On the Properties and Illustrative Examples of Soft SuperHypergraphs and Rough SuperHypergraphs

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Abstract

In graph theory, a hypergraph generalizes a classical graph by allowing each hyperedge to join any number of vertices, thereby modeling relationships beyond simple pairwise connections.[1] A superhypergraph takes this further by applying recursive powerset constructions to its hyperedge set, creating hierarchical and self-referential network layers.[2] A soft graph defines a family of subgraphs parameterized over a fixed universe of vertices and edges, while a rough graph uses lower and upper approximations to capture uncertainty in graph structure. In this paper, we revisit Soft SuperHypergraphs and Rough SuperHypergraphs—originally introduced in[3]—which integrate the flexibility of soft and rough graph frameworks with the layered complexity of superhypergraphs. We provide precise definitions, illustrative examples, and a detailed analysis of their fundamental properties, demonstrating their potential for modeling hierarchical and uncertain network systems.

Keywords: Superhypergraph; Hypergraph; Soft Graph; Rough Graph; Soft HyperGraph; Rough HyperGraph

1 Introduction

In mathematics, a hypergraph[1] is a generalization of the classical graph[4] structure. While a graph consists of a set of vertices (or nodes) and edges that connect pairs of vertices, a hypergraph allows for more flexible relationships. Specifically, a hyperedge can join any number of vertices, not just two; it may connect three, four, or even all of the vertices in the hypergraph. Smarandache[2] extended this concept to n-superhypergraphs and plithogenic n-superhypergraphs. These notions are believed to offer advantages, such as providing an intuitive way to represent hierarchical and complex networks in real-world systems.

Fujita[3] reviewed various classes of superhypergraphs. Wang et al.[5] and Feng et al.[6] examined the implementation of hypergraph energy functions in hypergraph neural networks. Alqahtani[7] introduced intuitionistic fuzzy quasi-supergraphs and applied them to decision making in social networks. Nalawade et al.[8] investigated structural properties, such as girth and Helly properties, of zero-divisor hypergraphs and superhypergraphs over $\mathbb{Z}_n$. Das et al.[9] conducted a comprehensive study of neutrosophic superhyper BCI-semigroups and their

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algebraic significance. Al-Odhari[10] discussed hyperstructures, superhyperstructures, and n-super superhyperstructures. Smarandache[11] explored the fundamentals of superhyperstructures and neutrosophic superhyperstructures. Extensions of superhypergraphs using fuzzy, neutrosophic, soft, and rough frameworks have been developed; like classical fuzzy graphs,[12] these extensions show promise for diverse applications, especially in decision-making contexts.

From these developments, research on hypergraphs, superhypergraphs, and related concepts has proven important. Meanwhile, studies on soft superhypergraphs and rough superhypergraphs remain in their early stages. By reexamining soft superhypergraphs and rough superhypergraphs and investigating their mathematical properties, we aim to promote the usefulness of superhypergraphs. In this paper, we explore soft superhypergraphs and rough superhypergraphs, along with their core properties. These frameworks extend soft graphs and rough graphs by incorporating the hierarchical structure of superhypergraphs.

The organization of this paper is as follows. In Section 2, we review the existing definitions of superhypergraphs and the powerset. Section 3 presents the definition and properties of soft n-superhypergraphs. Section 4 introduces the definition and properties of rough n-superhypergraphs. Section 5 concludes the paper and outlines directions for future work.

2 Preliminaries

In this section, we recall the basic notions and notation that will be used throughout this paper. Unless otherwise specified, all graphs are assumed to be finite, undirected, and simple.

2.1 Superhypergraphs

A *hypergraph* generalizes a graph by allowing each edge—called a *hyperedge*—to join any nonempty collection of vertices, thereby modeling higher-order relationships.[5,6] Building on this idea, a *superhypergraph* further extends the concept by iterating the powerset construction over a base set, yielding nested, multi-level structures.[7,8]

Definition 2.1 (Base Set). Let S be a nonempty set. We refer to S as a *base set* when it serves as the fundamental domain from which all subsequent powerset and higher-order constructions are derived. Formally,

$$S = \{ x : x \text{ lies in the underlying universe} \}.$$

All elements in iterative constructs such as $\mathrm{POWER}(S)$, $\mathrm{POWER}_n(S)$, or their nonempty variants are drawn from this S.

Definition 2.2 (Powerset). [11] For any set S, the *powerset* $\mathrm{POWER}(S)$ is the collection of all subsets of S, including both the empty set and S itself:

$$\mathrm{POWER}(S) = \{ A : A \subseteq S \}.$$

This construction enumerates every possible combination of elements in S.

Example 2.3 (Powerset of Pizza Toppings). For $S = \{\text{cheese}, \text{tomato}, \text{basil}\}$, $\mathrm{POWER}(S)$ contains all $2^3 = 8$ subsets, ranging from $\emptyset$ to S.

Definition 2.4 (n-th Iterated Powerset). [9–11] Let H be a nonempty set. The *n-th iterated powerset* of H, denoted $\mathrm{POWER}_n(H)$, is defined by

$$\mathrm{POWER}_1(H) = \mathrm{POWER}(H), \qquad \mathrm{POWER}_{n+1}(H) = \mathrm{POWER}\big(\mathrm{POWER}_n(H)\big) \quad (n \geq 1).$$

Dually, the *n-th iterated nonempty powerset*, written $\mathrm{POWER}^*_n(H)$, omits the empty set at each level:

$$\mathrm{POWER}^*_1(H) = \mathrm{POWER}(H) \setminus \{\emptyset\}, \quad \mathrm{POWER}^*_{n+1}(H) = \mathrm{POWER}\big(\mathrm{POWER}^*_n(H)\big) \setminus \{\emptyset\}.$$

These iterated constructions underlie the definition of superhypergraphs and their variants.

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Example 2.5 (Iterated Powerset of the Japanese Government Structure)**.** Consider the three main branches of government as the base set

$$H_0 = \{\text{Executive},\ \text{Legislative},\ \text{Judicial}\}.$$

Their first iterated powerset is

$$\begin{gathered}\mathrm{POWER}_1(H_0) = \mathcal{P}(H_0)\\ = \bigl\{\,\emptyset,\ \{\text{Executive}\},\ \{\text{Legislative}\},\ \{\text{Judicial}\},\ \{\text{Executive, Legislative}\},\ \{\text{Executive, Judicial}\},\\ \{\text{Legislative, Judicial}\},\ \{\text{Executive, Legislative, Judicial}\}\bigr\}.\end{gathered}$$

Omitting the empty set gives the nonempty first powerset $\mathrm{POWER}_1^*(H_0)$.

At the second level, we form all subsets of $\mathrm{POWER}_1(H_0)$:

$$\mathrm{POWER}_2(H_0) = \mathcal{P}\bigl(\mathrm{POWER}_1(H_0)\bigr),$$

which includes, for instance,

$$\bigl\{\{\text{Executive}\},\ \{\text{Legislative, Judicial}\}\bigr\},$$

$$\bigl\{\{\text{Executive, Judicial}\},\ \{\text{Legislative}\},\ \{\text{Executive, Legislative, Judicial}\}\bigr\},\ \ldots$$

One can interpret these second-level elements as "coalitions of coalitions," such as a joint committee of the Executive branch acting together with the combined Legislative–Judicial oversight body.

Proceeding further, the third iterated powerset is

$$\mathrm{POWER}_3(H_0) = \mathcal{P}\bigl(\mathrm{POWER}_2(H_0)\bigr),$$

whose members are collections of such second-level coalitions, modeling meta-governance structures (e.g. councils of inter-branch committees).

These nested constructions illustrate how the n-th iterated powerset can capture multi-scale, hierarchical groupings—from individual branches to committees of committees and beyond—providing the foundation for an n-superhypergraph representation of complex governmental networks.

Definition 2.6 (Graph)**.** [4] A *graph* G is an ordered pair $G = (V, E)$, where:

- V is a nonempty set of *vertices* (or nodes),
- $E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$ is a set of *edges*, each connecting two distinct vertices.

