Prospects for Applied Mathematics and Data Analysis

♦ ISSN: 2836-4449 Vol: 05, No. 01, 2025 ♦ PP. 01–07

♦ Article DOI: https://doi.org/10.54216/PAMDA.050101

FULL LENGTH ARTICLE

A Brief Study on Fuzzy Off-Group Theory

Takaaki Fujita[1,*] **Arif Mehmood Khattak**[2] **Arkan A. Ghaib**[3]

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan

[2] Department of Mathematics, Institute of Numerical Sciences, Gomal University, Dera Ismail Khan 29050, KPK, Pakistan

[3] Department of Information Technology, Management Technical College, Southern Technical University, Basrah 61004, Iraq

Emails: takaaki.fujita060@gmail.com; mehdaniyal@gmail.com; arkan.ghaib@stu.edu.iq

Received: November 20, 2024 Revised: December 24, 2024 Accepted: January 17, 2025 ⋆ Corresponding author

ABSTRACT

Uncertainty-handling frameworks such as fuzzy sets, rough sets, intuitionistic fuzzy sets, neutrosophic sets, Picture Fuzzy Sets, hyperneutrosophic sets, and plithogenic sets have attracted sustained research interest. These frameworks have been widely applied across various mathematical disciplines, including graph theory, topology, algebra, and group theory. More recently, the concept of the offset has emerged as a powerful and promising generalization of conventional uncertainty models. In this paper, we introduce a novel algebraic structure called the Fuzzy Off-Group and conduct an in-depth study of its fundamental mathematical properties. We hope that this framework will further advance research in group theory and uncertainty modeling with offsets, and that it will open up new avenues for application.

Keywords: Fuzzy Offset ▪ Fuzzy Set ▪ Fuzzy OffGroup ▪ Fuzzy Group

1. PRELIMINARIES

This section outlines the key concepts and definitions necessary for the discussions in this paper. The numbers considered in this paper are assumed to be finite.

1.1 Fuzzy Offset

The concept of a Fuzzy Offset extends the classical fuzzy set by allowing membership degrees outside the unit interval $[0,1]$ [1]. Although this paper focuses on offsets, one can similarly define Fuzzy Oversets (all memberships ≥ 0 with some > 1) and Fuzzy Undersets (all memberships ≤ 1 with some < 0) using the same principle. Related concepts such as Neutrosophic Offset are also known [1, 2].

Definition 1.1 (Fuzzy set)**.** A fuzzy set τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$ [3, 4]. A fuzzy relation on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a fuzzy relation on τ if

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y,z \in Y. \tag{1}$$

Example 1.2 (Fuzzy Set in HVAC Temperature Control)**.** Let $Y = [15,30]$ be the range of room temperatures (°C). Define the fuzzy set τ representing "Comfortable" temperature by

$$\tau(t) = \begin{cases} 0, & t \leq 18, \\ \dfrac{t-18}{4}, & 18 < t < 22, \\ 1, & 22 \leq t \leq 25, \\ \dfrac{30-t}{5}, & 25 < t < 30, \\ 0, & t \geq 30. \end{cases} \tag{2}$$

Here $\tau(t)$ measures the degree to which a temperature t is regarded as comfortable. This fuzzy set guides the HVAC control system to adjust heating or cooling gradually rather than using a simple on/off threshold.

Definition 1.3 (Fuzzy Offset)**.** Let X be a universe of dis-

© 2025 The Author(s). Published by ASPG. This article is licensed under CC BY 4.0.

course. A Fuzzy Offset $\widetilde{A}$ on X is a fuzzy set whose membership function [1, 5]

$$\mu_{\widetilde{A}} : X \longrightarrow [\Psi, \Omega] \tag{3}$$

takes values in an extended interval $[\Psi, \Omega]$ with $\Psi < 0$ and $\Omega > 1$. In particular, there must be elements $x, y \in X$ such that

$$\mu_{\widetilde{A}}(x) > 1 \quad \text{and} \quad \mu_{\widetilde{A}}(y) < 0. \tag{4}$$

