A Brief Study on Fuzzy Off-Group Theory

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Abstract

Uncertainty-handling frameworks such as fuzzy sets, rough sets, intuitionistic fuzzy sets, neutrosophic sets, Picture Fuzzy Sets, hyperneutrosophic sets, and plithogenic sets have attracted sustained research interest. These frameworks have been widely applied across various mathematical disciplines, including graph theory, topology, algebra, and group theory. More recently, the concept of the *offset* has emerged as a powerful and promising generalization of conventional uncertainty models. In this paper, we introduce a novel algebraic structure called the *Fuzzy Off-Group* and conduct an in-depth study of its fundamental mathematical properties. We hope that this framework will further advance research in group theory and uncertainty modeling with offsets, and that it will open up new avenues for application.

Keywords: Fuzzy Offset; Fuzzy Set; Fuzzy OffGroup; Fuzzy Group

1 Preliminaries

This section outlines the key concepts and definitions necessary for the discussions in this paper. The numbers considered in this paper are assumed to be finite.

1.1 Fuzzy Offset

The concept of a *Fuzzy Offset* extends the classical fuzzy set by allowing membership degrees outside the unit interval $[0, 1]$.[1] Although this paper focuses on offsets, one can similarly define *Fuzzy Oversets* (all memberships ≥ 0 with some > 1) and *Fuzzy Undersets* (all memberships ≤ 1 with some < 0) using the same principle. Related concepts such as Neutrosophic Offset are also known.[1,2]

Definition 1.1 (Fuzzy set). [3,4] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

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Example 1.2 (Fuzzy Set in HVAC Temperature Control)**.** Let $Y = [15, 30]$ be the range of room temperatures (°C). Define the fuzzy set τ representing "Comfortable" temperature by

$$\tau(t) = \begin{cases} 0, & t \le 18, \\ \dfrac{t-18}{4}, & 18 < t < 22, \\ 1, & 22 \le t \le 25, \\ \dfrac{30-t}{5}, & 25 < t < 30, \\ 0, & t \ge 30. \end{cases}$$

Here $\tau(t)$ measures the degree to which a temperature t is regarded as comfortable. This fuzzy set guides the HVAC control system to adjust heating or cooling gradually rather than using a simple on/off threshold.

Definition 1.3 (Fuzzy Offset)**.** [1,5] Let X be a universe of discourse. A *Fuzzy Offset* $\tilde{A}$ on X is a fuzzy set whose membership function

$$\mu_{\tilde{A}} : X \to [\Psi, \Omega]$$

takes values in an extended interval $[\Psi, \Omega]$ with $\Psi < 0$ and $\Omega > 1$. In particular, there must be elements $x, y \in X$ such that

$$\mu_{\tilde{A}}(x) > 1 \quad \text{and} \quad \mu_{\tilde{A}}(y) < 0.$$

Example 1.4 (Fuzzy Offset in Academic Performance)**.** Academic performance refers to a student's level of achievement in educational tasks, typically measured through grades, tests, and evaluations (cf.[6–8]). Let $X = \{\text{Sakura}, \text{Shinzo}, \text{Hiroko}, \text{Haruko}\}$ be a set of students. After including extra credit and penalties, each student's adjusted exam score s may lie in $[-10, 150]$. Define a membership function

$$\mu_{\tilde{A}} : X \longrightarrow [-0.1,\, 1.5], \qquad \mu_{\tilde{A}}(\text{student}) = \frac{s}{100}.$$

Concretely,

$$\mu_{\tilde{A}}(\text{Sakura}) = \frac{120}{100} = 1.2, \quad \mu_{\tilde{A}}(\text{Shinzo}) = \frac{80}{100} = 0.8,$$
$$\mu_{\tilde{A}}(\text{Hiroko}) = \frac{60}{100} = 0.6, \quad \mu_{\tilde{A}}(\text{Haruko}) = \frac{-10}{100} = -0.1.$$

Here $\mu_{\tilde{A}}(\text{Sakura}) > 1$ models over-achievement (extra credit) and $\mu_{\tilde{A}}(\text{Haruko}) < 0$ models a severe penalty. Therefore $\tilde{A}$ is a fuzzy offset on X.