Example 2.7 (Real-World Graph)**.** Vertices represent users in a social network; edges represent friendships between users.

Definition 2.8 (Hypergraph)**.** [1,13] A *hypergraph* $H = (V(H), E(H))$ consists of:

- A nonempty set $V(H)$ of vertices.
- A set $E(H)$ of hyperedges, where each hyperedge is a nonempty subset of $V(H)$, thereby allowing connections among multiple vertices.

Unlike standard graphs, hypergraphs are well-suited to represent higher-order relationships. In this paper, we restrict ourselves to the case where both $V(H)$ and $E(H)$ are finite.

Example 2.9 (Culinary Recipe Hypergraph)**.** A culinary recipe is a set of instructions detailing ingredients and steps to prepare a specific dish or food item (cf.[14]). We model a collection of recipes as a hypergraph on the set of basic ingredients:

Vertices (ingredients):

$$V = \{\text{Flour, Sugar, Eggs, Milk, Butter, Chocolate, Baking Powder, Vanilla}\}.$$

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Hyperedges (recipes):

$$\begin{aligned} e_{\text{Pancakes}} &= \{\text{Flour, Milk, Eggs, Baking Powder, Butter}\}, \\ e_{\text{Chocolate Cake}} &= \{\text{Flour, Sugar, Eggs, Butter, Chocolate, Vanilla, Baking Powder}\}, \\ e_{\text{Cookies}} &= \{\text{Flour, Sugar, Eggs, Butter, Vanilla}\}. \end{aligned}$$

Thus

$$E = \{\, e_{\text{Pancakes}},\ e_{\text{Chocolate Cake}},\ e_{\text{Cookies}} \,\}.$$

Then

$$H = (V, E)$$

is a hypergraph in which each recipe corresponds to a hyperedge connecting exactly the set of ingredients it requires. This representation captures higher-order relationships among ingredients—beyond pairwise co-occurrence—facilitating analysis of recipe similarity, substitution patterns, and ingredient networks in culinary data.

Definition 2.10 (n-SuperHyperGraph). [2,15]

Let V_0 be a finite base set of vertices. For each integer $k \geq 0$, define the iterative powerset by

$$\mathrm{POWER}^0(V_0) = V_0, \quad \mathrm{POWER}^{k+1}(V_0) = \mathrm{POWER}(\mathrm{POWER}^k(V_0)),$$

where $\mathrm{POWER}(\cdot)$ denotes the usual powerset operation. An *n-SuperHyperGraph* is then a pair

$$\mathrm{SuperHyG}^{(n)} = (V, E),$$

with

$$V \subseteq \mathrm{POWER}^n(V_0) \quad \text{and} \quad E \subseteq \mathrm{POWER}^n(V_0).$$

Each element of V is called an *n-supervertex* and each element of E an *n-superedge*.

Example 2.11 (Cloud Infrastructure as a 2-SuperHyperGraph)**.** Consider a simple cloud deployment whose *physical servers* form the base set

$$V_0 = \{S_1, S_2, S_3\}.$$

At the first level, we model *virtual machines* (VMs) as subsets of servers:

$$\mathrm{POWER}^1(V_0) = \{\{S_1\},\ \{S_2\},\ \{S_3\},\ \{S_1, S_2\}\}.$$

Select the VM-set

$$V_1 = \{\, v_1 = \{S_1\},\ v_2 = \{S_2\},\ v_3 = \{S_3\},\ v_4 = \{S_1, S_2\}\}.$$

At the second level, we view *container clusters* as subsets of these VMs:

$$\mathrm{POWER}^2(V_0) = \mathcal{P}\big(\mathrm{POWER}^1(V_0)\big),$$

and choose two clusters:

$$V_2 = \big\{C_1,\ C_2\big\}, \quad C_1 = \{v_1,\ v_2\}, \quad C_2 = \{v_2,\ v_3\}.$$

Finally, suppose we have a *microservice interaction* that spans both clusters, giving the single 2-superedge

$$E_2 = \big\{\, C_1,\ C_2 \big\}.$$

Thus

$$\mathrm{SuperHyG}^{(2)} = \big(V_2,\ \{E_2\}\big)$$

is a 2-SuperHyperGraph capturing the hierarchy

$$\begin{aligned} &\underbrace{\{S_i\}}_{\text{physical servers}} \to \underbrace{\{v_j\}}_{\text{VMs}} \\ &\to \underbrace{\{C_k\}}_{\text{container clusters}} \to \underbrace{E_2}_{\text{microservice interaction}} . \end{aligned}$$

In this model:

- Level-0 vertices S_i represent the physical hosts.
- Level-1 supervertices v_j are individual VMs (or small VM clusters).
- Level-2 supervertices C_k group VMs into container clusters.
- The 2-superedge E_2 encodes an application service that communicates across both clusters.

Example 2.12 (3-SuperHyperGraph: Multinational Company Structure)**.** A multinational company operates in multiple countries, managing production or delivering services across international borders with centralized control (cf.[16]). We illustrate a 3-SuperHyperGraph modeling the hierarchy of a multinational company.

Base employees:

$$V_0 = \{\text{Ayano, Masahiro, Tae, Shinya}\}.$$

Level-1 supervertices (project teams):

$$T_1 = \{\text{Ayano}, \text{Masahiro}\}, \quad T_2 = \{\text{Tae}, \text{Shinya}\} \subseteq \text{POWER}^1(V_0).$$

Level-2 supervertices (functional units):

$$U_1 = \{T_1\}, \quad U_2 = \{T_2\}, \quad U_3 = \{T_1, T_2\} \subseteq \text{POWER}^2(V_0).$$

Level-3 supervertices (corporate divisions):

$$D_1 = \{U_1, U_3\}, \quad D_2 = \{U_2, U_3\}, \quad D_3 = \{U_1, U_2\} \subseteq \text{POWER}^3(V_0).$$

Define

$$V = \{D_1,\, D_2,\, D_3\}, \quad E = \Big\{\{D_1, D_3\},\, \{D_2, D_3\}\Big\} \subseteq \text{POWER}^3(V_0).$$

Then

$$\text{SuperHyG}^{(3)} = (V, E)$$

is a 3-SuperHyperGraph capturing both the hierarchical grouping $(employees \to teams \to units \to divisions)$ and the inter-division collaborations within the company.

3 Soft n-Superhypergraphs

Soft-set theory provides a parameterized framework for modeling uncertainty without relying on membership functions or probabilities.[17] In graph theory, this idea gives rise to *soft graphs*, where each parameter selects a subgraph from a fixed host graph.[18] By analogy, *soft hypergraphs* associate each parameter with a collection of vertices and the induced hyperedges.[19,20] We now extend these notions to the hierarchical setting of n-superhypergraphs.

Definition 3.1 (Soft Graph)**.** Let $G = (V, E)$ be a simple undirected graph and C a nonempty set of parameters. A *soft graph* over G with parameter set C is a quadruple

$$\big(G,\, C,\, A,\, B\big),$$

where

$$A : C \longrightarrow \mathcal{P}(V), \qquad B : C \longrightarrow \mathcal{P}(E),$$

and for each $c \in C$,

$$B(c) \subseteq \big\{\{u, v\} \in E : u \in A(c),\ v \in A(c)\big\}.$$

The pair $\big(A(c), B(c)\big)$ is called the *soft subgraph* at parameter c.

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Example 3.2 (Soft Graph: Communication Modes in a Company)**.** Let $G = (V, E)$ represent the undirected communication network of a small company:

$$V = \{\text{Ayano, Masahiro, Tae, Shinya}\},$$

and

$$E = \big\{\{\text{Ayano}, \text{Masahiro}\},\ \{\text{Ayano}, \text{Tae}\},\ \{\text{Masahiro}, \text{Shinya}\},\ \{\text{Tae}, \text{Shinya}\}\big\},$$

where each edge indicates that the two employees can communicate.