Example 1.4 (Fuzzy Offset in Academic Performance)**.** Academic performance refers to a student's level of achievement in educational tasks, typically measured through grades, tests, and evaluations (cf. [6, 7, 8]). Let

$$X = \{\text{Sakura}, \text{Shinzo}, \text{Hiroko}, \text{Haruko}\}$$

be a set of students. After including extra credit and penalties, each student's adjusted exam score s may lie in $[-10, 150]$. Define a membership function

$$\mu_{\widetilde{A}} : X \longrightarrow [-0.1, 1.5], \qquad \mu_{\widetilde{A}}(\text{student}) = \frac{s}{100}. \tag{5}$$

Concretely,

$$\mu_{\widetilde{A}}(\text{Sakura}) = \frac{120}{100} = 1.2, \quad \mu_{\widetilde{A}}(\text{Shinzo}) = \frac{80}{100} = 0.8,$$
$$\mu_{\widetilde{A}}(\text{Hiroko}) = \frac{60}{100} = 0.6, \quad \mu_{\widetilde{A}}(\text{Haruko}) = \frac{-10}{100} = -0.1.$$

Here $\mu_{\widetilde{A}}(\text{Sakura}) > 1$ models over-achievement (extra credit) and $\mu_{\widetilde{A}}(\text{Haruko}) < 0$ models a severe penalty. Therefore $\widetilde{A}$ is a fuzzy offset on X.

Example 1.5 (Fuzzy Offset in Circuit Voltage Monitoring)**.** Let X be the set of measurement points in an electrical circuit, where each point x has a measured voltage v_x. Fix a nominal voltage $V_{\text{nom}} = 5\,\text{V}$ and a tolerance $\Delta = 2\,\text{V}$. Define the membership function

$$\mu_{\widetilde{A}}(x) = 1 + \frac{v_x - V_{\text{nom}}}{\Delta}. \tag{6}$$

Then

$$\mu_{\widetilde{A}}(x) > 1 \quad \text{if } v_x > V_{\text{nom}} + \Delta \qquad \text{(over-voltage)},$$
$$\mu_{\widetilde{A}}(x) < 0 \quad \text{if } v_x < V_{\text{nom}} - \Delta \qquad \text{(under-voltage)}.$$

Thus $\mu_{\widetilde{A}} : X \to [\Psi, \Omega]$ with $\Psi < 0 < 1 < \Omega$ captures both extreme under- and over-voltage conditions as a fuzzy offset.

1.2 Fuzzy Subgroup of a Group

Fuzzy Group Theory generalizes classical group theory by assigning membership degrees to group elements, capturing uncertainty and partial belonging [9, 10, 11, 12]. Related concepts such as the Neutrosophic Group are also well known [13, 14, 15]. The concept of a fuzzy subgroup of a group is presented below.

Definition 1.6 (Fuzzy Subgroup of a Group)**.** Let $(G, \cdot, e)$ be a group with identity e. A fuzzy subset of G is a function

$$\mu : G \longrightarrow [0, 1]. \tag{7}$$

Such a fuzzy subset μ is called a fuzzy subgroup (or fuzzy group) of G if, for all $x, y \in G$, it satisfies

1. $\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}$,
2. $\mu(x^{-1}) = \mu(x)$.

As a consequence, one shows that

$$\mu(e) = \sup_{x \in G} \mu(x), \tag{8}$$

so that the identity attains the maximum membership degree.

Example 1.7 (Fuzzy Subgroup of $\mathbb{Z}_4$)**.** Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ under addition modulo 4. Define a membership function $\mu : G \to [0, 1]$ by

$$\mu(k) = \begin{cases} 1.0, & k = 0, \\ 0.9, & k = 2, \\ 0.7, & k = 1, 3. \end{cases} \tag{9}$$

We verify the two axioms of Definition 1.6:

- **Closure under the group operation:** For all $i, j \in G$,

$$\mu(i + j) \geq \min\{\mu(i), \mu(j)\}.$$

  For instance,

$$\mu(1+3) = \mu(0) = 1.0 \geq \min\{0.7, 0.7\} = 0.7,$$
$$\mu(1+1) = \mu(2) = 0.9 \geq \min\{0.7, 0.7\} = 0.7,$$
$$\mu(2+2) = \mu(0) = 1.0 \geq \min\{0.9, 0.9\} = 0.9.$$

- **Inversion invariance:** For all $k \in G$,

$$\mu(-k) = \mu(k).$$

  Indeed, $\mu(-1) = \mu(3) = 0.7$ and $\mu(-2) = \mu(2) = 0.9$.