Example 1.5 (Fuzzy Offset in Circuit Voltage Monitoring)**.** Let X be the set of measurement points in an electrical circuit, where each point x has a measured voltage v_x. Fix a nominal voltage $V_{\text{nom}} = 5\text{V}$ and a tolerance $\Delta = 2\text{V}$. Define the membership function

$$\mu_{\tilde{A}}(x) \;=\; 1 \;+\; \frac{v_x - V_{\text{nom}}}{\Delta}.$$

Then

$$\mu_{\tilde{A}}(x) > 1 \quad \text{if } v_x > V_{\text{nom}} + \Delta \quad \text{(over-voltage)},$$
$$\mu_{\tilde{A}}(x) < 0 \quad \text{if } v_x < V_{\text{nom}} - \Delta \quad \text{(under-voltage)}.$$

Thus $\mu_{\tilde{A}} : X \to [\Psi, \Omega]$ with $\Psi < 0 < 1 < \Omega$ captures both extreme under- and over-voltage conditions as a fuzzy offset.

1.2 Fuzzy Subgroup of a Group

Fuzzy Group Theory generalizes classical group theory by assigning membership degrees to group elements, capturing uncertainty and partial belonging.[9–12] Related concepts such as the Neutrosophic Group are also well known.[13–15] The concept of a fuzzy subgroup of a group is presented below.

Definition 1.6 (Fuzzy Subgroup of a Group)**.** Let $(G, \cdot, e)$ be a group with identity e. A *fuzzy subset* of G is a function

$$\mu : G \longrightarrow [0, 1].$$

Such a fuzzy subset μ is called a *fuzzy subgroup* (or *fuzzy group*) of G if, for all $x, y \in G$, it satisfies

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1. $\mu(x \cdot y) \;\geq\; \min\{\mu(x),\, \mu(y)\}$,
2. $\mu(x^{-1}) \;=\; \mu(x)$.

As a consequence, one shows that

$$\mu(e) \;=\; \sup_{x \in G} \mu(x),$$

so that the identity attains the maximum membership degree.

Example 1.7 (Fuzzy Subgroup of $\mathbb{Z}_4$)**.** Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ under addition modulo 4. Define a membership function $\mu : G \to [0, 1]$ by

$$\mu(k) = \begin{cases} 1.0, & k = 0, \\ 0.9, & k = 2, \\ 0.7, & k = 1, 3. \end{cases}$$

We verify the two axioms of Definition 1.6:

- **Closure under the group operation:** For all $i, j \in G$,

  $$\mu(i + j) \;\geq\; \min\{\mu(i), \mu(j)\}.$$

  For instance,

  $$\mu(1+3) = \mu(0) = 1.0 \;\geq\; \min\{0.7, 0.7\} = 0.7, \quad \mu(1+1) = \mu(2) = 0.9 \;\geq\; \min\{0.7, 0.7\} = 0.7,$$

  $$\mu(2+2) = \mu(0) = 1.0 \;\geq\; \min\{0.9, 0.9\} = 0.9.$$

- **Inversion invariance:** For all $k \in G$,

  $$\mu(-k) \;=\; \mu(k).$$

  Indeed, $\mu(-1) = \mu(3) = 0.7$ and $\mu(-2) = \mu(2) = 0.9$.

Since both conditions hold, μ is a fuzzy subgroup of G.

2 Main Results: Fuzzy OffGroup

In this section, we present the main results of this paper.

2.1 Fuzzy OffGroup

A Fuzzy OffGroup extends fuzzy subgroups by permitting membership values outside $[0, 1]$, while preserving closure under operation and inversion invariance. As the main result of this paper, we present the definition of the Fuzzy OffGroup below.

