, \tag{173}$$
$$\overline{\psi}(D_2) = \{e \in E : [e]_\psi \cap D_2 \neq \varnothing\} = \{e_3\}. \tag{174}$$

Hence $(V, E, \underline{\varphi}, \overline{\varphi}, \underline{\psi}, \overline{\psi})$ is a rough 2-superhypergraph modeling uncertain neighborhood membership and collaboration types.

Example 4.10 (Rough 3-Superhypergraph: Global Supply-Chain Resilience). A supply chain is a system of organizations, people, activities, and resources involved in producing and delivering goods. We model a simplified global supply chain with four manufacturing sites:

$$V_0 = \{F_1, F_2, F_3, F_4\}. \tag{175}$$

Level-1 clusters (local factories) are

$$C_1 = \{F_1, F_2\}, \qquad C_2 = \{F_3, F_4\}, \tag{176}$$
$$C_3 = \{F_1, F_3\} \subseteq \text{POWER}(V_0). \tag{177}$$

Level-2 hubs (distribution centers) are

$$D_1 = \{C_1\}, \quad D_2 = \{C_2\}, \quad D_3 = \{C_1, C_3\} \subseteq \text{POWER}_2(V_0). \tag{178}$$

Level-3 regions (regional centers) are

$$R_1 = \{D_1\}, \qquad R_2 = \{D_2\}, \tag{179}$$
$$R_3 = \{D_1, D_3\} \subseteq \text{POWER}_3(V_0). \tag{180}$$

Thus $V = \{R_1, R_2, R_3\}$ and we define three 3-superedges capturing inter-region links:

$$e_1 = \{R_1, R_2\}, \quad e_2 = \{R_2, R_3\}, \quad e_3 = \{R_1, R_3\}, \tag{181}$$

$$E = \{e_1, e_2, e_3\}. \tag{182}$$

Equivalence on regions φ (by region type):

$$\varphi\text{-classes: } \{R_1, R_2\} \text{ (urban)}, \qquad \{R_3\} \text{ (mixed)}. \tag{183}$$

Equivalence on links ψ (by product category):

$$\psi\text{-classes: } \{e_1, e_2\} \text{ (electronics)}, \qquad \{e_3\} \text{ (automotive)}. \tag{184}$$

Vertex approximations. Let $X = \{R_1\}$. Then

$$\underline{\varphi}(X) = \{v \in V : [v]_\varphi \subseteq X\} = \varnothing, \tag{185}$$
$$\overline{\varphi}(X) = \{v \in V : [v]_\varphi \cap X \neq \varnothing\} = \{R_1, R_2\}. \tag{186}$$

Edge approximations.

- For the automotive link $D = \{e_3\}$:

$$\underline{\psi}(D) = \{e_3\}, \qquad \overline{\psi}(D) = \{e_3\}. \tag{187}$$

- For an electronics link $D' = \{e_1\}$:

$$\underline{\psi}(D') = \varnothing, \qquad \overline{\psi}(D') = \{e_1, e_2\}. \tag{188}$$

Hence the sextuple $(V, E, \underline{\varphi}, \overline{\varphi}, \underline{\psi}, \overline{\psi})$ is a rough 3-superhypergraph capturing uncertainty in regional classification and inter-region product links.

Example 4.11 (Rough 3-Superhypergraph: Healthcare Referral Network). A healthcare referral network connects providers to coordinate patient care, ensuring timely referrals and specialist consultations across facilities [28]. We model a healthcare system with four physicians:

$$V_0 = \{\mathrm{DrA}, \mathrm{DrB}, \mathrm{DrC}, \mathrm{DrD}\}. \tag{189}$$

Level-1 supervertices (clinics) are

$$Cl_1 = \{\mathrm{DrA}, \mathrm{DrB}\}, \qquad Cl_2 = \{\mathrm{DrC}, \mathrm{DrD}\} \subseteq \mathrm{POWER}(V_0). \tag{190}$$

Level-2 supervertices (hospitals) are

$$H_1 = \{Cl_1\}, \qquad H_2 = \{Cl_2\}, \tag{191}$$
$$H_3 = \{Cl_1, Cl_2\} \subseteq \mathrm{POWER}_2(V_0). \tag{192}$$

Level-3 supervertices (health regions) are

$$R_1 = \{H_1\}, \quad R_2 = \{H_2\}, \quad R_3 = \{H_1, H_3\} \subseteq \mathrm{POWER}_3(V_0). \tag{193}$$

Thus

$$V = \{R_1, R_2, R_3\}, \tag{194}$$
$$E = \{e_1 = \{R_1, R_2\}, e_2 = \{R_2, R_3\}, \tag{195}$$
$$e_3 = \{R_1, R_3\}\}. \tag{196}$$