Choose the set of communication media as parameters:

$$C = \{\text{Email, Slack, Meeting}\}.$$

Define the mappings $A\colon C \to \mathcal{P}(V)$ and $B\colon C \to \mathcal{P}(E)$ by

$$\begin{aligned} A(\text{Email}) &= \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}\},\\ A(\text{Slack}) &= \{\text{Ayano}, \text{Masahiro}, \text{Tae}\},\\ A(\text{Meeting}) &= \{\text{Masahiro}, \text{Tae}, \text{Shinya}\}, \end{aligned}$$

$$\begin{aligned} B(\text{Email}) &= \{\{\text{Ayano}, \text{Masahiro}\}, \{\text{Ayano}, \text{Tae}\}, \{\text{Masahiro}, \text{Shinya}\}\},\\ B(\text{Slack}) &= \{\{\text{Ayano}, \text{Masahiro}\}, \{\text{Masahiro}, \text{Tae}\}\},\\ B(\text{Meeting}) &= \{\{\text{Masahiro}, \text{Shinya}\}, \{\text{Tae}, \text{Shinya}\}\}. \end{aligned}$$

One checks that for each medium $c \in C$, every edge in $B(c)$ indeed connects two vertices in $A(c)$. Thus

$$\big(G,\, C,\, A,\, B\big)$$

is a valid soft graph modeling how different communication channels induce subgraphs of the overall employee network.

Definition 3.3 (Soft Hypergraph)**.** Let $H = (V, E)$ be a hypergraph with $E \subseteq \mathcal{P}(V)$, and let C be a nonempty parameter set. A *soft hypergraph* over H with parameters C is a quadruple

$$\big(H,\, C,\, A,\, B\big),$$

where

$$A : C \longrightarrow \mathcal{P}(V), \qquad B : C \longrightarrow \mathcal{P}(E),$$

and for every $c \in C$,

$$B(c) \ \subseteq\ \big\{\, e \in E : e \subseteq A(c) \big\}.$$

The substructure $\big(A(c), B(c)\big)$ is called the *soft subhypergraph* at c.

Example 3.4 (Soft Hypergraph: Community Book Clubs)**.** Community book clubs are local groups where members meet regularly to discuss shared readings, promoting literacy, dialogue, and social interaction. Consider a neighborhood of five readers:

$$V = \{\text{Ayano, Masahiro, Tae, Shinya, Eve}\}.$$

They form three informal book clubs:

$$\begin{aligned} e_1 &= \{\text{Ayano}, \text{Masahiro}\}, && \text{(Science Fiction Club)}\\ e_2 &= \{\text{Masahiro}, \text{Tae}, \text{Shinya}\}, && \text{(History Club)}\\ e_3 &= \{\text{Tae}, \text{Eve}\}. && \text{(Poetry Club)} \end{aligned}$$

Thus $E = \{e_1, e_2, e_3\}$ and $H = (V, E)$.

Define the parameter set of preferred genres:

$$C = \{\text{SciFi, History, Poetry}\}.$$

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For each genre $c \in C$, let

$$A(c) = \{\text{readers who enjoy } c\}, \quad B(c) = \{\text{clubs whose membership is entirely within } A(c)\}.$$

Concretely:

c	$A(c)$	$B(c)$
SciFi	{Ayano, Masahiro, Tae}	$\{e_1\}$
History	{Masahiro, Tae, Shinya}	$\{e_2\}$
Poetry	{Tae, Eve}	$\{e_3\}$

Here, for example, $B(\text{SciFi}) = \{e_1\}$ since only the Science Fiction Club e_1 has all members (Ayano, Masahiro) enjoying Science Fiction.

Hence the quadruple

$$\big(H, C, A, B\big)$$

is a soft hypergraph: each $(A(c), B(c))$ is a subhypergraph of H capturing the community's club structure filtered by genre preference.

Definition 3.5 (Soft n-Superhypergraph)**.** [3] Let $\mathrm{SuperHyG}^{(n)} = (V, E)$ be an n-superhypergraph, and let C be a nonempty set of parameters. A *soft n-superhypergraph* is a quadruple

$$\big(V, E, C, A, B\big),$$

where

$$A : C \longrightarrow \mathrm{POWER}(V), \quad B : C \longrightarrow \mathrm{POWER}(E),$$

such that for each $c \in C$, the pair $\big(A(c), B(c)\big)$ is a sub-superhypergraph of $\mathrm{SuperHyG}^{(n)}$; namely,

$$A(c) \subseteq V, \quad B(c) \subseteq \{\, e \in E : e \subseteq A(c)\}.$$

In other words, $(A(c), B(c))$ inherits the superhypergraph structure on the vertex set $A(c)$ and its induced edges $B(c)$.

Example 3.6 (Soft 2-Superhypergraph: Cross-Department Projects)**.** Cross-department projects involve collaboration between multiple departments, combining diverse expertise to achieve shared organizational goals efficiently and innovatively. We illustrate a soft 2-superhypergraph for a company with four employees:

$$V_0 = \{\text{Ayano}, \text{Masahiro}, \text{Tae}, \text{Shinya}\}.$$

Level-1 teams:

$$T_1 = \{\text{Ayano}, \text{Masahiro}\},\ T_2 = \{\text{Masahiro}, \text{Tae}\},\ T_3 = \{\text{Tae}, \text{Shinya}\} \subseteq \mathrm{POWER}(V_0).$$

Level-2 departments (2-supervertices):

$$D_1 = \{T_1, T_2\},\ D_2 = \{T_2, T_3\},\ D_3 = \{T_1, T_3\} \subseteq \mathrm{POWER}^2(V_0), \quad V = \{D_1, D_2, D_3\}.$$

We declare three 2-superedges representing inter-department collaborations:

$$e_1 = \{D_1, D_2\}, \quad e_2 = \{D_2, D_3\}, \quad e_3 = \{D_1, D_3\}, \quad E = \{e_1, e_2, e_3\},$$

so that $\mathrm{SuperHyG}^{(2)} = (V, E)$.

Parameters (project names):

$$C = \{\text{Alpha, Beta, Gamma}\}.$$

Soft vertex mapping $A \colon C \to \mathrm{POWER}(V)$:

$$A(\text{Alpha}) = \{D_1, D_2\}, \quad A(\text{Beta}) = \{D_2, D_3\}, \quad A(\text{Gamma}) = \{D_1, D_3\}.$$

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Soft edge mapping $B\colon C \to \mathrm{POWER}(E)$:

$$B(\text{Alpha}) = \{e_1\}, \quad B(\text{Beta}) = \{e_2\}, \quad B(\text{Gamma}) = \{e_3\}.$$

Here each $(A(c), B(c))$ is a sub-superhypergraph of $\mathrm{SuperHyG}^{(2)}$: for example,

$$B(\text{Alpha}) = \{e_1\} \subseteq \{\, e \in E : e \subseteq \{D_1, D_2\}\} = \{e_1\}.$$

Thus the quadruple

$$\big(V,\, E,\, C,\, A,\, B\big)$$

is a *soft 2-superhypergraph*, modeling how different cross-department projects select subsets of departments and their collaborations.

Example 3.7 (Soft 2-Superhypergraph: Community Skill-Sharing)**.** Skill-sharing is the collaborative exchange of expertise, where individuals teach and learn skills to enhance mutual growth and productivity (cf.[21]). We model a neighborhood of five participants:

$$V_0 = \{\text{Ayano, Masahiro, Tae, Shinya, Eve}\}.$$

Level-1 skill groups:

$$\begin{aligned} SG_1 &= \{\text{Ayano}, \text{Masahiro}\} && (\text{Gardening}),\\ SG_2 &= \{\text{Tae}, \text{Shinya}\} && (\text{Cooking}),\\ SG_3 &= \{\text{Masahiro}, \text{Tae}, \text{Eve}\} && (\text{Coding}). \end{aligned}$$

Level-2 community centers (2-supervertices):

$$C_1 = \{SG_1, SG_3\}, \quad C_2 = \{SG_1, SG_2\}, \quad C_3 = \{SG_2, SG_3\} \subseteq \mathrm{POWER}^2(V_0), \quad V = \{C_1, C_2, C_3\}.$$

Community events (2-superedges):

$$e_1 = \{C_1, C_2\}, \quad e_2 = \{C_2, C_3\}, \quad e_3 = \{C_1, C_3\}, \quad E = \{e_1, e_2, e_3\}.$$

Thus $\mathrm{SuperHyG}^{(2)} = (V, E)$ represents the skill-sharing network.