Definition 2.1 (Fuzzy OffGroup)**.** Let $(G, \cdot, e)$ be a group and let $[\Psi, \Omega]$ be an extended interval with $\Psi < 0 < 1 < \Omega$. A *fuzzy offgroup* of G is a mapping

$$\mu : G \;\longrightarrow\; [\Psi, \Omega]$$

which simultaneously

1. is a *fuzzy offset* in the sense of Definition ("Fuzzy Offset"), i.e. there exist $x, y \in G$ with $\mu(x) > 1$ and $\mu(y) < 0$,

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2. satisfies the usual fuzzy-subgroup conditions:

$$\mu(x \cdot y) \;\geq\; \min\{\mu(x), \mu(y)\} \quad \text{and} \quad \mu(x^{-1}) \;=\; \mu(x) \quad \text{for all } x, y \in G.$$

Thus a fuzzy offgroup extends the notion of a fuzzy subgroup by allowing membership values outside $[0, 1]$.

Example 2.2. Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ under addition mod 4, and choose $[\Psi, \Omega] = [-0.1, 1.2]$. Define

$$\mu(k) = \begin{cases} 1.2, & k = 0, \\ 0.7, & k = 1, 3, \\ -0.1, & k = 2. \end{cases}$$

Then:

- $\mu(1 + 3) = \mu(0) = 1.2 \geq \min\{0.7, 0.7\}$.
- $\mu(-2) = \mu(2) = -0.1$.
- We have $\mu(0) = \Omega > 1$ and $\mu(2) = \Psi < 0$,

so μ is a fuzzy offset and also satisfies $\mu(k + \ell) \geq \min\{\mu(k), \mu(\ell)\}$ and $\mu(-k) = \mu(k)$. Hence μ is a fuzzy offgroup of G.

Example 2.3 (Fuzzy OffGroup on $\mathbb{Z}_3$)**.** Let $G = \mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3, and choose the extended interval $[\Psi, \Omega] = [-0.2,\ 1.3]$. Define

$$\mu(k) \;=\; \begin{cases} 1.3, & k = 0, \\ -0.2, & k = 1, 2. \end{cases}$$

We check the conditions of Definition 2.1:

- **Offset condition:** $\mu(0) = 1.3 > 1$ and $\mu(1) = \mu(2) = -0.2 < 0$, so μ is a fuzzy offset.
- **Closure under the group operation:** For all $i, j \in G$,

$$\mu(i + j) \;=\; \begin{cases} \mu(0) = 1.3, & \text{if } i + j \equiv 0, \\ \mu(1) = -0.2, & \text{if } i + j \equiv 1, \\ \mu(2) = -0.2, & \text{if } i + j \equiv 2, \end{cases} \quad \min\{\mu(i), \mu(j)\} = -0.2.$$

  Since $1.3 \geq -0.2$ and $-0.2 \geq -0.2$, we have $\mu(i + j) \geq \min\{\mu(i), \mu(j)\}$.
- **Inversion invariance:** $\mu(-k) = \mu(3 - k) = \mu(k)$ for each $k \in \{1, 2\}$, and $\mu(0) = \mu(0)$.

Thus μ defines a fuzzy offgroup on $\mathbb{Z}_3$.

Example 2.4 (Fuzzy OffGroup on $\mathbb{Z}_2$)**.** Let $G = \mathbb{Z}_2 = \{0, 1\}$ under addition modulo 2, and choose the extended interval $[\Psi, \Omega] = [-0.5,\ 1.5]$. Define

$$\mu(k) \;=\; \begin{cases} 1.5, & k = 0, \\ -0.5, & k = 1. \end{cases}$$

We verify the conditions of Definition 2.1:

1. *Offset condition:* $\mu(0) = 1.5 > 1$ and $\mu(1) = -0.5 < 0$, so μ is a fuzzy offset.

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2. *Closure under the group operation:*

$$\mu(0+0) = \mu(0) = 1.5 \ \geq\ \min\{1.5, 1.5\} = 1.5,$$

$$\mu(1+0) = \mu(1) = -0.5 \ \geq\ \min\{-0.5, 1.5\} = -0.5,$$

$$\mu(1+1) = \mu(0) = 1.5 \ \geq\ \min\{-0.5, -0.5\} = -0.5.$$

3. *Inversion invariance:* Since $1^{-1} = 1$ and $0^{-1} = 0$, we have $\mu(-k) = \mu(k)$ for both $k = 0, 1$.