Since both conditions hold, μ is a fuzzy subgroup of G.

2. MAIN RESULTS: FUZZY OFFGROUP

In this section, we present the main results of this paper.

2.1 Fuzzy OffGroup

A Fuzzy OffGroup extends fuzzy subgroups by permitting membership values outside $[0, 1]$, while preserving closure under operation and inversion invariance. As the main result of this paper, we present the definition of the Fuzzy OffGroup below.

Definition 2.1 (Fuzzy OffGroup)**.** Let $(G, \cdot, e)$ be a group and let $[\Psi, \Omega]$ be an extended interval with $\Psi < 0 < 1 < \Omega$. A fuzzy offgroup of G is a mapping

$$\mu : G \longrightarrow [\Psi, \Omega] \tag{10}$$

which simultaneously

1. is a fuzzy offset in the sense of Definition 1.3, i.e., there exist $x, y \in G$ with $\mu(x) > 1$ and $\mu(y) < 0$;
2. satisfies the usual fuzzy-subgroup conditions:

$$\begin{aligned} &\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}, \\ &\mu(x^{-1}) = \mu(x), \end{aligned} \quad \text{for all } x, y \in G. \tag{11}$$

Thus a fuzzy offgroup extends the notion of a fuzzy subgroup by allowing membership values outside $[0,1]$.

Example 2.2. Let $G = \mathbb{Z}_4 = \{0,1,2,3\}$ under addition modulo 4, and choose $[\Psi,\Omega] = [-0.1, 1.2]$. Define

$$\mu(k) = \begin{cases} 1.2, & k = 0, \\ 0.7, & k = 1,3, \\ -0.1, & k = 2. \end{cases} \tag{12}$$

Then:

- $\mu(1+3) = \mu(0) = 1.2 \geq \min\{0.7, 0.7\}$.
- $\mu(-2) = \mu(2) = -0.1$.
- We have $\mu(0) = \Omega > 1$ and $\mu(2) = \Psi < 0$.

Thus μ is a fuzzy offset and also satisfies $\mu(k+\ell) \geq \min\{\mu(k), \mu(\ell)\}$ and $\mu(-k) = \mu(k)$. Hence μ is a fuzzy offgroup of G.

Example 2.3 (Fuzzy OffGroup on $\mathbb{Z}_3$)**.** Let $G = \mathbb{Z}_3 = \{0,1,2\}$ under addition modulo 3, and choose the extended interval $[\Psi,\Omega] = [-0.2, 1.3]$. Define

$$\mu(k) = \begin{cases} 1.3, & k = 0, \\ -0.2, & k = 1,2. \end{cases} \tag{13}$$

We check the conditions of Definition 2.1:

- **Offset condition:** $\mu(0) = 1.3 > 1$ and $\mu(1) = \mu(2) = -0.2 < 0$, so μ is a fuzzy offset.
- **Closure under the group operation:** For all $i, j \in G$,

$$\mu(i+j) = \begin{cases} 1.3, & i+j \equiv 0 \pmod 3, \\ -0.2, & i+j \equiv 1 \pmod 3, \\ -0.2, & i+j \equiv 2 \pmod 3, \end{cases} \tag{14}$$

  while $\min\{\mu(i), \mu(j)\} = -0.2$ whenever either input is nonzero. Since $1.3 \geq -0.2$ and $-0.2 \geq -0.2$, we have $\mu(i+j) \geq \min\{\mu(i), \mu(j)\}$.
- **Inversion invariance:** $\mu(-k) = \mu(3-k) = \mu(k)$ for each $k \in \{1,2\}$, and $\mu(0) = \mu(0)$.

Thus μ defines a fuzzy offgroup on $\mathbb{Z}_3$.