Parameters (workshop types):

$$C = \{\text{Gardening, Cooking, Coding}\}.$$

We define soft mappings $A\colon C \to \mathrm{POWER}(V)$ and $B\colon C \to \mathrm{POWER}(E)$ by:

Parameter c	$A(c) \subseteq V$	$B(c) \subseteq E$
Gardening	$\{C_1, C_2\}$	$\{e_1\}$
Cooking	$\{C_2, C_3\}$	$\{e_2\}$
Coding	$\{C_1, C_3\}$	$\{e_3\}$

Here, for example, $A(\text{Gardening}) = \{C_1, C_2\}$ since both centers C_1 and C_2 host gardening groups, and $B(\text{Gardening}) = \{e_1\}$ is the event linking those two centers.

Hence the quadruple

$$\big(V,\, E,\, C,\, A,\, B\big)$$

is a *soft 2-superhypergraph*, capturing which community centers and events are active under each workshop type.

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Example 3.8 (Soft 3-Superhypergraph: University Program Intakes)**.** A university program is a structured curriculum offering academic instruction, training, and certification in a specific field or discipline (cf.[22]). We model a small university's degree programs built over three base courses:

$$V_0 = \{\text{Math, Physics, History}\}.$$

Level-1 supervertices (modules):

$$M_1 = \{\text{Math}, \text{Physics}\}, \quad M_2 = \{\text{Physics}, \text{History}\} \subseteq \mathrm{POWER}(V_0).$$

Level-2 supervertices (curricula):

$$C_1 = \{M_1\}, \quad C_2 = \{M_2\}, \quad C_3 = \{M_1, M_2\} \subseteq \mathrm{POWER}^2(V_0).$$

Level-3 supervertices (degree programs):

$$P_1 = \{C_1\}, \quad P_2 = \{C_2\}, \quad P_3 = \{C_1, C_2\} \subseteq \mathrm{POWER}^3(V_0), \quad V = \{P_1, P_2, P_3\}.$$

We declare two 3-superedges representing program overlaps:

$$e_1 = \{P_1, P_3\}, \quad e_2 = \{P_2, P_3\}, \quad E = \{e_1, e_2\}.$$

Thus $\mathrm{SuperHyG}^{(3)} = (V, E)$.

Intake periods (parameters):

$$C = \{\text{Fall, Spring, Summer}\}.$$

Define

$$A : C \to \mathrm{POWER}(V), \qquad B : C \to \mathrm{POWER}(E),$$

by the table below:

Intake	A(Intake) $\subseteq V$	B(Intake) $\subseteq E$
Fall	$\{P_1, P_3\}$	$\{e_1\}$
Spring	$\{P_2, P_3\}$	$\{e_2\}$
Summer	$\{P_1, P_2, P_3\}$	$\{e_1, e_2\}$

For example, in the Fall intake only programs P_1 and P_3 are offered, and the overlap e_1 between them occurs.

Hence the quintuple

$$\big(V,\, E,\, C,\, A,\, B\big)$$

is a *soft 3-superhypergraph*, modeling which degree programs and their overlaps are active in each intake period.

Example 3.9 (Soft 3-Superhypergraph: Global Research Consortia)**.** Research consortia are collaborative alliances of institutions or researchers working jointly on shared scientific goals, projects, or innovations (cf.[23]). We model a global research network built over five researchers:

$$V_0 = \{\text{Ayano, Masahiro, Tae, Shinya, Eve}\}.$$

Level-1 supervertices (research groups):

$$G_1 = \{\text{Ayano}, \text{Masahiro}\}, \quad G_2 = \{\text{Tae}, \text{Shinya}\}, \quad G_3 = \{\text{Masahiro}, \text{Tae}, \text{Eve}\} \subseteq \mathrm{POWER}(V_0).$$

Level-2 supervertices (institutes):

$$I_1 = \{G_1\}, \quad I_2 = \{G_2\}, \quad I_3 = \{G_1, G_3\} \subseteq \mathrm{POWER}^2(V_0).$$

Level-3 supervertices (consortia):

$$C_1 = \{I_1, I_3\}, \quad C_2 = \{I_2, I_3\}, \quad C_3 = \{I_1, I_2\} \subseteq \mathrm{POWER}^3(V_0), \quad V = \{C_1, C_2, C_3\}.$$

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We declare three 3-superedges representing consortium collaborations:

$$e_1 = \{C_1, C_2\}, \quad e_2 = \{C_2, C_3\}, \quad e_3 = \{C_1, C_3\}, \quad E = \{e_1, e_2, e_3\},$$

so that $\mathrm{SuperHyG}^{(3)} = (V, E)$.

Funding calls (parameters):

$$C = \{\text{AI, Bio, Energy}\}.$$

Define

$$A : C \to \mathrm{POWER}(V), \qquad B : C \to \mathrm{POWER}(E),$$

by

Call	$A(\text{Call}) \subseteq V$	$B(\text{Call}) \subseteq E$
AI	$\{C_1, C_2\}$	$\{e_1\}$
Bio	$\{C_2, C_3\}$	$\{e_2\}$
Energy	$\{C_1, C_2, C_3\}$	$\{e_1, e_2, e_3\}$

For instance, under the AI call only consortia C_1 and C_2 apply, and the collaboration e_1 between them is active.

Thus the quadruple

$$\big(V,\, E,\, C,\, A,\, B\big)$$

is a *soft 3-superhypergraph*, modeling which research consortia and their collaborations participate in each funding call.

Theorem 3.10 (Closure under Union). *For any $c_1, c_2 \in C$, define a "union" parameter $c_\cup$ by*

$$A(c_\cup) \;=\; A(c_1) \,\cup\, A(c_2), \qquad B(c_\cup) \;=\; B(c_1) \,\cup\, B(c_2).$$

Then $\big(A(c_\cup), B(c_\cup)\big)$ is a sub-superhypergraph of $\mathrm{SuperHyG}^{(n)}$. *Consequently, extending C by $c_\cup$ yields a new soft n-superhypergraph.*

Proof. Since $A(c_1), A(c_2) \subseteq V$, clearly $A(c_\cup) \subseteq V$. Moreover, every edge in $B(c_\cup)$ lies in either $B(c_1)$ or $B(c_2)$, so if $e \in B(c_\cup)$ then $e \subseteq A(c_i) \subseteq A(c_1) \cup A(c_2) = A(c_\cup)$. Hence $B(c_\cup) \subseteq \{e \in E : e \subseteq A(c_\cup)\}$, and $(A(c_\cup), B(c_\cup))$ is an induced sub-superhypergraph. ☐

Theorem 3.11 (Closure under Intersection). *For any $c_1, c_2 \in C$, define an "intersection" parameter $c_\cap$ by*

$$A(c_\cap) \;=\; A(c_1) \,\cap\, A(c_2), \qquad B(c_\cap) \;=\; B(c_1) \,\cap\, B(c_2).$$

Then $\big(A(c_\cap), B(c_\cap)\big)$ is a sub-superhypergraph of $\mathrm{SuperHyG}^{(n)}$.

Proof. Clearly $A(c_\cap) \subseteq V$. If $e \in B(c_\cap)$, then $e \in B(c_1)$ and $e \in B(c_2)$, so $e \subseteq A(c_1)$ and $e \subseteq A(c_2)$. Hence $e \subseteq A(c_1) \cap A(c_2) = A(c_\cap)$, showing $B(c_\cap) \subseteq \{e : e \subseteq A(c_\cap)\}$. ☐

Theorem 3.12 (Induced Soft Sub-n-Superhypergraph). *If $C' \subseteq C$ is any nonempty subset of parameters, then*

$$\big(V,\, E,\, C',\, A|_{C'},\, B|_{C'}\big)$$

is also a soft n-superhypergraph.