Therefore μ defines a fuzzy offgroup on $\mathbb{Z}_2$.

We investigate the properties of the Fuzzy Off-Group. The following theorem holds.

Theorem 2.5. *A mapping $\mu : G \to [0,1]$ is a fuzzy subgroup of G if and only if it is a fuzzy offgroup whose image lies in $[0,1]$. In particular, every fuzzy subgroup is a special case of a fuzzy offgroup (with $\Psi = 0$, $\Omega = 1$).*

Proof. If $\mu : G \to [0,1]$ is a fuzzy subgroup, then trivially $\mu(G) \subseteq [0,1] = [\Psi, \Omega]$ with $\Psi = 0$, $\Omega = 1$. Since no element actually attains values outside $[0,1]$, the "offset" condition (existence of values > 1 or < 0) is vacuous when one sets $\Psi = 0$, $\Omega = 1$. Conditions on $\mu(x \cdot y)$ and $\mu(x^{-1})$ coincide exactly with those of Definition 2.1.

Conversely, if $\mu : G \to [\Psi, \Omega]$ is a fuzzy offgroup with $\mu(G) \subseteq [0,1]$, then the offset-specific requirement can be ignored, and the two inequalities

$$\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}, \quad \mu(x^{-1}) = \mu(x)$$

are exactly the axioms of a fuzzy subgroup. Hence μ is a fuzzy subgroup. □

Theorem 2.6 (Maximum Membership at the Identity)**.** *Let $\mu : G \to [\Psi, \Omega]$ be a fuzzy offgroup. Then*

$$\mu(e) \ =\ \sup_{x \in G} \mu(x).$$

In particular, $\mu(e) > 1$.

Proof. For any $x \in G$, since $x \cdot x^{-1} = e$ and μ satisfies the fuzzy-subgroup condition,

$$\mu(e) = \mu(x \cdot x^{-1}) \ \geq\ \min\{\mu(x), \mu(x^{-1})\} = \mu(x).$$

Hence $\mu(e) \geq \mu(x)$ for all x, so $\mu(e) = \sup_{x \in G} \mu(x)$. Because μ is a fuzzy offset, there exists $x_0 \in G$ with $\mu(x_0) > 1$, whence $\mu(e) \geq \mu(x_0) > 1$. □

Theorem 2.7 (Subgroups from α–cuts)**.** *For any λ with $\Psi < \lambda \leq \mu(e)$, define the λ–*cut

$$G_\lambda \ =\ \{\, x \in G : \mu(x) \geq \lambda \}.$$

Then G_λ is a (crisp) subgroup of G.

Proof. Since $\lambda \leq \mu(e)$ and $\mu(e) \geq \mu(e)$, the identity $e \in G_\lambda$. If $x, y \in G_\lambda$, then

$$\mu(x \cdot y) \ \geq\ \min\{\mu(x), \mu(y)\} \ \geq\ \lambda, \quad \mu(x^{-1}) = \mu(x) \ \geq\ \lambda.$$

Thus $x \cdot y \in G_\lambda$ and $x^{-1} \in G_\lambda$, so G_λ is closed under the group operation and inverses. □

Theorem 2.8 (Core as a Subgroup)**.** *Define the* core *of μ by*

$$\mathrm{Core}(\mu) \ =\ \{\, x \in G : \mu(x) = \mu(e) \}.$$

Then $\mathrm{Core}(\mu)$ is a subgroup of G.