Example 2.4 (Fuzzy OffGroup on $\mathbb{Z}_2$)**.** Let $G = \mathbb{Z}_2 = \{0,1\}$ under addition modulo 2, and choose the extended interval $[\Psi,\Omega] = [-0.5, 1.5]$. Define

$$\mu(k) = \begin{cases} 1.5, & k = 0, \\ -0.5, & k = 1. \end{cases} \tag{15}$$

We verify the conditions of Definition 2.1:

1. **Offset condition:** $\mu(0) = 1.5 > 1$ and $\mu(1) = -0.5 < 0$, so μ is a fuzzy offset.
2. **Closure under the group operation:**

$$\begin{aligned} \mu(0+0) &= \mu(0) = 1.5 \geq \min\{1.5, 1.5\} = 1.5, \\ \mu(1+0) &= \mu(1) = -0.5 \geq \min\{-0.5, 1.5\} = -0.5, \\ \mu(1+1) &= \mu(0) = 1.5 \geq \min\{-0.5, -0.5\} = -0.5. \end{aligned}$$

3. **Inversion invariance:** Since $1^{-1} = 1$ and $0^{-1} = 0$, we have $\mu(-k) = \mu(k)$ for both $k = 0, 1$.

Therefore μ defines a fuzzy offgroup on $\mathbb{Z}_2$.

We investigate the properties of the Fuzzy Off-Group. The following theorem holds.

Theorem 2.5. *A mapping $\mu : G \to [0,1]$ is a fuzzy subgroup of G if and only if it is a fuzzy offgroup whose image lies in $[0,1]$. In particular, every fuzzy subgroup is a special case of a fuzzy offgroup (with $\Psi = 0$, $\Omega = 1$).*

Proof. If $\mu : G \to [0,1]$ is a fuzzy subgroup, then trivially $\mu(G) \subseteq [0,1] = [\Psi,\Omega]$ with $\Psi = 0$, $\Omega = 1$. Since no element actually attains values outside $[0,1]$, the "offset" condition (existence of values > 1 or < 0) is vacuous when one sets $\Psi = 0$, $\Omega = 1$. Conditions on $\mu(x \cdot y)$ and $\mu(x^{-1})$ coincide exactly with those of Definition 2.1.

Conversely, if $\mu : G \to [\Psi,\Omega]$ is a fuzzy offgroup with $\mu(G) \subseteq [0,1]$, then the offset-specific requirement can be ignored, and the two conditions

$$\begin{aligned} &\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}, \\ &\mu(x^{-1}) = \mu(x) \end{aligned} \tag{16}$$

are exactly the axioms of a fuzzy subgroup. Hence μ is a fuzzy subgroup. □

Theorem 2.6 (Maximum Membership at the Identity)**.** *Let $\mu : G \to [\Psi,\Omega]$ be a fuzzy offgroup. Then*

$$\mu(e) = \sup_{x \in G} \mu(x). \tag{17}$$

In particular, $\mu(e) > 1$.

Proof. For any $x \in G$, since $x \cdot x^{-1} = e$ and μ satisfies the fuzzy-subgroup condition,

$$\mu(e) = \mu(x \cdot x^{-1}) \geq \min\{\mu(x), \mu(x^{-1})\} = \mu(x). \tag{18}$$

Hence $\mu(e) \geq \mu(x)$ for all x, so $\mu(e) = \sup_{x \in G} \mu(x)$. Because μ is a fuzzy offset, there exists $x_0 \in G$ with $\mu(x_0) > 1$, whence $\mu(e) \geq \mu(x_0) > 1$. □

Theorem 2.7 (Subgroups from α-cuts)**.** *For any λ with $\Psi < \lambda \leq \mu(e)$, define the λ-cut*

$$G_\lambda = \{x \in G : \mu(x) \geq \lambda\}. \tag{19}$$

Then G_λ is a (crisp) subgroup of G.

Proof. Since $\lambda \leq \mu(e)$, the identity $e \in G_\lambda$. If $x, y \in G_\lambda$, then

$$\begin{aligned} &\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \geq \lambda, \\ &\mu(x^{-1}) = \mu(x) \geq \lambda. \end{aligned} \tag{20}$$

Thus $x \cdot y \in G_\lambda$ and $x^{-1} \in G_\lambda$, so G_λ is closed under the group operation and inverses. □

Theorem 2.8 (Core as a Subgroup)**.** *Define the core of μ by*

$$\mathrm{Core}(\mu) = \{x \in G : \mu(x) = \mu(e)\}. \tag{21}$$

Then $\mathrm{Core}(\mu)$ *is a subgroup of* G.