Proof. Restriction of A and B to C' preserves the property that for each $c \in C'$, $(A(c), B(c))$ is a sub-superhypergraph, because no new parameters are introduced. ☐

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Theorem 3.13 (Crisp Parameter Reduction). *If there exists a parameter* $c^* \in C$ *with*

$$A(c^*) = V \quad \text{and} \quad B(c^*) = E,$$

then the soft n*-superhypergraph contains the exact superhypergraph* $\mathrm{SuperHyG}^{(n)}$ *as its* c^**-slice.*

Proof. By $A(c^*) = V$ and $B(c^*) = E$, the pair $\big(A(c^*), B(c^*)\big)$ equals (V, E), so it recovers the full n-superhypergraph. ☐

Theorem 3.14 (Complement Closure). *For any parameter* $c \in C$*, define its* complement *by*

$$A(\bar{c}) \;=\; V \setminus A(c), \qquad B(\bar{c}) \;=\; \{\, e \in E : e \subseteq V \setminus A(c)\}.$$

Then $\big(A(\bar{c}), B(\bar{c})\big)$ *is a sub-superhypergraph of* $\mathrm{SuperHyG}^{(n)}$.

Proof. Since $A(\bar{c}) \subseteq V$ by construction, we need only check that every edge $e \in B(\bar{c})$ satisfies $e \subseteq A(\bar{c})$. But $B(\bar{c})$ was defined exactly as those edges lying entirely in $V \setminus A(c) = A(\bar{c})$. Hence $\big(A(\bar{c}), B(\bar{c})\big)$ inherits the induced superhypergraph structure. ☐

Theorem 3.15 (Difference Closure). *For any* $c_1, c_2 \in C$*, define the* difference *parameter* $c_1 \setminus c_2$ *by*

$$A(c_1 \setminus c_2) = \; A(c_1) \setminus A(c_2), \qquad B(c_1 \setminus c_2) = \; \{\, e \in B(c_1) : e \subseteq A(c_1) \setminus A(c_2)\}.$$

Then $\big(A(c_1 \setminus c_2), B(c_1 \setminus c_2)\big)$ *is a sub-superhypergraph of* $\mathrm{SuperHyG}^{(n)}$.

Proof. Clearly $A(c_1 \setminus c_2) \subseteq A(c_1) \subseteq V$. If $e \in B(c_1 \setminus c_2)$, then $e \in B(c_1)$ and $e \subseteq A(c_1) \setminus A(c_2)$. In particular $e \subseteq A(c_1 \setminus c_2)$, so $(A(c_1 \setminus c_2), B(c_1 \setminus c_2))$ is an induced sub-superhypergraph. ☐

Theorem 3.16 (Boolean-Algebra Structure). *Extend the set of parameters* C *by adjoining complement symbols* $\bar{c}$ *for all* $c \in C$*, and close under union* $\cup$ *and intersection* $\cap$*. Then the collection* $\{\, A(d) \mid d \in \overline{C}\}$ *of vertex-sets, together with the corresponding edge-sets* $\{\, B(d) \mid d \in \overline{C}\}$*, forms a Boolean algebra of sub-superhypergraphs:*

$$\begin{aligned}
A(d_1 \cup d_2) &= A(d_1) \cup A(d_2), & B(d_1 \cup d_2) &= B(d_1) \cup B(d_2),\\
A(d_1 \cap d_2) &= A(d_1) \cap A(d_2), & B(d_1 \cap d_2) &= B(d_1) \cap B(d_2),\\
A(\overline{d}) &= V \setminus A(d), & B(\overline{d}) &= \{e : e \subseteq V \setminus A(d)\},
\end{aligned}$$

and all De Morgan and distributive laws hold.

Proof. De Morgan's laws $\overline{d_1 \cup d_2} = \bar{d}_1 \cap \bar{d}_2$, $\overline{d_1 \cap d_2} = \bar{d}_1 \cup \bar{d}_2$ follow directly from the set-theoretic counterparts on V. Distributivity of $\cup$ over $\cap$ (and vice versa) also holds because these operations on $A(\cdot)$ and $B(\cdot)$ reduce to the usual set-theoretic union, intersection, and complement on subsets of V and E. Therefore the parameterized sub-superhypergraphs form a Boolean algebra. ☐

4 Rough n-Superhypergraph

A rough graph models uncertainty in graph structures by using lower and upper approximations on vertices and edges. A rough graph can also be viewed as a graphical representation of a rough set.[24] A rough set models uncertainty by approximating a set using lower and upper bounds based on an equivalence relation.[25] A rough hypergraph extends rough set theory to hypergraphs, capturing imprecise relationships among groups of vertices via approximations.[26] A rough n-superhypergraph incorporates rough set approximations at multiple hierarchical levels of supervertices and superedges for uncertainty modeling.

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Definition 4.1 (Approximation Space)**.** [27] Let U be a nonempty set and R an equivalence relation on U. The pair (U, R) is called an *approximation space*. For any $X \subseteq U$, the *lower* and *upper* approximations of X are

$$\underline{R}(X) = \{\, x \in U : [x]_R \subseteq X \}, \qquad \overline{R}(X) = \{\, x \in U : [x]_R \cap X \neq \emptyset \},$$

where $[x]_R = \{\, y \in U : (x, y) \in R \}$ is the equivalence class of x.

Example 4.2 (Real-World Illustration of an Approximation Space)**.** Let U be the set of patients in a clinic and let R relate two patients if they share the same combination of symptoms (e.g., fever, cough, fatigue). For a target concept $X \subseteq U$ defined as "patients diagnosed with influenza,"

$$\underline{R}(X) = \{\, p \in U : \text{every patient with the same symptom profile as } p \text{ has influenza} \},$$

is the set of patients whose entire symptom-equivalence class is confirmed influenza, while

$$\overline{R}(X) = \{\, p \in U : \text{at least one patient in } p\text{'s symptom-equivalence class has influenza} \},$$

is the set of patients whose symptom group contains at least one confirmed influenza case.

Definition 4.3 (Rough Graph)**.** [28] Let $G = (V, E)$ be a simple graph and let R_E be an equivalence relation on the edge set E. For any $X \subseteq E$, define

$$\underline{R_E}(X) = \{\, e \in E : [e]_{R_E} \subseteq X \}, \qquad \overline{R_E}(X) = \{\, e \in E : [e]_{R_E} \cap X \neq \emptyset \}.$$

The pair $\big(\underline{G}, \overline{G}\big)$, where $\underline{G} = (V, \underline{R_E}(X))$ and $\overline{G} = (V, \overline{R_E}(X))$, is called the *rough graph* approximation of the subedge-set $X \subseteq E$. If X is not a union of equivalence classes under R_E, then G is said to be a *rough graph* with respect to R_E.

Example 4.4 (Rough Graph: Regional Road Network)**.** Consider a simplified road network connecting four towns:

$$V = \{\mathrm{T}_1,\ \mathrm{T}_2,\ \mathrm{T}_3,\ \mathrm{T}_4\},$$

with undirected roads (edges)

$$E = \big\{e_1 = \{\mathrm{T}_1, \mathrm{T}_2\},\ e_2 = \{\mathrm{T}_2, \mathrm{T}_3\},\ e_3 = \{\mathrm{T}_3, \mathrm{T}_4\},\ e_4 = \{\mathrm{T}_4, \mathrm{T}_1\},\ e_5 = \{\mathrm{T}_2, \mathrm{T}_4\}\big\}.$$

Define an equivalence relation R_E on E by grouping roads according to region:

$$[e]_{R_E} = \begin{cases} \{e_1,\ e_3\}, & \text{if } e \in \{e_1, e_3\} \quad \text{(Northern roads)}, \\ \{e_2,\ e_4,\ e_5\}, & \text{if } e \in \{e_2, e_4, e_5\} \quad \text{(Southern roads)}. \end{cases}$$

We wish to approximate the subset

$$X = \{e_1, e_2, e_4\}$$

of roads currently designated *under maintenance*.