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Proof. Because $\mu(e) = \sup_x \mu(x)$, we have $e \in \mathrm{Core}(\mu)$. If $x, y \in \mathrm{Core}(\mu)$ then

$$\mu(x \cdot y) \;\ge\; \min\{\mu(x), \mu(y)\} = \mu(e),$$

and by maximality of $\mu(e)$ this forces $\mu(x \cdot y) = \mu(e)$. Likewise,

$$\mu(x^{-1}) = \mu(x) = \mu(e).$$

Hence $\mathrm{Core}(\mu)$ is closed under products and inverses. □

Theorem 2.9 (Intersection of Fuzzy OffGroups). *Let $\mu_1, \mu_2 : G \to [\Psi, \Omega]$ be fuzzy offgroups such that there exist $a, b \in G$ with*

$$\mu_1(a), \mu_2(a) > 1 \quad \textit{and} \quad \mu_1(b), \mu_2(b) < 0.$$

Define

$$\mu(x) \;=\; \min\{\mu_1(x),\, \mu_2(x)\} \quad (x \in G).$$

Then μ is also a fuzzy offgroup on G.

Proof. For all $x, y \in G$,

$$\mu(x \cdot y) = \min\{\mu_1(x \cdot y), \mu_2(x \cdot y)\} \;\ge\; \min\{\min\{\mu_1(x), \mu_1(y)\}, \min\{\mu_2(x), \mu_2(y)\}\} = \min\{\mu(x), \mu(y)\},$$

and

$$\mu(x^{-1}) = \min\{\mu_1(x^{-1}), \mu_2(x^{-1})\} = \min\{\mu_1(x), \mu_2(x)\} = \mu(x).$$

Moreover, since $\mu_1(a), \mu_2(a) > 1$ and $\mu_1(b), \mu_2(b) < 0$, we have $\mu(a) > 1$ and $\mu(b) < 0$, so the offset condition holds. Thus μ satisfies all requirements of Definition 2.1. □

2.2 Fuzzy OverGroup and Fuzzy UnderGroup

The definitions of the Fuzzy OverGroup and Fuzzy UnderGroup are also provided below. These are subclasses of the Fuzzy OffGroup.

Definition 2.10 (Fuzzy OverGroup). Let $(G, \cdot, e)$ be a group and let $[0, \Omega]$ be an extended interval with $0 < 1 < \Omega$. A *fuzzy overgroup* on G is a mapping

$$\mu \colon G \;\longrightarrow\; [0, \Omega]$$

which simultaneously

1. is an *over-fuzzy set*, i.e. there exists $x \in G$ with

$$\mu(x) \;>\; 1,$$

2. satisfies the fuzzy-subgroup conditions:

$$\mu(x \cdot y) \;\ge\; \min\{\mu(x), \mu(y)\}, \quad \mu(x^{-1}) \;=\; \mu(x) \quad \text{for all } x, y \in G.$$

Thus a fuzzy overgroup extends the notion of a fuzzy subgroup by allowing membership values above 1.

Example 2.11 (Fuzzy OverGroup on $\mathbb{Z}_3$). Let $G = \mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3, and choose the extended interval $[0, \Omega] = [0, 1.4]$. Define

$$\mu(k) = \begin{cases} 1.4, & k = 0, \\ 0.8, & k = 1, 2. \end{cases}$$

We check the conditions of Definition 2.10:

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- **Over-fuzzy condition:** $\mu(0) = 1.4 > 1$, so there exists an element with membership above 1.
- **Closure under the group operation:**

$$1 + 2 \equiv 0: \quad \mu(0) = 1.4 \geq \min\{0.8, 0.8\} = 0.8,$$
$$2 + 2 \equiv 1: \quad \mu(1) = 0.8 \geq \min\{0.8, 0.8\} = 0.8,$$
$$1 + 1 \equiv 2: \quad \mu(2) = 0.8 \geq \min\{0.8, 0.8\} = 0.8.$$

- **Inversion invariance:** For each $k \in \{1, 2\}$,

$$-k \equiv (3 - k) \pmod 3, \quad \mu(-k) = \mu(3 - k) = 0.8 = \mu(k).$$

Hence μ defines a fuzzy overgroup on G.