Proof. Because $\mu(e) = \sup_x \mu(x)$, we have $e \in \mathrm{Core}(\mu)$. If $x, y \in \mathrm{Core}(\mu)$, then

$$\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} = \mu(e), \tag{22}$$

and by maximality of $\mu(e)$ this forces $\mu(x \cdot y) = \mu(e)$. Likewise,

$$\mu(x^{-1}) = \mu(x) = \mu(e). \tag{23}$$

Hence $\mathrm{Core}(\mu)$ is closed under products and inverses. □

Theorem 2.9 (Intersection of Fuzzy OffGroups)**.** *Let* $\mu_1, \mu_2 : G \to [\Psi, \Omega]$ *be fuzzy offgroups such that there exist* $a, b \in G$ *with*

$$\mu_1(a), \mu_2(a) > 1 \quad \textit{and} \quad \mu_1(b), \mu_2(b) < 0. \tag{24}$$

Define

$$\mu(x) = \min\{\mu_1(x), \mu_2(x)\} \qquad (x \in G). \tag{25}$$

Then μ *is also a fuzzy offgroup on* G.

Proof. For all $x, y \in G$,

$$\begin{aligned}\mu(x \cdot y) &= \min\{\mu_1(x \cdot y), \mu_2(x \cdot y)\} \\ &\geq \min\{\min\{\mu_1(x), \mu_1(y)\}, \min\{\mu_2(x), \mu_2(y)\}\} \\ &= \min\{\mu(x), \mu(y)\},\end{aligned}$$

and

$$\begin{aligned}\mu(x^{-1}) &= \min\{\mu_1(x^{-1}), \mu_2(x^{-1})\} \\ &= \min\{\mu_1(x), \mu_2(x)\} = \mu(x).\end{aligned}$$

Moreover, since $\mu_1(a), \mu_2(a) > 1$ and $\mu_1(b), \mu_2(b) < 0$, we have $\mu(a) > 1$ and $\mu(b) < 0$, so the offset condition holds. Thus μ satisfies all requirements of Definition 2.1. □

2.2 Fuzzy OverGroup and Fuzzy UnderGroup

The definitions of the Fuzzy OverGroup and Fuzzy UnderGroup are also provided below. These are subclasses of the Fuzzy OffGroup.

Definition 2.10 (Fuzzy OverGroup)**.** Let $(G, \cdot, e)$ be a group and let $[0, \Omega]$ be an extended interval with $0 < 1 < \Omega$. A fuzzy overgroup on G is a mapping

$$\mu : G \longrightarrow [0, \Omega] \tag{26}$$

which simultaneously

1. is an over-fuzzy set, i.e., there exists $x \in G$ with $\mu(x) > 1$;
2. satisfies the fuzzy-subgroup conditions:

$$\begin{aligned}&\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}, \\ &\mu(x^{-1}) = \mu(x),\end{aligned} \qquad \text{for all } x, y \in G. \tag{27}$$

Thus a fuzzy overgroup extends the notion of a fuzzy subgroup by allowing membership values above 1.

Example 2.11 (Fuzzy OverGroup on $\mathbb{Z}_3$)**.** Let $G = \mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3, and choose the extended interval $[0, \Omega] = [0, 1.4]$. Define

$$\mu(k) = \begin{cases} 1.4, & k = 0, \\ 0.8, & k = 1, 2. \end{cases} \tag{28}$$

We check the conditions of Definition 2.10:

- **Over-fuzzy condition:** $\mu(0) = 1.4 > 1$, so there exists an element with membership above 1.
- **Closure under the group operation:**

$$\begin{aligned}1 + 2 \equiv 0 : &\quad \mu(0) = 1.4 \geq \min\{0.8, 0.8\} = 0.8, \\ 2 + 2 \equiv 1 : &\quad \mu(1) = 0.8 \geq \min\{0.8, 0.8\} = 0.8, \\ 1 + 1 \equiv 2 : &\quad \mu(2) = 0.8 \geq \min\{0.8, 0.8\} = 0.8.\end{aligned}$$

- **Inversion invariance:** For each $k \in \{1, 2\}$,

$$-k \equiv (3 - k) \pmod 3, \tag{29}$$

$$\mu(-k) = \mu(3 - k) = 0.8 = \mu(k). \tag{30}$$

Hence μ defines a fuzzy overgroup on G.