Equivalence on regions φ (by setting):

$$\varphi\text{-classes: } \{R_1, R_3\} \text{ (rural)}, \qquad \{R_2\} \text{ (urban)}. \tag{197}$$

Equivalence on referral types ψ:

$$\psi\text{-classes: } \{e_1, e_2\} \quad \text{(routine referrals)}, \tag{198}$$
$$\{e_3\} \quad \text{(emergency transfers)}. \tag{199}$$

Vertex approximations. Take $X = \{R_2\}$. Then

$$\underline{\varphi}(X) = \{R_2\}, \qquad \overline{\varphi}(X) = \{R_2\}. \tag{200}$$

Edge approximations. For $D = \{e_2\}$ (routine referral),

$$\underline{\psi}(D) = \varnothing, \qquad \overline{\psi}(D) = \{e_1, e_2\}. \tag{201}$$

Hence the sextuple $(V, E, \underline{\varphi}, \overline{\varphi}, \underline{\psi}, \overline{\psi})$ is a rough 3-superhypergraph capturing uncertainty in regional classification and referral types in a healthcare network.

Theorem 4.12 (Monotonicity of Approximations). For any $X \subseteq Y \subseteq V$,

$$\underline{\varphi}(X) \subseteq \underline{\varphi}(Y), \qquad \overline{\varphi}(X) \subseteq \overline{\varphi}(Y). \tag{202}$$

Similarly, for any $D \subseteq D' \subseteq E$,

$$\underline{\psi}(D) \subseteq \underline{\psi}(D'), \qquad \overline{\psi}(D) \subseteq \overline{\psi}(D'). \tag{203}$$

Proof. If $X \subseteq Y$, then any equivalence class $[v]_\varphi$ contained in X is also contained in Y, so $\underline{\varphi}(X) \subseteq \underline{\varphi}(Y)$. Moreover, if $[v]_\varphi \cap X \neq \varnothing$, then $[v]_\varphi \cap Y \neq \varnothing$, yielding $\overline{\varphi}(X) \subseteq \overline{\varphi}(Y)$. The same arguments apply to $\underline{\psi}$ and $\overline{\psi}$ on E. □

Theorem 4.13 (Duality of Approximations). For any $X \subseteq V$,

$$V \setminus \overline{\varphi}(X) = \underline{\varphi}(V \setminus X), \qquad V \setminus \underline{\varphi}(X) = \overline{\varphi}(V \setminus X). \tag{204}$$

Similarly, for any $D \subseteq E$,

$$E \setminus \overline{\psi}(D) = \underline{\psi}(E \setminus D), \qquad E \setminus \underline{\psi}(D) = \overline{\psi}(E \setminus D). \tag{205}$$

Proof. By definition,

$$\underline{\varphi}(V \setminus X) = \{v : [v]_\varphi \subseteq V \setminus X\} \tag{206}$$
$$= \{v : [v]_\varphi \cap X = \varnothing\} \tag{207}$$
$$= V \setminus \{v : [v]_\varphi \cap X \neq \varnothing\} = V \setminus \overline{\varphi}(X). \tag{208}$$

The other identities follow similarly, and the edge-case dualities mirror these arguments. □

Theorem 4.14 (Crisp Reduction). If φ and ψ are the identity relations on V and E, then for all $X \subseteq V$, $\underline{\varphi}(X) = \overline{\varphi}(X) = X$, and for all $D \subseteq E$, $\underline{\psi}(D) = \overline{\psi}(D) = D$. Hence the rough n-superhypergraph collapses to the exact superhypergraph $\mathrm{SuperHyG}(n) = (V, E)$.

Proof. Under the identity relation, each class $[v]_\varphi = \{v\}$. Thus $\underline{\varphi}(X) = \{v : \{v\} \subseteq X\} = X$ and $\overline{\varphi}(X) = \{v : \{v\} \cap X \neq \varnothing\} = X$. Identical reasoning applies to $\underline{\psi}$ and $\overline{\psi}$. □

Theorem 4.15 (Induced Sub-Rough Superhypergraph). Let $V' \subseteq V$ and $E' = \{e \in E : e \subseteq V'\}$. Restrict φ and ψ to V' and E'. Then

$$\left(V', E', \underline{\varphi}|_{V'}, \overline{\varphi}|_{V'}, \underline{\psi}|_{E'}, \overline{\psi}|_{E'}\right) \tag{209}$$

is a rough n-superhypergraph on the induced substructure.