The *lower approximation* of X is

$$\underline{R_E}(X) = \{\, e \in E : [e]_{R_E} \subseteq X \},$$

but neither equivalence class lies entirely in X, so

$$\underline{R_E}(X) = \varnothing.$$

The *upper approximation* of X is

$$\overline{R_E}(X) = \{\, e \in E : [e]_{R_E} \cap X \neq \varnothing \},$$

and since both Northern and Southern classes meet X, we get

$$\overline{R_E}(X) = \{e_1, e_2, e_3, e_4, e_5\} = E.$$

Hence the rough-graph approximation of X is the pair

$$\big(\underline{G},\ \overline{G}\big)$$

with $\underline{G} = (V, \varnothing)$ and $\overline{G} = (V, E)$. This models the uncertainty in which entire regional road classes are under maintenance: no region is entirely contained in the maintenance set (empty lower approximation), yet both regions are partially affected (full upper approximation).

Definition 4.5 (Rough Hypergraph)**.** [26] Let $H = (V, E)$ be a hypergraph, R_V an equivalence relation on V, and R_E an equivalence relation on E. For any $A \subseteq V$ and $D \subseteq E$, define

$$\underline{R_V}(A) = \{\, v \in V : [v]_{R_V} \subseteq A\}, \quad \overline{R_V}(A) = \{\, v \in V : [v]_{R_V} \cap A \neq \emptyset\},$$

$$\underline{R_E}(D) = \{\, e \in E : [e]_{R_E} \subseteq D\}, \quad \overline{R_E}(D) = \{\, e \in E : [e]_{R_E} \cap D \neq \emptyset\}.$$

The tuple

$$\big(V,\, E,\, \underline{R_V},\, \overline{R_V},\, \underline{R_E},\, \overline{R_E}\big)$$

is called a *rough hypergraph*, capturing uncertainty on both vertices and hyperedges via rough set approximations.

Example 4.6 (Rough Hypergraph: E-Commerce Customer–Product Network)**.** E-commerce is the buying and selling of goods or services through electronic platforms, especially the internet, enabling global transactions. We model an online store with six customers and four products:

Vertices (customers):

$$V = \{\text{Ayano, Masahiro, Tae, Shinya, Eve, Frank}\}.$$

Hyperedges (product purchasers):

$$\begin{aligned} e_1 &= \{\text{Ayano}, \text{Masahiro}, \text{Tae}\}, \\ e_2 &= \{\text{Masahiro}, \text{Shinya}, \text{Eve}\}, \\ e_3 &= \{\text{Tae}, \text{Frank}\}, \\ e_4 &= \{\text{Ayano}, \text{Eve}, \text{Frank}\}, \end{aligned} \qquad E = \{e_1, e_2, e_3, e_4\}.$$

Equivalence on customers (R_V): Customers are grouped by age:

Class	Members
18–24	Ayano, Masahiro, Eve
25–34	Tae, Shinya
35+	Frank

Equivalence on products (R_E): Products are grouped by category:

Category	Hyperedges
Electronics	e_1, e_2
Books	e_3, e_4

Vertex approximations: Let $A = \{\text{Ayano}, \text{Masahiro}, \text{Eve}\}$. Then

$$\underline{R_V}(A) = \{v \in V : [v]_{R_V} \subseteq A\} = \{\text{Ayano}, \text{Masahiro}, \text{Eve}\},$$

$$\overline{R_V}(A) = \{v \in V : [v]_{R_V} \cap A \neq \emptyset\} = \{\text{Ayano}, \text{Masahiro}, \text{Eve}\}.$$

Edge approximations: 1. For $D_1 = \{\, e_3, e_4\}$ (Books category):

$$\underline{R_E}(D_1) = \{e \in E : [e]_{R_E} \subseteq D_1\} = \{e_3, e_4\}, \quad \overline{R_E}(D_1) = \{e \in E : [e]_{R_E} \cap D_1 \neq \emptyset\} = \{e_3, e_4\}.$$

2. For $D_2 = \{\, e_1\}$ (single Electronics product):

$$\underline{R_E}(D_2) = \emptyset, \quad \overline{R_E}(D_2) = \{e_1, e_2\}.$$

Hence the tuple

$$\big(V,\, E,\, \underline{R_V},\, \overline{R_V},\, \underline{R_E},\, \overline{R_E}\big)$$

is a rough hypergraph capturing uncertainty in both customer segmentation and product-category membership.

Definition 4.7 (Rough n-Superhypergraph)**.** [3] Let $\mathrm{SuperHyG}^{(n)} = (V, E)$ be an n-superhypergraph. Suppose φ is an equivalence relation on V and ψ an equivalence relation on E. For any $X \subseteq V$, define the *lower* and *upper* φ-approximations by

$$\underline{\varphi}(X) \;=\; \{\, v \in V : [v]_\varphi \subseteq X\}, \qquad \overline{\varphi}(X) \;=\; \{\, v \in V : [v]_\varphi \cap X \neq \varnothing\},$$

where $[v]_\varphi = \{\, u \in V : (u, v) \in \varphi\}$. Similarly, for any $D \subseteq E$ set

$$\underline{\psi}(D) \;=\; \{\, e \in E : [e]_\psi \subseteq D\}, \qquad \overline{\psi}(D) \;=\; \{\, e \in E : [e]_\psi \cap D \neq \varnothing\},$$

where $[e]_\psi = \{\, f \in E : (f, e) \in \psi\}$. The *rough n-superhypergraph* is the sextuple

$$\bigl(V,\, E,\;\; \underbrace{(\underline{\varphi}, \overline{\varphi})}_{\varphi\text{-approx. on } V} \;,\;\; \underbrace{(\underline{\psi}, \overline{\psi})}_{\psi\text{-approx. on } E} \;\;\bigr).$$

This structure captures uncertainty by replacing each exact vertex or edge set with its rough approximations under φ and ψ.

Example 4.8 (Rough 2-Superhypergraph: Corporate Division Collaborations)**.** Corporate Division Collaborations involve multiple business units within a company working together to achieve shared objectives and strategic goals. Consider a company with four employees:

$$V_0 = \{\text{Ayano},\ \text{Masahiro},\ \text{Tae},\ \text{Shinya}\}.$$

Level-1 supervertices (teams) are

$$T_1 = \{\text{Ayano}, \text{Masahiro}\}, \quad T_2 = \{\text{Tae}, \text{Shinya}\},$$

so that $\{T_1, T_2\} \subseteq \mathrm{POWER}(V_0)$. Level-2 supervertices (divisions) are

$$D_1 = \{T_1\}, \quad D_2 = \{T_2\}, \quad D_3 = \{T_1, T_2\},$$

hence $\{D_1, D_2, D_3\} \subseteq \mathrm{POWER}^2(V_0)$ and we set

$$V = \{D_1, D_2, D_3\}.$$

Define three collaborations (2-superedges):

$$e_1 = \{D_1, D_3\}, \quad e_2 = \{D_2, D_3\}, \quad e_3 = \{D_1, D_2\}, \quad E = \{e_1, e_2, e_3\}.$$

Equivalence on divisions (φ)—by size:

$$\varphi\text{-classes:} \quad \{D_1, D_2\} \quad \text{(single-team)}, \quad \{D_3\} \quad \text{(two-team)}.$$

Equivalence on collaborations (ψ)—by project type:

$$\psi\text{-classes:} \quad \{e_1, e_2\} \quad \text{(R\&D)}, \quad \{e_3\} \quad \text{(Administrative)}.$$

Vertex approximations. Let $X = \{D_1\}$. Then

$$\underline{\varphi}(X) = \{\, v \in V : [v]_\varphi \subseteq X\} = \varnothing, \qquad \overline{\varphi}(X) = \{\, v \in V : [v]_\varphi \cap X \neq \emptyset\} = \{D_1, D_2\}.$$

Edge approximations. Let $D = \{e_1\}$. Then

$$\underline{\psi}(D) = \{\, e \in E : [e]_\psi \subseteq D\} = \varnothing, \qquad \overline{\psi}(D) = \{\, e \in E : [e]_\psi \cap D \neq \emptyset\} = \{e_1, e_2\}.$$

Thus

$$\bigl(V,\, E,\, \underline{\varphi},\, \overline{\varphi},\, \underline{\psi},\, \overline{\psi}\bigr)$$

is a rough 2-superhypergraph capturing uncertain division membership and project classification in the company.