Theorem 2.12 (Maximum Membership at the Identity)**.** *Let* $\mu : G \to [0, \Omega]$ *be a fuzzy overgroup on* G. *Then*

$$\mu(e) = \sup_{x \in G} \mu(x), \tag{31}$$

and in particular $\mu(e) > 1$.

Proof. For any $x \in G$, using $x \cdot x^{-1} = e$ and the subgroup condition,

$$\mu(e) = \mu(x \cdot x^{-1}) \geq \min\{\mu(x), \mu(x^{-1})\} = \mu(x). \tag{32}$$

Hence $\mu(e) \geq \mu(x)$ for all x, so equality holds as the supremum. Since μ is an over-fuzzy set, there exists $x_0 \in G$ with $\mu(x_0) > 1$, whence $\mu(e) \geq \mu(x_0) > 1$. □

Theorem 2.13 (α-cut Subgroups)**.** *For any* α *with* $1 < \alpha \leq \mu(e)$, *define the "*α*-cut"*

$$G_\alpha = \{x \in G : \mu(x) \geq \alpha\}. \tag{33}$$

Then G_α *is a (crisp) subgroup of* G.

Proof. Since $\alpha \leq \mu(e)$, we have $e \in G_\alpha$. If $x, y \in G_\alpha$, then

$$\begin{aligned}&\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\} \geq \alpha, \\ &\mu(x^{-1}) = \mu(x) \geq \alpha.\end{aligned} \tag{34}$$

Thus $x \cdot y \in G_\alpha$ and $x^{-1} \in G_\alpha$, so G_α is closed under the group operation and inverses. □

Theorem 2.14 (Intersection of Fuzzy OverGroups)**.** *Let* $\mu_1, \mu_2 : G \to [0, \Omega]$ *be fuzzy overgroups, each with some element exceeding* 1. *Define*

$$\mu(x) = \min\{\mu_1(x), \mu_2(x)\} \qquad (x \in G). \tag{35}$$

Then μ *is also a fuzzy overgroup on* G.

Proof. Since each μ_i has some a_i with $\mu_i(a_i) > 1$, their minimum still satisfies $\mu(a) > 1$ for a chosen appropriately. Moreover, for all $x, y \in G$,

$$\begin{aligned}\mu(x\cdot y) &= \min\{\mu_1(x\cdot y), \mu_2(x\cdot y)\}\\ &\geq \min\{\min\{\mu_1(x), \mu_1(y)\}, \min\{\mu_2(x), \mu_2(y)\}\}\\ &= \min\{\mu(x), \mu(y)\},\end{aligned}$$

and

$$\mu(x^{-1}) = \min\{\mu_1(x^{-1}), \mu_2(x^{-1})\} = \mu(x). \tag{36}$$

Thus μ satisfies all requirements of Definition 2.10. □

Definition 2.15 (Fuzzy UnderGroup)**.** Let $(G, \cdot, e)$ be a group and let $[\Psi, 1]$ be an extended interval with $\Psi < 0 < 1$. A fuzzy undergroup on G is a mapping

$$\mu : G \longrightarrow [\Psi, 1] \tag{37}$$

which simultaneously

1. is an under-fuzzy set, i.e., there exists $y \in G$ with $\mu(y) < 0$;
2. satisfies the fuzzy-subgroup conditions:

$$\begin{aligned}&\mu(x\cdot y) \geq \min\{\mu(x), \mu(y)\},\\ &\mu(x^{-1}) = \mu(x),\end{aligned} \quad \text{for all } x, y \in G. \tag{38}$$

Thus a fuzzy undergroup extends the notion of a fuzzy subgroup by allowing membership values below 0.