Proof. Restricting the equivalence relations preserves their properties and the definitions of lower and upper approximations. For any $X' \subseteq V'$, the original inclusion tests remain

valid in V', and similarly for edges, so all axioms hold for the induced tuple. □

Theorem 4.16 (Distribution of Approximations). For any $X_1, X_2 \subseteq V$ and $D_1, D_2 \subseteq E$, we have

$$\underline{\varphi}(X_1 \cap X_2) = \underline{\varphi}(X_1) \cap \underline{\varphi}(X_2), \tag{210}$$
$$\overline{\varphi}(X_1 \cup X_2) = \overline{\varphi}(X_1) \cup \overline{\varphi}(X_2), \tag{211}$$
$$\underline{\psi}(D_1 \cap D_2) = \underline{\psi}(D_1) \cap \underline{\psi}(D_2), \tag{212}$$
$$\overline{\psi}(D_1 \cup D_2) = \overline{\psi}(D_1) \cup \overline{\psi}(D_2). \tag{213}$$

Proof. We prove the first identity; the others are analogous:

$$\begin{aligned} v \in \underline{\varphi}(X_1 \cap X_2) &\iff [v]_\varphi \subseteq X_1 \cap X_2 \\ &\iff [v]_\varphi \subseteq X_1 \text{ and } [v]_\varphi \subseteq X_2 \\ &\iff v \in \underline{\varphi}(X_1) \cap \underline{\varphi}(X_2). \end{aligned}$$

Similarly,

$$\begin{aligned} v \in \overline{\varphi}(X_1 \cup X_2) &\iff [v]_\varphi \cap (X_1 \cup X_2) \neq \varnothing \\ &\iff [v]_\varphi \cap X_1 \neq \varnothing \text{ or } [v]_\varphi \cap X_2 \neq \varnothing \\ &\iff v \in \overline{\varphi}(X_1) \cup \overline{\varphi}(X_2). \end{aligned}$$

□

Definition 4.17 (Definable Set). A subset $X \subseteq V$ is called φ-definable if it is a union of φ-equivalence classes:

$$X = \bigcup_{v \in X} [v]_\varphi. \tag{214}$$

Similarly, $D \subseteq E$ is ψ-definable if D is a union of ψ-classes.

Theorem 4.18 (Characterization of Definable Sets). A subset $X \subseteq V$ is φ-definable if and only if $\underline{\varphi}(X) = X = \overline{\varphi}(X)$. Likewise, $D \subseteq E$ is ψ-definable if and only if $\underline{\psi}(D) = D = \overline{\psi}(D)$.

Proof. If X is a union of whole equivalence classes, then every class meeting X lies entirely in X, so both approximations coincide with X. Conversely, if $\overline{\varphi}(X) = X$, then any class intersecting X must lie in X, so X is a union of classes; the equality $\underline{\varphi}(X) = X$ is equivalent. □

Theorem 4.19 (Best Definable Approximations). For any $X \subseteq V$:

1. $\underline{\varphi}(X)$ is the greatest φ-definable subset of X.
2. $\overline{\varphi}(X)$ is the least φ-definable superset of X.

Analogous statements hold for $\underline{\psi}$ and $\overline{\psi}$ on E.

Proof. By definition, $\underline{\varphi}(X)$ is the union of all equivalence classes contained in X, so any definable subset of X (being a union of classes) must lie within $\underline{\varphi}(X)$. Similarly, $\overline{\varphi}(X)$ is the union of all classes meeting X, hence it is the minimal definable superset. □

5. CONCLUSION AND FUTURE WORKS

In this paper, we have examined Soft SuperHypergraphs and Rough SuperHypergraphs, including their key properties. For future work, we plan to investigate further extensions of SuperHypergraphs using Hypersoft Sets, Hesitant Fuzzy Sets, Plithogenic Sets, Picture Fuzzy Sets, and Neutrosophic Sets. We also intend to explore potential applications of these extended models in computer science and decision-making.

FUNDING

ACKNOWLEDGMENTS

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

REFERENCES

[1] C. Berge, *Hypergraphs: Combinatorics of Finite Sets*. Elsevier, 1984, vol. 45.

[2] F. Smarandache, *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[3] F. Smarandache, "Introduction to the n-superhypergraph—the most general form of graph today," *Neutrosophic Sets and Systems*, vol. 48, no. 1, pp. 483–485, 2022.

[4] S. M. Cioabă and M. R. Murty, *A First Course in Graph Theory and Combinatorics*, 2nd ed., ser. Texts and Readings in Mathematics. Springer Singapore, 2022, vol. 55.

[5] Y. Wang, Q. Gan, X. Qiu, X. Huang, and D. Wipf, "From hypergraph energy functions to hypergraph neural networks," in *International Conference on Machine Learning*. PMLR, 2023, pp. 35 605–35 623.

[6] Y. Feng, H. You, Z. Zhang, R. Ji, and Y. Gao, "Hypergraph neural networks," in *Proceedings of the AAAI Conference on Artificial Intelligence*, vol. 33, 2019, pp. 3558–3565.