Theorem 2.12 (Maximum Membership at the Identity). *Let $\mu\colon G \to [0, \Omega]$ be a fuzzy overgroup on G. Then*

$$\mu(e) \;=\; \sup_{x \in G} \mu(x), \quad \textit{and in particular} \quad \mu(e) > 1.$$

Proof. For any $x \in G$, using $x \cdot x^{-1} = e$ and the subgroup condition,

$$\mu(e) = \mu(x \cdot x^{-1}) \;\geq\; \min\{\mu(x), \mu(x^{-1})\} = \mu(x).$$

Hence $\mu(e) \geq \mu(x)$ for all x, so equality holds as the supremum. Since μ is an over-fuzzy set, there exists $x_0 \in G$ with $\mu(x_0) > 1$, whence $\mu(e) \geq \mu(x_0) > 1$. □

Theorem 2.13 (α–cut Subgroups). *For any α with $1 < \alpha \leq \mu(e)$, define the "α–cut"*

$$G_\alpha \;=\; \{\, x \in G : \mu(x) \geq \alpha \}.$$

Then G_α is a (crisp) subgroup of G.

Proof. Since $\alpha \leq \mu(e)$ we have $e \in G_\alpha$. If $x, y \in G_\alpha$, then

$$\mu(x \cdot y) \;\geq\; \min\{\mu(x), \mu(y)\} \;\geq\; \alpha, \quad \mu(x^{-1}) = \mu(x) \;\geq\; \alpha.$$

Thus $x \cdot y \in G_\alpha$ and $x^{-1} \in G_\alpha$, so G_α is closed under the group operation and inverses. □

Theorem 2.14 (Intersection of Fuzzy OverGroups). *Let $\mu_1, \mu_2\colon G \to [0, \Omega]$ be fuzzy overgroups, each with some element exceeding 1. Define*

$$\mu(x) \;=\; \min\{\mu_1(x),\, \mu_2(x)\} \quad (x \in G).$$

Then μ is also a fuzzy overgroup on G.

Proof. Since each μ_i has some a_i with $\mu_i(a_i) > 1$, their minimum still satisfies $\mu(a) > 1$ for a chosen appropriately. Moreover, for all $x, y \in G$,

$$\mu(x \cdot y) = \min\{\mu_1(x \cdot y), \mu_2(x \cdot y)\} \;\geq\; \min\{\min\{\mu_1(x), \mu_1(y)\}, \min\{\mu_2(x), \mu_2(y)\}\} = \min\{\mu(x), \mu(y)\},$$

and

$$\mu(x^{-1}) = \min\{\mu_1(x^{-1}), \mu_2(x^{-1})\} = \min\{\mu_1(x), \mu_2(x)\} = \mu(x).$$

Thus μ satisfies all requirements of Definition 2.10. □

Definition 2.15 (Fuzzy UnderGroup). Let $(G, \cdot, e)$ be a group and let $[\Psi, 1]$ be an extended interval with $\Psi < 0 < 1$. A *fuzzy undergroup* on G is a mapping

$$\mu\colon G \;\longrightarrow\; [\Psi, 1]$$

which simultaneously

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1. is an *under-fuzzy set*, i.e. there exists $y \in G$ with

$$\mu(y) \ < \ 0,$$

2. satisfies the fuzzy-subgroup conditions:

$$\mu(x \cdot y) \ \geq \ \min\{\mu(x), \mu(y)\}, \quad \mu(x^{-1}) \ = \ \mu(x) \quad \text{for all } x, y \in G.$$

Thus a fuzzy undergroup extends the notion of a fuzzy subgroup by allowing membership values below 0.