Example 2.16 (Fuzzy UnderGroup on $\mathbb{Z}_3$)**.** Let $G = \mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3, and choose the extended interval $[\Psi, 1] = [-0.3, 1]$. Define

$$\mu(k) = \begin{cases} 1, & k = 0,\\ -0.3, & k = 1, 2.\end{cases} \tag{39}$$

We check the conditions of Definition 2.15:

- **Under-fuzzy condition:** $\mu(1) = \mu(2) = -0.3 < 0$, so there exists an element with membership below 0.
- **Closure under the group operation:**

$$\begin{aligned}1+2 \equiv 0: &\quad \mu(0) = 1 \geq \min\{-0.3, -0.3\} = -0.3,\\ 2+2 \equiv 1: &\quad \mu(1) = -0.3 \geq \min\{-0.3, -0.3\} = -0.3,\\ 1+1 \equiv 2: &\quad \mu(2) = -0.3 \geq \min\{-0.3, -0.3\} = -0.3.\end{aligned}$$

- **Inversion invariance:** For each $k \in \{1, 2\}$,

$$-k \equiv (3-k) \pmod 3, \tag{40}$$

$$\mu(-k) = \mu(3-k) = -0.3 = \mu(k). \tag{41}$$

Thus μ defines a fuzzy undergroup on G.

Theorem 2.17 (Maximum Membership at the Identity)**.** *Let $\mu : G \to [\Psi, 1]$ be a fuzzy undergroup on G. Then*

$$\mu(e) = \sup_{x\in G} \mu(x). \tag{42}$$

Proof. Exactly as in Theorem 2.12, using $x \cdot x^{-1} = e$ and noting $\mu(e) \leq 1$ by the range. □

Theorem 2.18 (α-cut Subgroups)**.** *For any α with $\Psi < \alpha \leq \mu(e)$, define*

$$G_\alpha = \{x \in G : \mu(x) \geq \alpha\}. \tag{43}$$

Then G_α is a (crisp) subgroup of G.

Proof. Identical to the proof of Theorem 2.13. □

Theorem 2.19 (Intersection of Fuzzy UnderGroups)**.** *Let $\mu_1, \mu_2 : G \to [\Psi, 1]$ be fuzzy undergroups, each with some element below 0. Define*

$$\mu(x) = \min\{\mu_1(x), \mu_2(x)\} \qquad (x \in G). \tag{44}$$

Then μ is also a fuzzy undergroup on G.

Proof. Since each μ_i has some b_i with $\mu_i(b_i) < 0$, their minimum still satisfies $\mu(b) < 0$. The subgroup conditions follow just as in Theorem 2.14. □

3. CONCLUSION AND FUTURE WORKS

In this paper, we examined the Fuzzy OffGroup and clarified its key properties. In future work, we aim to explore extensions of the Off-Group structure by incorporating the frameworks of hypergroups [16, 17] and superhypergroups [18]. We also plan to investigate further extensions using plithogenic sets [19, 20], hesitant fuzzy sets [21, 22], bipolar fuzzy sets [23, 24], picture fuzzy sets [25], and hyperfuzzy sets [12, 26, 27]. Furthermore, we hope that future research will develop more concrete mathematical structures for the concepts introduced in this paper.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

REFERENCES

[1] F. Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[2] F. Smarandache, M. A. Quiroz-Martínez, J. Estupiñán, N. B. Hernández, and M. Y. L. Vazquez, "Application of neutrosophic offsets for digital image processing," 2020.

[3] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[4] H.-J. Zimmermann, *Fuzzy Set Theory—and Its Applications*. Springer Science & Business Media, 2011.

[5] F. Smarandache, "Practical applications of the independent neutrosophic components and of the neutrosophic offset components," *Neutrosophic Sets and Systems*, vol. 47, pp. 558–572, 2021.

[6] M. Richardson, C. Abraham, and R. Bond, "Psychological correlates of university students' academic performance: A systematic review and meta-analysis," *Psychological Bulletin*, vol. 138, no. 2, p. 353, 2012.

[7] A. E. Poropat, "A meta-analysis of the five-factor model of personality and academic performance," *Psychological Bulletin*, vol. 135, no. 2, p. 322, 2009.

[8] S. Sudha, M. Lathamaheswari, S. Broumi, and F. Smarandache, "Neutrosophic fuzzy magic labeling graph with its application in academic performance of the students," *Neutrosophic Sets and Systems*, vol. 60, no. 1, p. 8, 2023.