[7] A. Razzaque, I. Masmali, L. Latif, U. Shuaib, A. Razaq, G. Alhamzi *et al.*, "On t-intuitionistic fuzzy graphs: A comprehensive analysis and application in poverty reduction," *Scientific Reports*, vol. 13, p. 17027, 2023.

[8] H. Garrett, "Properties of superhypergraph and neutrosophic superhypergraph," *Neutrosophic Sets and Systems*, vol. 49, no. 1, pp. 531–561, 2022.

[9] M. Ghods, Z. Rostami, and F. Smarandache, "Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties," *Neutrosophic Sets and Systems*, vol. 50, no. 1, pp. 480–487, 2022.

[10] F. Smarandache, "Superhyperfunction, superhyperstruc-

ture, neutrosophic superhyperfunction and neutrosophic superhyperstructure: Current understanding and future directions," *Neutrosophic Systems with Applications*, vol. 12, no. 1, pp. 68–76, 2023.

[11] A. Rosenfeld, "Fuzzy graphs," in *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*. Elsevier, 1975, pp. 77–95.

[12] A. Bretto, *Hypergraph Theory: An Introduction*, ser. Mathematical Engineering. Springer, Cham, 2013, vol. 1.

[13] R. Mulcki, F. Rachel, and H. Laksana, "Preservation of west java culinary recipe using augmented reality, photogrammetry, and storytelling illustration," *Indonesian Journal of Computing, Engineering and Design (IJoCED)*, 2023.

[14] M. Hamidi, F. Smarandache, and E. Davneshvar, "Spectrum of superhypergraphs via flows," *Journal of Mathematics*, vol. 2022, no. 1, p. 9158912, 2022.

[15] N. H. Parmenas, "Employee engagement: Turnover prevention strategies and the key to improving performance management in a multinational company," *Journal of Economics, Management, Entrepreneurship, and Business (JEMEB)*, 2022.

[16] D. Molodtsov, "Soft set theory—first results," *Computers & Mathematics with Applications*, vol. 37, no. 4–5, pp. 19–31, 1999.

[17] S. Hussain, J. Hussain, I. Rosyida, and S. Broumi, "Quadripartitioned neutrosophic soft graphs," in *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*. IGI Global, 2022, pp. 771–795.

[18] A. Amini, N. Firouzkouhi, A. Gholami, A. R. Gupta, C. Cheng, and B. Davvaz, "Soft hypergraph for modeling global interactions via social media networks," *Expert Systems with Applications*, vol. 203, p. 117466, 2022.

[19] B. George, K. T. Rajesh, and J. Jinta, "A study on soft hypergraphs and their and & or operations," *Journal of the Calcutta Mathematical Society*, vol. 19, no. 1, pp. 29–44, 2023.

[20] H. T. Maddali and A. Lazar, "Sociality and skill sharing in the garden," in *Proceedings of the 2020 CHI Conference on Human Factors in Computing Systems*. Association for Computing Machinery, 2020, pp. 1–13.

[21] C. Gobec, F. Rillotta, and P. Raghavendra, "Where to next? experiences of adults with intellectual disability after they complete a university program," *Journal of Applied Research in Intellectual Disabilities*, 2022.

[22] M. Morrison, M. Mourby, H. Gowans, S. Coy, and J. Kaye, "Governance of research consortia: Challenges of implementing responsible research and innovation within europe," *Life Sciences, Society and Policy*, vol. 16, no. 1, p. 13, 2020.

[23] R. Noor, I. Irshad, and I. Javaid, "Soft rough graphs," 2017, arXiv: General Mathematics.

[24] Z. Pawlak, "Rough sets," *International Journal of Computer & Information Sciences*, vol. 11, pp. 341–356, 1982.

[25] M. Sarwar, "A theoretical investigation based on the rough approximations of hypergraphs," *Journal of Mathematics*, vol. 2022, no. 1, p. 1540004, 2022.

[26] Z. Pawlak, L. Polkowski, and A. Skowron, "Rough set theory," *KI*, vol. 15, no. 3, pp. 38–39, 2001.

[27] M. Akram, M. Arshad, and Shumaiza, "Fuzzy rough graph theory with applications," *International Journal of Computational Intelligence Systems*, vol. 12, pp. 90–107, 2018.

[28] S. K. Chan, S. C. Chau, S. Y. Chan, C. C. Tong, K.-O. Lam, D. L. W. Kwong, T.-W. Leung, M. Y. Luk, A. W.-M. Lee, H. C. W. Choi, and V. H. F. Lee, "Incidence and demographics of nasopharyngeal carcinoma in cheung chau island of hong kong—a distinct geographical area with minimal residential mobility and restricted public healthcare referral network," *Cancer Control: Journal of the Moffitt Cancer Center*, vol. 28, 2021.