Example 2.16 (Fuzzy UnderGroup on $\mathbb{Z}_3$)**.** Let $G = \mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3, and choose the extended interval $[\Psi, 1] = [-0.3, 1]$. Define

$$\mu(k) = \begin{cases} 1, & k = 0, \\ -0.3, & k = 1, 2. \end{cases}$$

We check the conditions of Definition 2.15:

- **Under-fuzzy condition:** $\mu(1) = \mu(2) = -0.3 < 0$, so there exists an element with membership below 0.
- **Closure under the group operation:**

$$1 + 2 \equiv 0 : \quad \mu(0) = 1 \geq \min\{-0.3, -0.3\} = -0.3,$$

$$2 + 2 \equiv 1 : \quad \mu(1) = -0.3 \geq \min\{-0.3, -0.3\} = -0.3,$$

$$1 + 1 \equiv 2 : \quad \mu(2) = -0.3 \geq \min\{-0.3, -0.3\} = -0.3.$$

- **Inversion invariance:** For each $k \in \{1, 2\}$,

$$-k \equiv (3 - k) \pmod 3, \quad \mu(-k) = \mu(3 - k) = -0.3 = \mu(k).$$

Thus μ defines a fuzzy undergroup on G.

Theorem 2.17 (Maximum Membership at the Identity)**.** *Let $\mu\colon G \to [\Psi, 1]$ be a fuzzy undergroup on G. Then*

$$\mu(e) \ = \ \sup_{x \in G} \mu(x).$$

Proof. Exactly as in Theorem 2.12, using $x \cdot x^{-1} = e$ and noting $\mu(e) \leq 1$ by the range. ☐

Theorem 2.18 (α–cut Subgroups)**.** *For any α with $\Psi < \alpha \leq \mu(e)$, define*

$$G_\alpha \ = \ \{ x \in G : \mu(x) \geq \alpha \}.$$

Then G_α is a (crisp) subgroup of G.

Proof. Identical to the proof of Theorem 2.13. ☐

Theorem 2.19 (Intersection of Fuzzy UnderGroups)**.** *Let $\mu_1, \mu_2\colon G \to [\Psi, 1]$ be fuzzy undergroups, each with some element below 0. Define*

$$\mu(x) \ = \ \min\{\mu_1(x), \, \mu_2(x)\} \quad (x \in G).$$

Then μ is also a fuzzy undergroup on G.

Proof. Since each μ_i has some b_i with $\mu_i(b_i) \ < \ 0$, their minimum still satisfies $\mu(b) \ < \ 0$. The subgroup conditions follow just as in Theorem 2.14. ☐

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3 Conclusion and Future Works

In this paper, we examined the Fuzzy OffGroup and clarified its key properties. In future work, we aim to explore extensions of the Off-Group structure by incorporating the frameworks of hypergroups[16,17] and superhypergroups.[18] We also plan to investigate further extensions using plithogenic sets,[19,20] hesitant fuzzy sets,[21,22] bipolar fuzzy sets,[23,24] picture fuzzy sets,[25] and hyperfuzzy sets.[12,26,27] Furthermore, we hope that future research will develop more concrete mathematical structures for the concepts introduced in this paper.

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Considerations

This work does not involve any experiments or studies involving human participants or animals, and therefore no ethical approvals were required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Research Integrity

The authors hereby confirm that, to the best of their knowledge, this manuscript is their original work, has not been published in any other journal, and is not currently under consideration for publication elsewhere at this stage.

Disclaimer (Note on Computational Tools)

No computer-assisted proof, symbolic computation, or automated theorem proving tools (e.g., Mathematica, SageMath, Coq, etc.) were used in the development or verification of the results presented in this paper. All proofs and derivations were carried out manually and analytically by the authors.

Disclaimer (Limitations and Claims)

The theoretical concepts presented in this paper have not yet been subject to practical implementation or empirical validation. Future researchers are invited to explore these ideas in applied or experimental settings. Although every effort has been made to ensure the accuracy of the content and the proper citation of sources, unintentional errors or omissions may persist. Readers should independently verify any referenced materials.

To the best of the authors' knowledge, all mathematical statements and proofs contained herein are correct and have been thoroughly vetted. Should you identify any potential errors or ambiguities, please feel free to contact the authors for clarification.

The results presented are valid only under the specific assumptions and conditions detailed in the manuscript. Extending these findings to broader mathematical structures may require additional research. The opinions and conclusions expressed in this work are those of the authors alone and do not necessarily reflect the official positions of their affiliated institutions.

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