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Example 4.9 (Rough 2-Superhypergraph: Neighborhood Resource Sharing)**.** We model a city with six residents:

$$V_0 = \{\text{Ayano, Masahiro, Tae, Shinya, Eve, Frank}\}.$$

Level-1 supervertices (households):

$$H_1 = \{\text{Ayano}, \text{Masahiro}\}, \quad H_2 = \{\text{Tae}, \text{Shinya}\}, \quad H_3 = \{\text{Eve}, \text{Frank}\} \subseteq \text{POWER}(V_0).$$

Level-2 supervertices (neighborhoods):

$$N_1 = \{H_1, H_2\}, \quad N_2 = \{H_2, H_3\}, \quad N_3 = \{H_1, H_3\} \subseteq \text{POWER}^2(V_0), \quad V = \{N_1, N_2, N_3\}.$$

We define three neighborhood-collaboration superedges:

$$e_1 = \{N_1, N_2\}, \quad e_2 = \{N_2, N_3\}, \quad e_3 = \{N_1, N_3\}, \quad E = \{e_1, e_2, e_3\}.$$

Equivalence on neighborhoods (φ): Group by socio-economic status:

$$\varphi\text{-classes:} \quad \{N_1, N_2\} \text{ (residential)}, \quad \{N_3\} \text{ (commercial)}.$$

Equivalence on collaborations (ψ): Group by event type:

$$\psi\text{-classes:} \quad \{e_1, e_2\} \text{ (food distribution)}, \quad \{e_3\} \text{ (emergency response)}.$$

Vertex approximations. Let $X = \{N_2\}$. Then

$$\underline{\varphi}(X) = \{v \in V : [v]_\varphi \subseteq X\} = \varnothing, \qquad \overline{\varphi}(X) = \{v \in V : [v]_\varphi \cap X \neq \emptyset\} = \{N_1, N_2\}.$$

Edge approximations. 1. For $D_1 = \{e_1\}$ (food distribution):

$$\underline{\psi}(D_1) = \{e \in E : [e]_\psi \subseteq D_1\} = \varnothing, \quad \overline{\psi}(D_1) = \{e \in E : [e]_\psi \cap D_1 \neq \emptyset\} = \{e_1, e_2\}.$$

2. For $D_2 = \{e_3\}$ (emergency response):

$$\underline{\psi}(D_2) = \{e \in E : [e]_\psi \subseteq D_2\} = \{e_3\}, \quad \overline{\psi}(D_2) = \{e \in E : [e]_\psi \cap D_2 \neq \emptyset\} = \{e_3\}.$$

Hence

$$\left(V,\, E,\, \underline{\varphi},\, \overline{\varphi},\, \underline{\psi},\, \overline{\psi}\right)$$

is a rough 2-superhypergraph modeling uncertain neighborhood membership and collaboration types.

Example 4.10 (Rough 3-Superhypergraph: Global Supply-Chain Resilience)**.** A supply chain is a system of organizations, people, activities, and resources involved in producing and delivering goods. We model a simplified global supply chain with four manufacturing sites:

$$V_0 = \{\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3, \mathrm{F}_4\}.$$

Level-1 clusters (local factories):

$$C_1 = \{\mathrm{F}_1, \mathrm{F}_2\}, \quad C_2 = \{\mathrm{F}_3, \mathrm{F}_4\}, \quad C_3 = \{\mathrm{F}_1, \mathrm{F}_3\} \subseteq \text{POWER}(V_0).$$

Level-2 hubs (distribution centers):

$$D_1 = \{C_1\}, \quad D_2 = \{C_2\}, \quad D_3 = \{C_1, C_3\} \subseteq \text{POWER}^2(V_0).$$

Level-3 regions (regional centers):

$$R_1 = \{D_1\}, \quad R_2 = \{D_2\}, \quad R_3 = \{D_1, D_3\} \subseteq \text{POWER}^3(V_0).$$

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Thus $V = \{R_1, R_2, R_3\}$ and we define three 3-superedges capturing inter-region links:

$$e_1 = \{R_1, R_2\}, \quad e_2 = \{R_2, R_3\}, \quad e_3 = \{R_1, R_3\}, \quad E = \{e_1, e_2, e_3\}.$$

Equivalence on regions φ (by region type):

$$\varphi\text{-classes:} \quad \{R_1, R_2\} \text{ (urban)}, \quad \{R_3\} \text{ (mixed)}.$$

Equivalence on links ψ (by product category):

$$\psi\text{-classes:} \quad \{e_1, e_2\} \text{ (electronics)}, \quad \{e_3\} \text{ (automotive)}.$$

Vertex approximations. Let $X = \{R_1\}$. Then

$$\underline{\varphi}(X) = \{v \in V : [v]_\varphi \subseteq X\} = \varnothing, \quad \overline{\varphi}(X) = \{v \in V : [v]_\varphi \cap X \neq \emptyset\} = \{R_1, R_2\}.$$

Edge approximations.

- For the automotive link $D = \{e_3\}$:

$$\underline{\psi}(D) = \{e \in E : [e]_\psi \subseteq D\} = \{e_3\}, \quad \overline{\psi}(D) = \{e \in E : [e]_\psi \cap D \neq \emptyset\} = \{e_3\}.$$

- For an electronics link $D' = \{e_1\}$:

$$\underline{\psi}(D') = \{e \in E : [e]_\psi \subseteq D'\} = \varnothing, \quad \overline{\psi}(D') = \{e \in E : [e]_\psi \cap D' \neq \emptyset\} = \{e_1, e_2\}.$$

Hence the sextuple

$$\big(V,\, E,\, \underline{\varphi},\, \overline{\varphi},\, \underline{\psi},\, \overline{\psi}\big)$$

is a rough 3-superhypergraph capturing uncertainty in regional classification and inter-region product links.

Example 4.11 (Rough 3-Superhypergraph: Healthcare Referral Network)**.** A healthcare referral network connects providers to coordinate patient care, ensuring timely referrals and specialist consultations across facilities (cf.[29]). We model a healthcare system with four physicians:

$$V_0 = \{\text{DrA, DrB, DrC, DrD}\}.$$

Level-1 supervertices (clinics):

$$\mathrm{Cl}_1 = \{\mathrm{DrA}, \mathrm{DrB}\}, \quad \mathrm{Cl}_2 = \{\mathrm{DrC}, \mathrm{DrD}\} \;\subseteq\; \mathrm{POWER}(V_0).$$

Level-2 supervertices (hospitals):

$$H_1 = \{\mathrm{Cl}_1\}, \quad H_2 = \{\mathrm{Cl}_2\}, \quad H_3 = \{\mathrm{Cl}_1, \mathrm{Cl}_2\} \;\subseteq\; \mathrm{POWER}^2(V_0).$$

Level-3 supervertices (health regions):

$$R_1 = \{H_1\}, \quad R_2 = \{H_2\}, \quad R_3 = \{H_1, H_3\} \;\subseteq\; \mathrm{POWER}^3(V_0).$$

Thus

$$V = \{R_1, R_2, R_3\}, \qquad E = \big\{e_1 = \{R_1, R_2\},\, e_2 = \{R_2, R_3\},\, e_3 = \{R_1, R_3\}\big\}.$$

Equivalence on regions φ (by setting):

$$\varphi\text{-classes:} \quad \{R_1, R_3\} \text{ (rural)}, \quad \{R_2\} \text{ (urban)}.$$

Equivalence on referral-types ψ:

$$\psi\text{-classes:}\quad \{e_1,e_2\}\ \text{(routine referrals)},\quad \{e_3\}\ \text{(emergency transfers)}.$$

Vertex approximations. Take $X=\{R_2\}$. Then

$$\underline{\varphi}(X)=\{v\in V:[v]_\varphi\subseteq X\}=\{R_2\},\quad \overline{\varphi}(X)=\{v\in V:[v]_\varphi\cap X\neq\emptyset\}=\{R_2\}.$$

Edge approximations. For $D=\{e_2\}$ (routine referral):

$$\underline{\psi}(D)=\{e\in E:[e]_\psi\subseteq D\}=\varnothing,\quad \overline{\psi}(D)=\{e\in E:[e]_\psi\cap D\neq\emptyset\}=\{e_1,e_2\}.$$

Hence the sextuple

$$\bigl(V,\,E,\,\underline{\varphi},\,\overline{\varphi},\,\underline{\psi},\,\overline{\psi}\bigr)$$

is a *rough 3-superhypergraph* capturing uncertainty in regional classification and referral types in a healthcare network.