[9] P. S. Das, "Fuzzy groups and level subgroups," *Journal of Mathematical Analysis and Applications*, vol. 84, no. 1, pp. 264–269, 1981.

[10] J. N. Mordeson, K. R. Bhutani, and A. Rosenfeld, *Fuzzy Group Theory*. Springer, 2005, vol. 182.

[11] F. E. Boran, S. Genç, M. Kurt, and D. Akay, "A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method," *Expert Systems with Applications*, vol. 36, no. 8, pp. 11 363–11 368, 2009.

[12] J. Ghosh and T. K. Samanta, "Hyperfuzzy sets and hyperfuzzy group," *International Journal of Advanced Science and Technology*, vol. 41, pp. 27–37, 2012.

[13] B. Basumatary and N. Wary, "A study on neutrosophic bitopological group," *Neutrosophic Sets and Systems*, vol. 38, pp. 164–178, 2020.

[14] A. Elrawy, M. Abdalla, and M. A. Saleem, "Some results on neutrosophic group," *Journal of Mathematics*, vol. 2023, pp. 1–6, 2023.

[15] M. Abdel-Basset, M. Mohamed, and F. Smarandache, "A hybrid neutrosophic group ANP-TOPSIS framework for supplier selection problems," *Symmetry*, vol. 10, p. 226, 2018.

[16] G. Muhiuddin, E. Mohammadzadeh, F. Mohammadzadeh, and R. A. Borzooei, "Construction of fuzzy hyper BCI-algebras using fuzzy hypergroupoid," *New Mathematics and Natural Computation*, vol. 19, pp. 557–577, 2022.

[17] M. Al Tahan, S. Hoskova-Mayerova, and B. Davvaz, "An overview of topological hypergroupoids," *Journal of Intelligent & Fuzzy Systems*, vol. 34, pp. 1907–1916, 2018.

[18] A. Kargın, F. Smarandache, and M. Şahin, *New Type Hyper Groups, New Type SuperHyper Groups and Neutro-New Type SuperHyper Groups*. Infinite Study, 2023.

[19] F. Sultana, M. Gulistan, M. Ali, N. Yaqoob, M. Khan, T. Rashid, and T. Ahmed, "A study of plithogenic graphs: Applications in spreading coronavirus disease (COVID-19) globally," *Journal of Ambient Intelligence and Humanized Computing*, vol. 14, no. 10, pp. 13 139–13 159, 2023.

[20] S. G. Quek, G. Selvachandran, F. Smarandache, J. Vimala, S. H. Le, Q.-T. Bui, and V. C. Gerogiannis, "Entropy measures for plithogenic sets and applications in multi-attribute decision making," *Mathematics*, vol. 8, no. 6, p. 965, 2020.

[21] Z. Xu, *Hesitant Fuzzy Sets Theory*. Springer, 2014, vol. 314.

[22] V. Torra and Y. Narukawa, "On hesitant fuzzy sets and decision," in *2009 IEEE International Conference on Fuzzy Systems*. IEEE, 2009, pp. 1378–1382.

[23] M. Akram, "Bipolar fuzzy graphs," *Information Sciences*, vol. 181, no. 24, pp. 5548–5564, 2011.

[24] W.-R. Zhang, "Bipolar fuzzy sets and relations: A computational framework for cognitive modeling and multiagent decision analysis," in *NAFIPS/IFIS/NASA '94: Proceedings of the First International Joint Conference of the North American Fuzzy Information Processing Society Biannual Conference, the Industrial Fuzzy Control and Intelligent Systems Conference, and the NASA Joint Technology Workshop*, 1994, pp. 305–309.

[25] B. C. Cuong and V. Kreinovich, "Picture fuzzy sets—a new concept for computational intelligence problems," in *2013 Third World Congress on Information and Communication Technologies (WICT 2013)*. IEEE, 2013, pp. 1–6.

[26] Y. L. Liu, H. S. Kim, and J. Neggers, "Hyperfuzzy subsets and subgroupoids," *Journal of Intelligent & Fuzzy Systems*, vol. 33, pp. 1553–1562, 2017.

[27] Z. Nazari and B. Mosapour, "The entropy of hyperfuzzy sets," *Journal of Dynamical Systems and Geometric Theories*, vol. 16, no. 2, pp. 173–185, 2018.