Theorem 4.12 (Monotonicity of Approximations)**.** *For any $X\subseteq Y\subseteq V$,*

$$\underline{\varphi}(X)\ \subseteq\ \underline{\varphi}(Y),\qquad \overline{\varphi}(X)\ \subseteq\ \overline{\varphi}(Y).$$

Similarly, for any $D\subseteq D'\subseteq E$,

$$\underline{\psi}(D)\ \subseteq\ \underline{\psi}(D'),\qquad \overline{\psi}(D)\ \subseteq\ \overline{\psi}(D').$$

Proof. If $X\subseteq Y$, then any equivalence class $[v]_\varphi$ contained in X is also contained in Y, so $\underline{\varphi}(X)\subseteq\underline{\varphi}(Y)$. Moreover, if $[v]_\varphi\cap X\neq\varnothing$, then $[v]_\varphi\cap Y\neq\varnothing$, yielding $\overline{\varphi}(X)\subseteq\overline{\varphi}(Y)$. The same arguments apply to $\underline{\psi}$ and $\overline{\psi}$ on E. ☐

Theorem 4.13 (Duality of Approximations)**.** *For any $X\subseteq V$,*

$$V\setminus\overline{\varphi}(X)=\underline{\varphi}\bigl(V\setminus X\bigr),\quad V\setminus\underline{\varphi}(X)=\overline{\varphi}\bigl(V\setminus X\bigr).$$

Similarly, for any $D\subseteq E$,

$$E\setminus\overline{\psi}(D)=\underline{\psi}\bigl(E\setminus D\bigr),\quad E\setminus\underline{\psi}(D)=\overline{\psi}\bigl(E\setminus D\bigr).$$

Proof. By definition,

$$\underline{\varphi}(V\setminus X)=\{v:[v]_\varphi\subseteq V\setminus X\}=\{v:[v]_\varphi\cap X=\varnothing\}=V\setminus\{v:[v]_\varphi\cap X\neq\varnothing\}=V\setminus\overline{\varphi}(X).$$

The other identities follow similarly, and the edge-case dualities mirror these arguments. ☐

Theorem 4.14 (Crisp Reduction)**.** *If φ and ψ are the identity relations on V and E, then for all $X\subseteq V$, $\underline{\varphi}(X)=\overline{\varphi}(X)=X$, and for all $D\subseteq E$, $\underline{\psi}(D)=\overline{\psi}(D)=D$. Hence the rough n-superhypergraph collapses to the exact superhypergraph $\mathrm{SuperHyG}^{(n)}=(V,E)$.*

Proof. Under the identity relation, each class $[v]_\varphi=\{v\}$. Thus $\underline{\varphi}(X)=\{v:\{v\}\subseteq X\}=X$ and $\overline{\varphi}(X)=\{v:\{v\}\cap X\neq\varnothing\}=X$. Identical reasoning applies to $\underline{\psi},\overline{\psi}$. ☐

Theorem 4.15 (Induced Sub-Rough Superhypergraph)**.** *Let $V'\subseteq V$ and $E'=\{e\in E:e\subseteq V'\}$. Restrict φ and ψ to V' and E'. Then*

$$\bigl(V',\,E',\,\underline{\varphi}|_{V'},\,\overline{\varphi}|_{V'},\,\underline{\psi}|_{E'},\,\overline{\psi}|_{E'}\bigr)$$

is a rough n-superhypergraph on the induced substructure.

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Proof. Restricting the equivalence relations preserves their properties and the definitions of lower/upper approximations. For any $X' \subseteq V'$, the original inclusion tests remain valid in V', and similarly for edges, so all axioms hold for the induced tuple. ☐

Theorem 4.16 (Distribution of Approximations)**.** *For any* $X_1, X_2 \subseteq V$ *and* $D_1, D_2 \subseteq E$*, we have*

$$\underline{\varphi}(X_1 \cap X_2) = \underline{\varphi}(X_1) \cap \underline{\varphi}(X_2), \quad \overline{\varphi}(X_1 \cup X_2) = \overline{\varphi}(X_1) \cup \overline{\varphi}(X_2),$$

$$\underline{\psi}(D_1 \cap D_2) = \underline{\psi}(D_1) \cap \underline{\psi}(D_2), \quad \overline{\psi}(D_1 \cup D_2) = \overline{\psi}(D_1) \cup \overline{\psi}(D_2).$$

Proof. We prove the first identity; the others are analogous.

$$v \in \underline{\varphi}(X_1 \cap X_2) \iff [v]_\varphi \subseteq X_1 \cap X_2 \iff [v]_\varphi \subseteq X_1 \text{ and } [v]_\varphi \subseteq X_2$$

$$\iff v \in \underline{\varphi}(X_1) \text{ and } v \in \underline{\varphi}(X_2) \iff v \in \underline{\varphi}(X_1) \cap \underline{\varphi}(X_2).$$

Similarly,

$$v \in \overline{\varphi}(X_1 \cup X_2) \iff [v]_\varphi \cap (X_1 \cup X_2) \neq \varnothing \iff \text{either } [v]_\varphi \cap X_1 \neq \varnothing \text{ or } [v]_\varphi \cap X_2 \neq \varnothing$$

$$\iff v \in \overline{\varphi}(X_1) \cup \overline{\varphi}(X_2).$$

☐

Definition 4.17 (Definable Set)**.** A subset $X \subseteq V$ is called φ*-definable* if it is a union of φ-equivalence classes:

$$X = \bigcup_{v \in X} [v]_\varphi.$$

Similarly, $D \subseteq E$ is ψ*-definable* if D is a union of ψ-classes.

Theorem 4.18 (Characterization of Definable Sets)**.** *A subset* $X \subseteq V$ *is* φ*-definable if and only if* $\underline{\varphi}(X) = X = \overline{\varphi}(X)$*. Likewise,* $D \subseteq E$ *is* ψ*-definable if and only if* $\underline{\psi}(D) = D = \overline{\psi}(D)$*.*

Proof. If X is a union of whole equivalence classes, then every class meeting X lies entirely in X, so both approximations coincide with X. Conversely, if $\underline{\varphi}(X) = X$, then any class intersecting X must lie in X, so X is a union of classes; the equality $\overline{\varphi}(X) = X$ is equivalent. ☐

Theorem 4.19 (Best Definable Approximations)**.** *For any* $X \subseteq V$*:*

1. $\underline{\varphi}(X)$ *is the greatest* φ*-definable subset of* X*.*
2. $\overline{\varphi}(X)$ *is the least* φ*-definable superset of* X*.*

Analogous statements hold for $\underline{\psi}, \overline{\psi}$ *on* E*.*

Proof. By definition, $\underline{\varphi}(X)$ is the union of all equivalence classes contained in X, so any definable subset of X (being a union of classes) must lie within $\underline{\varphi}(X)$. Similarly, $\overline{\varphi}(X)$ is the union of all classes meeting X, hence it is the minimal definable superset. ☐

5 Conclusion and Future Works

In this paper, we have examined Soft SuperHypergraphs and Rough SuperHypergraphs, including their key properties. For future work, we plan to investigate further extensions of SuperHypergraphs using Hypersoft Sets, Hesitant Fuzzy Sets, Plithogenic Sets, Picture Fuzzy Sets, and Neutrosophic Sets. We also intend to explore potential applications of these extended models in computer science and decision-making.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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